Self-Dual and Decomposable Geometric Goppa Codes

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Abstract

In this paper, a necessary and sufficient criterion for self-duality of geometric Goppa codes is given.

Resumé

On donne une condition nécessaire et suffisante pour l'autodualité d'un code géométrique de Goppa.

1 Introduction

The aim of this note is to establish a necessary and sufficient criterion for self-duality of geometric Goppa codes. Self-dual codes play an important role in coding theory, so it is of interest to find conditions under which a code is self-dual.

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Here we consider geometric Goppa codes, defined as follows:

Let $\mathcal{X}$ be a projective, nonsingular, absolutely irreducible curve of genus g defined over the finite field $\mathbf{F}_q$. Let $\mathbf{F}_q(\mathcal{X})$ be the function field of $\mathcal{X}$ over $\mathbf{F}_q$ and $\Omega_{\mathcal{X}}$ the vector space of rational differential forms on $\mathcal{X}$ over $\mathbf{F}_q$.

Let $\mathcal{P} = \{P_1, \dots, P_n\}$ be a set of n distinct rational points on the curve $\mathcal{X}$. We fix the order of the P_i and denote by D the divisor $P_1 + \dots + P_n$. For a rational divisor G on $\mathcal{X}$ of degree m and support disjoint from $\mathcal{P}$ we consider the vector spaces $\mathcal{L}(G) = \{f \in \mathbf{F}_q(\mathcal{X})^* \mid (f) \geq -G\} \cup \{0\}$ and $\Omega(G) = \{\omega \in \Omega_{\mathcal{X}} \setminus \{0\} \mid (\omega) \geq G\} \cup \{0\}$. The algebraic-geometric or geometric Goppa codes associated to $\mathcal{P}$ and G over $\mathbf{F}_q$ are defined by

$$\begin{aligned} C_L(\mathcal{X}, \mathcal{P}, G, \mathbf{F}_q) &= \{(f(P_1), \dots, f(P_n)) \mid f \in \mathcal{L}(G)\}, \\ C_{\Omega}(\mathcal{X}, \mathcal{P}, G, \mathbf{F}_q) &= \{(\text{res}_{P_1}(\omega), \dots, \text{res}_{P_n}(\omega)) \mid \omega \in \Omega(G - D)\}, \end{aligned}$$

see Goppa [5], [6], [7] .

For the properties of geometric Goppa codes we refer to the textbook [11]. In order to be able to say something about the dimension and the minimum distance of these codes we restrict to the case $2g - 2 < \deg(G) < n$, in which $C_L(\mathcal{X}, \mathcal{P}, G, \mathbf{F}_q)$ has dimension $\deg(G) + 1 - g$ and at least minimum distance $n - \deg(G)$; and $C_{\Omega}(\mathcal{X}, \mathcal{P}, G, \mathbf{F}_q)$ has dimension $n - \deg(G) - 1 + g$ and at least minimum distance $\deg(G) - 2g + 2$.

In this paper we obtain a necessary and sufficient condition for self-duality of a geometric Goppa code $C_L(\mathcal{X}, \mathcal{P}, G, \mathbf{F}_q)$ in terms of G and D . From now on we write, for short, $C(\mathcal{P}, G)$ instead of $C_L(\mathcal{X}, \mathcal{P}, G, \mathbf{F}_q)$ whenever it is clear which curve and field are meant.

Let us remember that a code $C(\mathcal{P}, G)$ is self-dual if it coincides with its dual $C(\mathcal{P}, G)^{\perp}$, that is with its orthogonal for the bilinear form

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum x_i y_i.$$

Let ω be a differential form on $\mathcal{X}$, and $W = (\omega)$. Since there is an isomorphism $\Omega(G - D) \cong \mathcal{L}(D + W - G)$, we have the following

Proposition 1.1 *There exists a differential form ω on $\mathcal{X}$ with simple poles and residue 1 at every $P_i \in \mathcal{P}$ such that $C(\mathcal{P}, G)^\perp = C(\mathcal{P}, D + W - G)$, where W is the divisor of ω .*

