# On special divisors and the two variable zeta function of algebraic curves over finite fields

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**Abstract**

The gonality sequence of a plane curve is computed. A two variable zeta function for curves over a finite field is defined and the rationality and a functional equation are proved.

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## 1 Introduction

Geometric Goppa codes, also called algebraic-geometric codes, have larger dimension if one takes special divisors. Over an algebraically closed field one has the Brill-Noether bound for the existence of special divisors, but this bound is no longer valid over a finite field [12]. Abundant codes were introduced to improve the miminum distance of algebraic-geometric codes by the notion of the gonality of a curve [12]. This notion was extended to the gonality sequence and applied to generalized Hamming weights of AG codes [14, 10]. In Section 2 we compute the gonality sequence of a non-singular plane curve with at least one rational point using a theorem of M. Noether.

The zeta function of a curve over a finite field can be viewed as the generating function counting the number of effective divisors. We introduce a two variable zeta function $Z(t, u)$ as a generating function counting divisor classes of

a given degree and dimension. It is straightforward to prove that this two variable zeta function is a rational function with $(1-t)(1-ut)$ in the denominator and a two variable polynomial in the numerator, which has integer coefficients, has degree $2g$ in $t$ and degree $g$ in $u$, where $g$ is the genus of the curve. By the correct choice of the definition it satisfies a functional equation. It is not clear what the analog of the "Riemann hypothesis" should be, nor whether it has a cohomological interpretation. Substituting $q$ for $u$ in the two variable zeta funtion gives the one variable zeta function, where $q$ is the number of elements of the finite field. In general the two variable zeta function is not determined by the ordinary zeta function, but it does for the class of hyperelliptic curves.

In this paper we raise more questions than giving answers. But we think that it is of importance in the study of codes and curves, since many questions can be settled if more is known about special divisors as we mentioned at the begining.

## 2 Special divisors

**Definition 2.1.** The dimension $l(G)$ of a divisor $G$ of degree $m$ on a curve of genus $g$ is at least $m+1-g$, by the Theorem of Riemann. An effective divisor is called *special* in case $l(G) > m+1-g$, and $i(G) = l(G)-m-1+g$ is called the *index of speciality*, furthermore $i(G) = l(K-G)$ for every canonical divisor $K$, by the Theorem of Riemann-Roch. The *gonality* [12] of a curve over a field $\mathbb{F}$ is the smallest degree of a non-constant morphism from the curve to the projective line and which is defined over $\mathbb{F}$; in other words, it is the smallest degree of a divsor $G$ such that $l(G) > 1$. More generally, the *gonality sequence* $(\gamma_k | k \in \mathbb{N})$ of a curve is defined by [14]:

$$\gamma_k = \min \{\deg(G) \mid l(G) \geq k\}.$$

Thus $\gamma_1 = 0$ and $\gamma_2$ is the gonality of the curve.

In order to compute the gonality sequence of plane curves we need the following theorem.

**Theorem 2.2** *Let $\mathcal{X}$ be a non-singular plane curve of degree $d$ and genus $g$ over a perfect field $\mathbb{F}$. Let $G$ be a divisor of degree $m$ and dimension $k$ which is defined over $\mathbb{F}$. Then:*
*(i) If $m > d(d-3)$, then $k = m+1-g$.*
*(ii) If $0 \leq m \leq d(d-3)$, then write $m = jd-i$ with $0 \leq j < d-3$ and $0 \leq i < d$, one has:*

$$k \leq \begin{cases} \frac{1}{2}j(j+1) & if \quad i > j \\ \frac{1}{2}(j+1)(j+2) - i & if \quad 0 \leq i \leq j. \end{cases}$$

*Proof.* See [11, 4, 7]. □

**Remark 2.3.** Theorem 2.2 was claimed by M. Noether [11] with an incomplete proof and later proved by Ciliberto [4] over the complex numbers and in greater generality for integral Gorenstein plane curves over any algebraically closed field by Hartshorne [7]. The theorem holds over perfect fields, so in particular over finite fields, because for perfect fields $\mathbb{F}$ the dimension of a divisor $G$ which is defined over $\mathbb{F}$ does not change if we consider the dimension of $G$ over the algebraic closure of $\mathbb{F}$, see [5, 13]. Noether's Theorem gives the maximal dimension of a divisor of a given degree $m$ on a non-singular plane curve of degree $d$ in terms of $m$ and $d$. In the following corollary we consider the inverse function, we give the minimal degree of a divisor of a given dimension on a plane curve. Thus the gonality sequence is completely determined for plane curves.

**Corollary 2.4** *Let $\mathcal{X}$ be a non-singular plane curve of degree $d$ and genus $g$ over a perfect field with at least one rational point. Write $k = \frac{1}{2}(j+1)(j+2) - i$ with $0 \leq i \leq j$. Then:*

$$\gamma_k = \begin{cases} jd - i & if \quad 1 \leq k \leq g \\ \\ k + g - 1 & if \quad k > g \end{cases}$$

*Proof.* If $k > g$, then all divisors of dimension $k$ have degree $k + g - 1$, so $\gamma_k = k + g - 1$. The genus $g$ of a non-singular plane curve of degree $d$ is equal to $\frac{1}{2}(d-1)(d-2)$.

Now we consider the case $1 \leq k \leq g$. We can write $k$ in a unique way as $k = \frac{1}{2}(j+1)(j+2) - i$ with $0 \leq i \leq j \leq d-3$. Suppose there exists a divisor $G$ of degree $m$ and dimension $k$ which is at least 1 and at most $g$. We can write $m = j'd - i'$ with $0 \leq j' \leq d-3$ and $0 \leq i' < d$.

Consider the total ordering on pairs $(a, b)$ of integers as follows: $(a, b) \leq (a', b')$ is and only if $a < a'$ or $a = a'$ and $b \geq b'$. Remark that the function $\varphi$, defined by $\varphi(a, b) = \frac{1}{2}(a+1)(a+2) - b$, is strictly increasing on the set $\{(a, b) | 0 \leq b \leq a \leq d-3\}$ with respect to the total order. Furthermore the function $\psi$, defined by $\psi(a, b) = ad - b$, is stricly increasing on the set $\{(a, b) | 0 \leq a, 0 \leq b < d\}$ with respect to the total order.

If $i' \leq j'$, then

$$\frac{1}{2}(j+1)(j+2) - i = k \leq \frac{1}{2}(j'+1)(j'+2) - i',$$

by Noether's Theorem. So $j < j'$ or $j = j'$ and $i \geq i'$, by the above remark on the function $\varphi$. Thus $jd - i \leq j'd - i' = m$, by the remark on the function $\psi$.

If $i' > j'$, then

$$\frac{1}{2}j(j+1) < \frac{1}{2}(j+1)(j+2) - i = k \leq \frac{1}{2}j'(j'+1),$$

by Noether's Theorem. So $j < j'$ and therefore $jd - i < j'd - i' = m$.

Thus $\gamma_k \geq jd - i$. Now we prove that in fact $\gamma_k$ is equal to $jd - i$. By assumption there exists a rational point $P$. We may assume, after a projective change of coordinates, that $P = (0 : 1 : 0)$. Let $x$ and $y$ be the rational functions on the curve defined by $x = X/Z$ and $y = Y/Z$, where $X, Y$ and $Z$ are the homogeneous coordinates of the projective plane. Let $H$ be the intersection divisor of the curve with the line defined by the equation $Z = 0$. Then

$$\{x^a y^b \mid a, b \in \mathbb{N}_0,\ a + b \leq j\ \}$$

is a set of linear independent elements of the vector space $L(jH)$ in case $j < d$. So the dimension of the divisor $jH$ is at least $\frac{1}{2}(j+1)(j+2)$. Hence the dimension of the divisor $jH - iP$ is at least $\frac{1}{2}(j+1)(j+2) - i$. The degree of $jH - iP$ is $jd - i$. Therefore $\gamma_k \leq jd - i$.