A sufficient condition for self-duality follows from Proposition 1.1

Theorem 1.2 *If there exists a differential form η with simple poles and residue 1 at every $P_i \in \mathcal{P}$ such that $2G = D + K$, where $K = (\eta)$, then $C(\mathcal{P}, G)$ is self dual.*

Proof: Let ω be the differential form cited in 1.1. There exists a rational function f such that $\eta = f\omega$; since both η and ω have simple poles at every P_i it is $v_{P_i}(f) = 0$ and $f(P_i) = \text{res}_{P_i}(\eta)/\text{res}_{P_i}(\omega) = \text{res}_{P_i}(\eta) = 1$. Then $K = W + (f)$, so $D + W - G \sim D + K - G$ and $C(\mathcal{P}, G) = C(\mathcal{P}, D + K - G) = C(\mathcal{P}, D + W - G) = C(\mathcal{P}, G)^\perp$; thus $C(\mathcal{P}, G)$ is self dual. $\square$

This result is due to H. Stichtenoth (see [9]). In some cases the condition stated there is also necessary. For example, in the $g > 1$ case, Y. Driencourt and Stichtenoth (see [3]) gave the following

Theorem 1.3 *Let $g > 1$. Suppose G is a divisor with:*

- a) $G \geq 0$;*
 - b) $\deg(G) \geq 4g - 1$; and*
 - c) $v_P(G) \geq \deg(G) - 2g + 1$ for all places $P \leq G$;*
- then, the converse of 1.2 holds. $\square$*

Later this result has been generalized by C.P. Xing (see [12]) by assuming $n > 6g - 4$ only.

However the converse of 1.2 is not always true. For instance, in the same paper [3] a counterexample with $g = 1$ and $n = 4 = 2g + 2$ is found. Moreover, as shown in [8], it is possible to get a similar example over $\mathbf{F}_q$, where q is an even power of 2, with a code arising from an hyperelliptic curve of genus $g = \frac{1}{2}q - 1$, and of length $n = 2g + 2$.

Furthermore, the converse of Theorem 1.2 is not true for so-called *decomposable* codes, which we study in the next section. In section 3 we prove the converse of (1.2). Both sections, 2 and 3, are based on the paper [8] by C. Munuera and R. Pellikaan. Self-dual Goppa codes have been studied in [1], [2], [3] and [9] too.

2 Decomposable Codes

Definition 2.1 If C_1 is an $[n_1, k_1]$ code, and C_2 is an $[n_2, k_2]$ code, then we say that C is the *direct sum* of C_1 and C_2 if (up to reordering of coordinates)

$$C = \{(\mathbf{x}, \mathbf{y}) \mid \mathbf{x} \in C_1, \mathbf{y} \in C_2\}.$$

We denote this by $C = C_1 \oplus C_2$. If moreover C_1 and C_2 are nonzero, then we say that C *decomposes into* C_1 and C_2 . We call a linear code C *decomposable* if there exist nonzero codes C_1 and C_2 such that C decomposes into C_1 and C_2 .

Remark 2.2 1) The code C decomposes into C_1 and C_2 if and only if (up to reordering of coordinates) C has a generator matrix of the form

$$M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix},$$

where M_1 and M_2 are nonempty generator matrices for C_1 and C_2 , respectively.

2) If C decomposes into an $[n_1, k_1, d_1]$ and an $[n_2, k_2, d_2]$ code, then obviously C is an $[n_1 + n_2, k_1 + k_2, d]$ code, where $d = \min(d_1, d_2)$. Hence

$$d \leq \frac{n - k}{2} + 1$$

by the Singleton bound. Thus there are no MDS decomposable codes.