Thus $\gamma_k = jd - i$. □

# 3 Two variable zeta function

The zeta function $Z(\mathcal{X}, \mathbb{F}_q; t)$ of a curve $\mathcal{X}$ over the finite field $\mathbb{F}_q$ can be viewed as the generating function counting the effective divisors on $\mathcal{X}$ which are defined over $\mathbb{F}_q$, that is to say

$$Z(\mathcal{X}, \mathbb{F}_q; t) = \sum_{n=0}^{\infty} a_n t^n,$$

where $a_n$ is the number of effective divisors on $\mathcal{X}$ of degree $n$. We have seen in Section 1 that special divisors play an increasing role in algebraic-geometric codes. If one tries to introduce a two variable zeta function as the generating function of effective divisors of given degree and dimension by:

$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty} a_{n,k} t^n v^k,$$

where $a_{n,k}$ is the number of effective divisors on $\mathcal{X}$ of degree $n$ and dimension $k$, then it is easy to show that it is a rational function, but it is difficult to prove other properties for this two variable zeta function, in particular there seems to be no straightforward generalization of the functional equation. Two divisors are called *equivalent* if their difference is a principal divisor, that is to say the divisor of a rational function. We can define the degree and the dimension of a

divisor class as the degree and the dimension of one of the divisors in its class, since these numbers are the same for two equivalent divisors. The number of all effective divisors in a given divisor class $\mathcal{D}$ is equal to $(q^{l(\mathcal{D})}-1)/(q-1)$. The number of divisor classes of a given degree is constant and is called the *class number* and denoted by $h$. The original zeta function, which we denote by $Z(t)$, can be rewritten as follows:

$$Z(t)=\sum_{\mathcal{D}}\frac{q^{l(\mathcal{D})}-1}{q-1}t^{\deg(\mathcal{D})},$$

where $\mathcal{D}$ runs over all divisor classes of non-negative degree. The zeta function is a rational function

$$Z(t)=\frac{P(t)}{(1-t)(1-qt)},$$

where $P(t)$ is a polynomial in the variable $t$ with integer coefficients of degree $2g$. Moreover $Z(t)$ satifies the following functional equation:

$$Z(t)=q^{g-1}t^{2g-2}Z(\frac{1}{qt}).$$

One can write

$$P(t)=\sum_{i=0}^{2g}p_it^i,$$

such that $p_i$ is an integer such that $p_0=1, p_{2g}=q^g$ and $p_{2g-i}=q^{g-i}p_i$ for all $0\leq i\leq 2g$. Furtermore $P(1)=h$.

If one looks at the proof of the functional equation for curves of the ordinary zeta function in for instance [9, 13], then one sees that treating $q$ as a variable gives a new zeta function which is rational and satisfies the functional equation.

**Definition 3.1.** Let $\mathcal{X}$ be a non-singular absolutely irreducible curve over the finite field $\mathbb{F}_q$. Define the two variable zeta function $Z(\mathcal{X},\mathbb{F}_q;t,u)$ as follows:

$$Z(\mathcal{X},\mathbb{F}_q;t,u)=\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}b_{n,k}\frac{u^k-1}{u-1}t^n,$$

where $b_{n,k}$ is the number of divisor classes of $\mathcal{X}$ of degree $n$ and dimension $k$. If there is no risk of confusion we denote $Z(\mathcal{X},\mathbb{F}_q;t,u)$ by $Z(t,u)$.

**Remark 3.2.** (1) By the above remark we have that $Z(\mathcal{X},\mathbb{F}_q;t)=Z(\mathcal{X},\mathbb{F}_q;t,q)$ and

$$a_{n,k}=\frac{q^k-1}{q-1}b_{n,k}.$$

(2) The dimension $l(\mathcal{D})$ of a divisor class $\mathcal{D}$ is equal to $n+1-g$ in case $\deg(\mathcal{D}) = n > 2g-2$. Thus $b_{n,n+1-g} = h$ and $b_{n,k} = 0$ for all $k \geq 1$ and $k \neq n+1-g$.

(3) Let $\mathcal{K}$ be the canonical class, then $\deg(\mathcal{K}-\mathcal{D}) = (2g-2) - \deg(\mathcal{D})$ and $l(\mathcal{K}-\mathcal{D}) = l(\mathcal{D}) + g - 1 - \deg(\mathcal{D})$. The operation which sends $\mathcal{D}$ to $\mathcal{K}-\mathcal{D}$ is an involution on the set of all divisor classes of degree between 0 and $2g-2$. Thus $b_{n,k} = b_{2g-2-n,k-n-1+g}$.

**Remark 3.3.** It follows directly from the definition that the gonality sequence can be obtained from $Z(t,u)$ as follows:

$$\gamma_k = \min\{n \mid w_{n,k} > 0\}, \text{ where } w_{n,k} = \sum_{k' \geq k} b_{n,k'}.$$

We will give an example which shows that the original zeta function does not give the gonality sequence.

**Example 3.4.** The two variable zeta function of the projective line is equal to

$$\frac{1}{(1-t)(1-ut)},$$

since all divisors of degree $m$ are equivalent and have dimension $m+1$.
The two variable zeta function of an elliptic curve over a finite field with $N$ rational points over this field is equal to

$$Z(t,u) = \frac{1 + (N-1-u)t + ut^2}{(1-t)(1-ut)},$$

since the class number is equal to $N$ and all divisors of degree $m > 0$ have dimension $m$.

**Proposition 3.5** *If $\mathcal{X}$ is a non-singular, absolutely irreducible curve over the finite field $\mathbb{F}_q$ of genus $g$, then its two variable zeta function is a rational function*

$$Z(t,u) = \frac{P(t,u)}{(1-t)(1-ut)},$$

*where $P(t,u)$ is a polynomial in the variables $t$ and $u$ with integer coefficients of degree $2g$ in $t$ and degree $g$ in $u$. Moreover $Z(t,u)$ satifies the following functional equation:*

$$Z(t,u) = u^{g-1}t^{2g-2}Z(\frac{1}{tu}, u).$$

*One can write*

$$P(t,u) = \sum_{i=0}^{2g} P_i(u)t^i,$$

*such that $P_i(u)$ is a polynomial in the variable $u$ with integer coefficients, $P_0(u) = 1, P_{2g}(u) = u^g$, $deg(P_i(u)) \leq 1+\frac{1}{2}i$ and $P_{2g-i}(u) = u^{g-i}P_i(u)$ for all $0 \leq i \leq 2g$. Furtermore $P(1,u) = h$.*

*Proof.* Example 3.4 shows that the proposition is correct in case $g = 0$. So we may assume that $g > 0$, that is to say $2g - 2 \geq 0$. The proof is verbatim the same as for the ordinary zeta function for curves over the finite field $\mathbb{F}_q$, where one treats $q$ as a variable and denotes it by $u$, see [9, 13], except for the bound on the degree of $P_i(u)$. We will sketch the proof of the rationality and the functional equation. Define

$$F(t,u) = \sum_{\deg(\mathcal{D})>2g-2} u^{l(\mathcal{D})}t^{\deg(\mathcal{D})} - \sum_{\mathcal{D}} t^{\deg(\mathcal{D})}$$

$$G(t,u) = \sum_{0\leq\deg(\mathcal{D})\leq 2g-2} u^{l(\mathcal{D})}t^{\deg(\mathcal{D})}.$$

Then

$$(u-1)Z(t,u) = F(t,u) + G(t,u).$$

Remark 3.2.2 implies that we can rewrite $F(t,u)$ as follows:

$$F(t,u) = hu^{1-g}\sum_{n>2g-2}(ut)^n - h\sum_{n=0}^{\infty}t^n =$$

$$\frac{hu^g t^{2g-1}}{1-ut} - \frac{h}{1-t}.$$

From this and a straightforward calculation follows that $F(t,u)$ satisfies the functional equation.