Example 2.3 Let C be a code of minimum distance one and length greater than one. If $\mathbf{x}$ is a codeword of weight one and $\{\mathbf{x}, \mathbf{y}_1, \dots, \mathbf{y}_{k-1}\}$ is a basis of C , then so is $\{\mathbf{x}, \mathbf{y}_1 - \lambda_1 \mathbf{x}_1, \dots, \mathbf{y}_{k-1} - \lambda_{k-1} \mathbf{x}_1\}$. Thus C has a generator matrix of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & M' \end{pmatrix},$$

where M' is an $(n - 1) \times (k - 1)$ matrix, so C is decomposable. We say that codes of minimum distance one and length greater than one are *trivial decomposable*.

In the following we will discuss nontrivial decomposable geometric Goppa codes.

Proposition 2.4 *Let G be a divisor such that $\deg(G) < n$. Then the following statements are equivalent:*

- 1) $C(\mathcal{P}, G)$ is a decomposable code.
- 2) There are two nonzero effective divisors D_1 and D_2 such that $D_1 + D_2 = D$, both $\mathcal{L}(G - D_1)$ and $\mathcal{L}(G - D_2)$ are not zero, and $\mathcal{L}(G) = \mathcal{L}(G - D_1) \oplus \mathcal{L}(G - D_2)$.

Proof: We have $\mathcal{L}(G - D_1) \cap \mathcal{L}(G - D_2) = \mathcal{L}(G - D) = (0)$, since $\deg(G) < n$. Both $\mathcal{L}(G - D_1)$ and $\mathcal{L}(G - D_2)$ are subspaces of $\mathcal{L}(G)$. If C is decomposable, then there exist two effective divisors D_1 and D_2 such that $D_1 + D_2 = D$, and a basis $\{f_1, \dots, f_s, g_1, \dots, g_t\}$ with $(f_i) \geq D_2$ for $i = 1, \dots, s$ and $(g_i) \geq D_1$ for $i = 1, \dots, t$, so $\{f_1, \dots, f_s\} \subseteq \mathcal{L}(G - D_2)$ and $\{g_1, \dots, g_t\} \subseteq \mathcal{L}(G - D_1)$ and (2) is proved. Conversely, if $\mathcal{L}(G - D_1) = \langle g_1, \dots, g_s \rangle$ and $\mathcal{L}(G - D_2) = \langle f_1, \dots, f_t \rangle$, then $\mathcal{L}(G) = \langle f_1, \dots, f_s, g_1, \dots, g_t \rangle$ so $C(\mathcal{P}, G)$ decomposes into $C(\text{supp}(D_1), G - D_2)$ and $C(\text{supp}(D_2), G - D_1)$. $\square$

Corollary 2.5 *If $\deg(G) < n$ and $C(\mathcal{P}, G)$ is a decomposable code of dimension k , then $k \leq \ell(2G - D) + 1$.*

Proof: For two effective divisors E_1, E_2 we have $\ell(E_1) + \ell(E_2) \leq \ell(E_1 + E_2) + 1$, see [11]. According to Proposition 2.4 there are two effective divisors D_1 and D_2 such that $D = D_1 + D_2$ and $\ell(G - D_1) > 0$ and $\ell(G - D_2) > 0$, so

$$k = \ell(G - D_1) + \ell(G - D_2) \leq \ell(2G - D) + 1.$$

$\square$

Corollary 2.6 *If $\deg(G) < n$ and $C(\mathcal{P}, G)$ is nontrivial decomposable, then $n \leq 2g + 2$.*

Proof: Let $m = \deg(G)$. Suppose $C(\mathcal{P}, G)$ decomposes into $C(\text{supp}(D_1), G)$ and $C(\text{supp}(D_2), G)$, with $n_1 = \deg(D_1)$ and $n_2 = \deg(D_2)$. We may assume that $n_1 \leq n_2$. It is not possible that both $m - n_1 < 0$ and $m - n_2 < 0$, otherwise $\ell(G - D_i) = 0$ for $i = 1, 2$, so $\ell(G) = 0$, by Proposition 2.4. But the code is not zero. Now there are several cases.