The operation which sends $\mathcal{D}$ to $\mathcal{K} - \mathcal{D}$ is an involution on the set of all divisor classes of degree between 0 and $2g-2$, by Remark 3.2.3. Thus

$$u^{g-1}t^{2g-2}G(\frac{1}{tu},u) =$$

$$u^{g-1}t^{2g-2}\sum_{\mathcal{D}} u^{l(\mathcal{D})}(ut)^{-\deg(\mathcal{D})} = \sum_{\mathcal{D}} u^{l(\mathcal{D})+g-1-\deg(\mathcal{D})}t^{2g-2-\deg(\mathcal{D})} =$$

$$\sum_{\mathcal{D}} u^{l(\mathcal{K}-\mathcal{D})}t^{\deg(\mathcal{K}-\mathcal{D})} = \sum_{\mathcal{D}'} u^{l(\mathcal{D}')}t^{\deg(\mathcal{D}')} = G(t,u).$$

The summation over $\mathcal{D}$ and $\mathcal{D}'$ is taken over all divisor classes of degree between 0 and $2g-2$.

Therefore $F(t,u)$ and $G(t,u)$ both satisfy the functional equation, so $Z(t,u)$ satisfies the functional equation.

It is immediate from the definition of $F(t,u)$ and $G(t,u)$ that $F(t,1)+G(t,1)=0$, moreover $G(t,u)$ is a polynomial of degree $2g-2$, and from the above expression of $F(t,u)$ as a rational function it follows that $F(t,u)+G(t,u)$ is a rational function with denominator $(1-t)(1-ut)$ and numerator a polynomial $Q(t,u)$ of degree $2g$ which is divisible by $u-1$. Therefore $Q(t,u)=(u-1)P(t,u)$.

The functional equation for $Z(t,u)$ is equivalent with the property $P_{2g-i}(u)=u^{g-i}P_i(u)$ for all $0\leq i\leq 2g$. If we write $G(t,u)=\sum_{i=0}^{2g-2}G_i(u)t^i$, then $G_0(u)=(h-1)+u$, since 0 is the only divisor class of degree 0 and dimension 1, all the other divisor classes of deree 0 have dimension 0. So $P_0(u)=1$ and $P_{2g}(u)=u^{2g}$. The fact that $\deg(P_i(u))\leq 1+\frac{1}{2}i$ follows from Clifford's Theorem [13].

We have that

$$P(t,u)=\frac{(1-t)(1-tu)}{u-1}(F(t,u)+G(t,u))=$$

$$h(1-t)u^g t^{2g-1}-\frac{h(1-ut)}{u-1}+\frac{(1-t)(1-tu)}{u-1}G(t,u)$$

and $G(t,u)$ is a polynomial in $t$ and $u$. Substituting 1 for $t$ gives $P(1,u)=h$. □

**Remark 3.6.** (1) It is not clear what the analog of the Riemann hypothesis for the two variable zeta function should look like. Duursma [6] considers the $L$-series $L(t,\chi)$ of the Hilbert class field of the curve, with $\chi$ a character on the class group of the curve. These satisfy the analog of the Riemann hypothesis, and they determine the two variable zeta function $Z(t,u)$. The trivial character gives the original zeta function $Z(t)$ of the curve. Weight enumerators of AG codes on a given curve can be computed from the $L$-series. Computations have been carried out in extenso for the Klein quartic.

(2) The one variable zeta function over $\mathbb{F}_q$ determines the one variable zeta function over any extension of $\mathbb{F}_q$. Is this also true for the two variable zeta function ?

(3) If we have a ramified cover of $\mathcal{Y}$ over $\mathcal{X}$, then the one variable zeta function of $\mathcal{Y}$ is divisible by the one of $\mathcal{X}$, by [8]. Is there a similar result for the two variable zeta function ? The following example shows that the divisibility property is not valid for the two variable zeta function.

**Example 3.7.** Consider the curves $\mathcal{X}$ and $\mathcal{Y}$ over $\mathbb{F}_2$ defined by

$$\begin{array}{lcl}\mathcal{X} & : & y^2+y=x^3+1,\\ \mathcal{Y} & : & y^4+y=x^3+1.\end{array}$$

Then $\mathcal{Y}$ is an Artin-Schreier cover of $\mathcal{X}$ by the map $(y,x) \mapsto (y^2+y,x)$. But $Z(\mathcal{X},\mathbb{F}_2;t,u)$ has numerator $1+(2-u)t+ut^2$ and $Z(\mathcal{Y},\mathbb{F}_2;t,u)$ has numerator

$$1+(2-u)t+(8-2u)t^2+(10-5u)t^3+(8-2u)ut^4+(2-u)u^2t^5+u^3t^6.$$

# 4 The two variable zeta function of hyperelliptic curves

An absolutely irreducible non-singular curve $\mathcal{X}$ is called *hyperelliptic* if its genus is at least 2 and there exists a morphism $\varphi$ of degree 2 to the projective line, that is to say the gonality of the curve is 2. This morphism is unique and the inverse images of the rational points on the projective line define $q+1$ effective divisors of degree 2, which are called the *hyperelliptic divisors* of the curve. In the following lemma we assume for simplicity that there exists a rational point $P_\infty$ such that $\varphi$ ramifies at this place.

**Lemma 4.1** *Every divisor $D$ on a hyperelliptic curve of genus $g$ is equivalent with a divisor $T+sP_\infty$, where $T$ is an effective divisor of degree $t$ which does not contain a hyperelliptic divisor and does not have $P_\infty$ in its support, and*

$$l(D)=\begin{cases} \lfloor \frac{s}{2} \rfloor +1 & \text{if} \quad 2t+s\le 2g-2,\\ t+s+1-g & \text{if} \quad 2t+s>2g-2.\end{cases}$$

*If moreover $deg(T)\le g$, then $l(T)=1$ and $T$ is unique.*

*Proof.* See [1, 2, 3] □

The following reformulation does not assume the existence of a rational point $P_\infty$ such that $\varphi$ ramifies at this place.

**Lemma 4.2** *Let $H$ be a hyperelliptic divisor on a hyperelliptic curve $\mathcal{X}$ of genus $g$. Every divisor $D$ on $\mathcal{X}$ is equivalent with a divisor $T+rH$, where $T$ is an effective divisor of degree $t$ which does not contain a hyperelliptic divisor, and*

$$l(D)=\begin{cases} r+1 & \text{if} \quad t+r\le g-1,\\ t+2r+1-g & \text{if} \quad t+r>g-1.\end{cases}$$

*If moreover $deg(T)\le g$, then $l(T)=1$ and $T$ is unique.*

□

**Proposition 4.3** *The two variable zeta function of a hyperelliptic curve of genus $g$ is determined by the one variable zeta function in the following way:*

$$b_{n,k}=\begin{cases} a_{n-2k+2}-(q+1)a_{n-2k}+qa_{n-2k-2} & \text{if } 0\le n\le 2g-2,\ k\ge 1,\\ h & \text{if } n>2g-2,\ k=n+1-g,\\ 0 & \text{otherwise.}\end{cases}$$