Case 1a: If $m - n_1 > 2g - 2$ and $m - n_2 < 0$, then $m + 1 - g \leq m - n_1 + 1 - g$, by Proposition 2.4 and the Riemann-Roch Theorem. Hence $n_1 = 1$. Thus the code is trivial decomposable, which is a contradiction.

Case 1b: If $m - n_1 > 2g - 2$ and $0 \leq m - n_2 \leq 2g - 2$, then

$$m + 1 - g \leq m - n_1 + 1 - g + \frac{m - n_2}{2} + 1,$$

by Proposition 2.4, the Riemann-Roch theorem and Clifford's theorem. So $n + n_1 \leq m + 2$. Moreover $\deg(G) < n$, hence $m = n - 1$ and $n_1 = 1$. Thus the code is trivial decomposable, which is a contradiction.

Case 2: If $m - n_2 > 2g - 2$ then

$$m + 1 - g = m - n_1 + 1 - g + m - n_2 + 1 - g,$$

so $n + g = m + 1$. Moreover $\deg(G) < n$, hence $m = n - 1$ and $g = 0$. Thus the code is MDS and decomposable, so it is trivial decomposable, which is a contradiction.

Case 3a: If $0 \leq m - n_1 \leq 2g - 2$ and $m - n_2 < 0$.

Case 3b: If $m - n_1 \leq 2g - 2$ and $0 \leq m - n_2$, then $0 \leq m - n_i \leq 2g - 2$ for $i = 1, 2$, since $n_1 \leq n_2$ and either $0 \leq m - n_1$ or $0 \leq m - n_2$. Thus

$$m + 1 - g \leq \frac{m - n_1}{2} + 1 + \frac{m - n_2}{2} + 1,$$

so $n \leq 2g + 2$. $\square$

Example 2.7 Let G be a divisor of degree m on a curve $\mathcal{X}$ of genus g . Suppose $m < n = 2g + 2$. If

1) G has degree $2g$ and there are two nonzero effective divisors D_1 and D_2 such that $D_1 + D_2 = D$ and $G - D_1$ is canonical and $G - D_2$ is principal, or
2) G is not special and $\mathcal{X}$ is hyperelliptic and there are two nonzero effective divisors D_1 and D_2 such that $D_1 + D_2 = D$ and both $G - D_1$ and $G - D_2$ are hyperelliptic divisors,
then $C(D, G)$ is decomposable.

This is a direct consequence of Proposition 2.4.

Conversely, if $C(\mathcal{P}, G)$ is a decomposable code of length $n = 2g + 2$ and dimension $k = g + 1$, it is $\deg(G) = 2g$, so according to 2.5 it is

$$\begin{aligned} \deg(2G - D) &= 2g - 2 \\ \ell(2G - D) &= g \end{aligned}$$

so $2G - D$ is a canonical divisor with simple poles at every point in $\mathcal{P}$, $2G - D = (\omega)$. If $C(\mathcal{P}, G)$ decomposes about the codes made from the points in $\mathcal{P}_1, \mathcal{P}_2$ ($\mathcal{P}_1 \cap \mathcal{P}_2 = \emptyset, \mathcal{P}_1 \cup \mathcal{P}_2 = \mathcal{P}$), in such a way that the residues of ω are equal to λ_1 at every point in $\mathcal{P}_1$, and the residues of ω are equal to λ_2 at every point in $\mathcal{P}_2$, with $\lambda_1 \neq \lambda_2$, then according to 1.2, the code is self-dual but the converse of 1.2 fails.

In the same way the converse of 1.2 is not true, in general, for self-dual decomposable geometric Goppa codes.

3 Equality of codes and equivalence of divisors

In order to prove the converse of 1.2 and according to the results obtained in the last section, we must restrict ourselves to $n > 2g + 2$ (and $\deg(G) < n$), in which case there are no decomposable nontrivial geometric Goppa codes.

The problem we want to solve, can be viewed as a particular case of the following: given two divisors G, H on $\mathcal{X}$ with support disjoint from $\mathcal{P}$, and such that $C(\mathcal{P}, G) = C(\mathcal{P}, H)$, then, which relationship between G and H is there?

This problem has been treated by C.P. Xing in [12]. In his work he assumes $2g - 1 < \deg(G) = \deg(H) < n - 1$ and $n > 2g + 2$. In this case he obtains

Proposition 3.1 *Suppose $n > 2g + 2$. Let G and H be two effective divisors with support disjoint from $\mathcal{P}$ on the curve $\mathcal{X}$. If $2g - 1 < \deg(G), \deg(H) < n - 1$, then $C(\mathcal{P}, G) = C(\mathcal{P}, H)$ if and only if $G = H$.*

However equality is not the most general situation in which two divisors provide the same code as we shall see. Firstly we define a special case of linear equivalence of divisors

Definition 3.2 1) Two divisors G, H on $\mathcal{X}$ are called *equivalent about $\mathcal{P}$* , denoted $G \sim_{\mathcal{P}} H$, if there exists a rational function f such that $f(P_i) = 1$

for every $P_i \in \mathcal{P}$ and $H - G = (f)$.

2) Let C be a linear code in $\mathbf{F}_q^n$ and $\lambda = (\lambda_1, \dots, \lambda_n)$ an n -tuple of nonzero elements in $\mathbf{F}_q$. For every $\mathbf{x} = (x_1, \dots, x_n) \in C$, define $\lambda\mathbf{x} = (\lambda_1 x_1, \dots, \lambda_n x_n)$ and $\lambda C = \{\lambda \mathbf{x} \mid \mathbf{x} \in C\}$.

Remark 3.3 We can view λ as a linear map in $\mathbf{F}_q^n$ leaving the Hamming metric invariant. Note that a linear map of $\mathbf{F}_q^n$ leaves the Hamming metric invariant if and only if it is of the form $\lambda\sigma$ for some permutation σ of $\{1, \dots, n\}$.

The relation between the notion of equivalence about $\mathcal{P}$ and our problem is given by the following property (see [10])

Proposition 3.4 *Let G, H be two divisors on $\mathcal{X}$ with support disjoint from $\mathcal{P}$. If $G \sim_{\mathcal{P}} H$ then $C(\mathcal{P}, G) = C(\mathcal{P}, H)$.*

Proof: If $H - G = (f)$ then the map

$$\mathcal{L}(H) \longrightarrow \mathcal{L}(G) \quad ; \quad h \mapsto fh$$

is an isomorphism of vector spaces. Thus $\mathcal{L}(G) = \{fh \mid h \in \mathcal{L}(H)\}$, so $C(\mathcal{P}, G) = (f(P_1), \dots, f(P_n))C(\mathcal{P}, H) = (1, \dots, 1)C(\mathcal{P}, H) = C(\mathcal{P}, H) \quad \square$

In the rest of this section we shall prove the converse of 3.4 .

Proposition 3.5 *Assume $n > 2g + 2$ and $2g - 1 < \deg(G) < n - 1$. If there is a word in $C(\mathcal{P}, G)$ with weight n then $C(\mathcal{P}, G) = C(\mathcal{P}, H)$ if and only if $G \sim_{\mathcal{P}} H$*

Proof: After 3.4 we have to prove only one direction of the assertion. Let $(x_1, \dots, x_n) \in C(\mathcal{P}, H)$ be a codeword of weight n and let $f \in \mathcal{L}(G), h \in \mathcal{L}(H)$ be the functions such that $f(P_i) = x_i = h(P_i)$ for every $P_i \in \mathcal{P}$. Let us consider the divisors $G' = G + (f), H' = H + (h)$. It is clear that G' and H' are effective divisors with support disjoint with $\mathcal{P}$. Furthermore, as we have seen in 3.4 it is

$$C(\mathcal{P}, G) = (f(P_1), \dots, f(P_n))C(\mathcal{P}, G') = (x_1, \dots, x_n)C(\mathcal{P}, G')$$