*Proof.* (i) Suppose $0 \leq n \leq g$. Let $D$ be a divisor of degree $n$ and dimension $k$. Then $D$ is equivalent with a divisor $T+(k-1)H$, where $T$ is an effective divisor of degree $n-2k+2$ and dimension 1 which does not contain a hyperelliptic divisor and $T$ is unique, by Lemma 4.2. Thus $a_{n,k} = (q^k-1)/(q-1)a_{n-2k+2,1}$. We have seen in Remark 3.2.1 that $a_{n,k} = (q^k-1)/(q-1)b_{n,k}$. So $b_{n,k} = a_{n-2k+2,1}$ which is equal to $b_{n-2k+2,1}$. Now we prove with induction on $n$ that $b_{n,1} = a_n - (q+1)a_{n-2} + qa_{n-4}$ for $0 \leq n \leq 2g-2$. If $n=0$, then $b_{0,1} = a_0$, so the formula for $b_{n,1}$ is correct in case $n=0$. Assume that the formula is correct for all $n' < n$. Then

$$a_n = \sum_{k\geq 1} a_{n,k} = \sum_{k\geq 1} \frac{q^k-1}{q-1} b_{n-2k+2,1}.$$

From the induction hypothesis follows that

$$\begin{aligned} a_n &= b_{n,1} + \sum_{k\geq 2}\{a_{n-2k+2} - (q+1)a_{n-2k} + qa_{n-2k-2}\} \\ &= b_{n,1} + (q+1)a_{n-2} - qa_{n-4}. \end{aligned}$$

Thus the formula for $b_{n,1}$ holds. Therefore

$$b_{n,k} = b_{n-2k+2,1} = a_{n-2k+2} - (q+1)a_{n-2k} + qa_{n-2k-2}.$$

(ii) If $g < n \leq 2g-2$, then $b_{n,k} = b_{2g-2-n,k-n-1+g}$, by Remark 3.2.3, which is again equal to $a_{n-2k+2} - (q+1)a_{n-2k} + qa_{n-2k-2}$ by the previous case.
(iii) If $n > 2g-2$, then $b_{n,k}$ is equal to $h$ if $k = n+1-g$ and zero otherwise, by Remark 3.2.2. □

**Example 4.4.** We have seen that for hyperelliptic curves the two variable zeta function is determined by the one varible zeta function. This property is not longer valid for arbitrary curves. Consider the following two maximal curves over $\mathbb{F}_q$ where $q = 7^2$ of genus 3:

$$\begin{array}{lcl} \mathcal{X}_1 & : & x^4 + y^4 + 1 = 0, \\ \mathcal{X}_2 & : & y^2 = x^7 - x. \end{array}$$

The first is an example of a Fermat curve $x^n + y^n + 1 = 0$, and it is maximal, since in this case $n$ divides $\sqrt{q}+1$. The second is maximal because it has the Hermitian curve $y^8 = x^7 - x$ as a fourfold cover, by [8]. Both have the same one variable zeta function with numerator $(1-7t)^6$. The first curve is a plane curve of degree 4 and has gonality 3 by Corollary 2.4, the second curve is hyperelliptic and has gonality 2. The gonality is determined by the two variable zeta function. Thus this example shows that the one variable zeta function does not give enough information to compute the gonality nor the two variable zeta function.

**Update and acknowledgement**

1. (July 1, 1997) In Remark 3.2 the misprint $b_{n,n+1-k}$ is changed into $b_{n,n+1-g}$ after a remark of Gerhard Schiffels.

2. (February 26, 1998) Example 3.7 has been corrected after a remark by Manasé Bezara that the original curves with equations $y^2 = x^3 + 1$ and $y^4 = x^3 + 1$ have both genus zero in characteristic two.

3. (January 6, 1999) The proof of $\gamma_k \geq jd - i$ in Corollary 2.4 has been corrected after a remark by Fernando Torres.

4. (March 28, 2000) Remark 3.3 has been corrected after Gerhard Schiffels. Note that if the curve has a rational point, then $b_{n,k} > 0$ if and only if $w_{n,k} > 0$.

5. (November 21, 2008) Naumann [15] showed that the polynomial $P(t, u)$ is irreducible and pointed out that in Proposition 3.5 and its proof the first 1 in $\deg(P_i(u)) \leq 1 + \frac{1}{2}i$ was missing.

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