and

$$C(\mathcal{P}, H) = (h(P_1), \dots, h(P_n))C(\mathcal{P}, H') = (x_1, \dots, x_n)C(\mathcal{P}, H')$$

so

$$(x_1, \dots, x_n)C(\mathcal{P}, G') = (x_1, \dots, x_n)C(\mathcal{P}, H')$$

that is $C(\mathcal{P}, G') = C(\mathcal{P}, H')$. Now, according to 3.1, it is $G' = H'$, so $G - H = (h/f)$, and $G \sim_{\mathcal{P}} H$ $\square$

Remark 3.6 The above proposition is not true without the restriction on the degree of the divisors $2g - 1 < \deg(G) < n - 1$ as we can see as follows: let K be a canonical divisor and P, Q two rational points not in $\mathcal{P}$. Since $\ell(K + P) = \ell(K + Q) = \ell(K)$ it is $\mathcal{L}(K + P) = \mathcal{L}(K + Q)$ so $C(\mathcal{P}, K + P) = C(\mathcal{P}, K + Q)$ but $K + P \not\sim K + Q$.

The result stated in 3.5 seems to depend on the existence of codewords with weight large enough but, in fact, this condition is superfluous.

Proposition 3.7 *If C is a code in $\mathbf{F}_q^n$ not contained in a coordinate hyperplane, and $n < q$, then there is a codeword in C with weight n .*

Proof: For every $i = 1, \dots, n$ let $C_i = C \cap (x_i = 0)$. Since C is not contained in any hyperplane of the form $x_i = 0$, it is $C_i \neq C$ for all i . If there is not a codeword with weight n then $C \subseteq \cup_{i=1}^n C_i$, so, by taking cardinalities we have $q^k \leq nq^{k-1}$ (where $k = \dim C$), that is $q \leq n$ $\square$

Corollary 3.8 *If $2g - 1 < \deg(G) < n - 1$ and $n < q$, then there exists in $C(\mathcal{P}, G)$ a codeword with weight n .*

Proof: If $C(\mathcal{P}, G)$ is contained in a hyperplane $x_i = 0$, then P_i is a base point for G , but since $2g - 1 < \deg(G)$, G is base point free $\square$

For given a positive integer r let us consider the vector space over $\mathbf{F}_{q^r}$

$$\mathcal{L}(G, \mathbf{F}_{q^r}) = \{f \in \mathbf{F}_{q^r}(\mathcal{X})^* \mid (f) \geq -G\} \cup \{0\}$$

and the code $C(\mathcal{X}, \mathcal{P}, G, \mathbf{F}_{q^r})$, image of the map

$$\mathcal{L}(G, \mathbf{F}_{q^r}) \longrightarrow \mathbf{F}_{q^r}^n ; f \mapsto (f(P_1), \dots, f(P_n)).$$

Lemma 3.9 *For every divisor G rational over $\mathbf{F}_q$, it is $\dim_{\mathbf{F}_q}(\mathcal{L}(G, \mathbf{F}_q)) = \dim_{\mathbf{F}_{q^r}}(\mathcal{L}(G, \mathbf{F}_{q^r}))$.*

Proof: See [11], chap. (2.31) $\square$

Lemma 3.10 *Let G, H be two divisors on $\mathcal{X}$, rational over $\mathbf{F}_q$. If $C(\mathcal{X}, \mathcal{P}, G, \mathbf{F}_q) = C(\mathcal{X}, \mathcal{P}, G, \mathbf{F}_q)$ then $C(\mathcal{X}, \mathcal{P}, G, \mathbf{F}_{q^r}) = C(\mathcal{X}, \mathcal{P}, G, \mathbf{F}_{q^r})$.*

Proof: Let $\{f_1, \dots, f_k\}$ be a basis for $\mathcal{L}(G, \mathbf{F}_q)$. Since $\mathcal{L}(G, \mathbf{F}_q) \subseteq \mathcal{L}(G, \mathbf{F}_{q^r})$, according to 3.9, it is a basis for $\mathcal{L}(G, \mathbf{F}_{q^r})$ too, so both, $C(\mathcal{X}, \mathcal{P}, G, \mathbf{F}_q)$ and $C(\mathcal{X}, \mathcal{P}, G, \mathbf{F}_{q^r})$ have the same generator matrix $\square$

At this moment we are able to prove the following

Theorem 3.11 *Assume $n > 2g + 2$. If $2g - 1 < \deg(G), \deg(H) < n - 1$ then $C(\mathcal{P}, G) = C(\mathcal{P}, H)$ if and only if $G \sim_{\mathcal{P}} H$.*

Proof: After 3.4 it is enough to prove the 'only if' part. Assume $C(\mathcal{P}, G) = C(\mathcal{P}, H)$ and take an integer r such that $q^r > n$. According to 3.7 there is a codeword of weight n in $C(\mathcal{X}, \mathcal{P}, G, \mathbf{F}_{q^r})$, so from 3.5 it is $G \sim_{\mathcal{P}} H$ over $\mathbf{F}_{q^r}$. Thus $\ell(G - H, \mathbf{F}_{q^r}) = 1$, and there is a function $f \in \mathcal{L}(G - H, \mathbf{F}_{q^r})$ such that $f(P_1) = \dots = f(P_n) = 1$. Since G and H are rational over $\mathbf{F}_q$ it is $\ell(G - H, \mathbf{F}_q) = 1$ according to 3.9, so there exists a nonzero function $f' \in \mathcal{L}(G - H, \mathbf{F}_q)$, so $f = \lambda f'$ for some $\lambda \in \mathbf{F}_{q^r}^*$, so $f'(P_1) = 1/\lambda$, and (after scaling f' if necessary) we can assume $f'(P_1) = 1$, so $\lambda = 1$, so $f = f'$, and finally $G \sim_{\mathcal{P}} H$ over $\mathbf{F}_q$. $\square$

Let us return now to our problem of settling the converse of 1.2 for the self-duality of geometric Goppa codes.

As we know, the dimension of a code $C(\mathcal{P}, G)$ is $k = \ell(G) - \ell(G - D)$, thus,

- a) if $\deg(G) \geq n$ then $k \geq n - g$;
- b) if $2g - 2 < \deg(G) < n$ then $g \geq k \geq n - g$; and
- c) if $\deg(G) \leq 2g - 2$ then $k \leq g$;

(these results follow from Riemann-Roch and Clifford's theorems) so then, taking into account that $k = \frac{n}{2}$ when $C(\mathcal{P}, G)$ is self dual, it is $k = \frac{n}{2}$ $2g - 2 < \deg(G) < n$, and then $\frac{n}{2} = k = \ell(G) = \deg(G) + 1 - g$. Since we are assuming $n > 2g + 2$, in fact it is $2g - 1 < \deg(G) < n - 1$ and we can apply theorem 3.11 (note that when $n > 2g + 2$ every self-dual geometric Goppa code has minimum distance $d > 1$, so it is not trivial decomposable). Thus we obtain the following

Theorem 3.12 *Assume $n > 2g + 2$. The code $C(\mathcal{P}, G)$ is self dual if and only if there exists a differential form η with simple poles and residue 1 at every $P_i \in \mathcal{P}$ such that $2G = D + K$, where $K = (\eta)$.*

Proof: Since $C^\perp(\mathcal{P}, G) = C(\mathcal{P}, D + W - G)$, where $W = (\omega)$, ω with simple poles and residue 1 at every $P_i \in \mathcal{P}$, according to 3.11 there is a rational function f such that $f(P_i) = 1$ for all $P_i \in \mathcal{P}$ and $2G = D + W + (f)$. Then it is enough to take $\eta = f\omega$ $\square$

When $n \leq 2g + 2$ the conclusion of the theorem fails as said in sections 1 and 2.

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