A Short Introduction to Nonstandard Analysis

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November 1989

Nonstandard analysis was invented in the early sixties by the logician Abraham Robinson. It is a powerful application of logic (in particular, model theory) to mathematical analysis. Any theorem that can be proved using nonstandard analysis, can also be proved by classical methods. However, nonstandard methods may yield (much) shorter and/or intuitive proofs.

It is not necessary to be acquainted with logic in order to be able to use nonstandard analysis. This situation is analogous to the following: it suffices to know the properties of the real number system in order to do real analysis.

To illustrate how nonstandard analysis works we begin with an elementary problem from calculus. Suppose we are to find a primitive of the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^{ax} \cos bx$ $(a, b \in \mathbb{R})$. One way to solve this problem is as follows: consider the function ${}^*f : \mathbb{C} \to \mathbb{C}$ defined by ${}^*f(z) = e^{(a+bi)z}$ and note that $f = \mathrm{Re}\,{}^*f$. Obviously ${}^*F(z) := \frac{1}{a+bi}\, e^{(a+bi)z}$ is a primitive of ${}^*f$ and $\mathrm{Re}\,{}^*F$ is a primitive of f.
Let us reconsider this method of solution. In order to find a primitive of the function $f : \mathbb{R} \to \mathbb{R}$ we extended this function to a function ${}^*f : \mathbb{C} \to \mathbb{C}$. The problem was easy to solve for the function ${}^*f$ on $\mathbb{C}$ and by taking the real part of ${}^*f$ we found the solution for f. Nonstandard analysis works in a similar way. A field ${}^*\mathbb{R}$, which is a proper extension of $\mathbb{R}$, is constructed. This field ${}^*\mathbb{R}$ has a much richer structure than $\mathbb{R}$. The construction of ${}^*\mathbb{R}$ is such that statements on $\mathbb{R}$ (including e.g. functions on $\mathbb{R}$) can be transferred to ${}^*\mathbb{R}$ and vice-versa.

Definition 1 *Let S be any set. The* ***superstructure*** *$V(S)$ over S is defined to be $\bigcup_{n \in \mathbb{N}} V_n(S)$, where $V_1(S) := S$ and $V_{n+1}(S) := \mathcal{P}\left(\bigcup_{k=1}^{n} V_k(S)\right)$ ($\mathcal{P}$ denotes the power set).*

We will be mainly interested in the superstructure $V(\mathbb{R})$ over $\mathbb{R}$. The superstructure $V(\mathbb{R})$ is sufficiently large to do 'ordinary analysis in it. E.g., subsets of $\mathbb{R}$ are in $V_2(\mathbb{R})$ and real-valued functions on $\mathbb{R}$ are in $V_3(\mathbb{R})$, since functions can be seen as ordered pairs. Banach spaces are also in the superstructure $V(\mathbb{R})$.

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Definition 2 *Let ${}^*\mathbb{R}$ be a set and ${}^*V(\mathbb{R}) \to V({}^*\mathbb{R})$ be a mapping. If ${}^*A \in V({}^*\mathbb{R})$, then*

- *A is called **standard** if $A = {}^*B$ for some $B \in V(\mathbb{R})$.*
- *A is called **internal** if $A \in {}^*B$ for some $B \in V(\mathbb{R})$.*
- *A is called **external** if A is not internal.*

We now introduce a formal language for $V(S)$. Only statements in this language can be transferred from $V(\mathbb{R})$ to $V({}^*\mathbb{R})$ (cf. the example above).
A precise description of this language can be found in the books listed in the bibliography. Loosely speaking, any statement containing the symbols $=$, $\in$, $\wedge$, $\vee$, $\neg$, $\Rightarrow$, $\exists$, $\forall$ and constants and variables denoting elements of $V(S)$ is a sentence of $L(V(S))$. The use of $\in$, $\exists$ and $\forall$ is restricted to $X \in Y$, $\forall X \in Y$ and $\exists X \in Y$, where X and Y are constants or variables for arbitrary elements or sets in $V(S)$. E.g., the sentence $\forall X \in V(S)$ is not in $L(V(S))$. Examples of sentences in $L(V(S))$ are :

$$\forall\, A \subset \mathbb{N}\, \exists\, a \in A\, \forall\, b \in A : a \geq b$$

and if $f : \mathbb{R} \to \mathbb{R}$ is a function:

$$\forall\, x \in \mathbb{R}\, \forall\, \varepsilon > 0\, \exists\, \delta > 0\, \forall\, y \in \mathbb{R} : |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

We are now able to state the key theorem of nonstandard analysis.

Theorem 3 *There exists for each cardinal κ a set ${}^*\mathbb{R}$ and an embedding ${}^* : V(\mathbb{R}) \to V({}^*\mathbb{R})$ such that*

*a) ${}^*\mathbb{R}$ contains $\mathbb{R}$ and ${}^*r = r$ for all $r \in \mathbb{R}$ (**Extension Principle**).*

*b) If $A_1, \ldots, A_n \in V(\mathbb{R})$, then any $L(V(\mathbb{R}))$ statement Φ that is true for $A_1, \ \ldots, A_n$ in $V(\mathbb{R})$ is true for ${}^*A_1, \ldots, {}^*A_n$ in $V({}^*\mathbb{R})$, and conversely (**Transfer Principle**).*

*c) If $\{A_i\}_{i \in I}$ ($\mathrm{card}(I < \kappa)$) is a family of internal sets with the finite intersection property, then $\bigcap_{i \in I} A_i \neq \emptyset$ (κ-**Saturation Principle**).*

Proof: See any book mentioned in the bibliography.

The following corollary intends to give some feeling for the structure of ${}^*\mathbb{R}$ if ${}^*\mathbb{R}$ is κ-saturated with $\kappa > card(\mathbb{N})$.

Corollary 4 *Let ${}^*\mathbb{R}$ be as in Theorem 3 with $\kappa > card(\mathbb{N})$. Then*

a) $< {}^*\mathbb{R}, <, +, . >$ is a complete ordered field.

b) There exists $\omega \in {}^*\mathbb{R}$ such that $\omega >^* x$ for all $x \in \mathbb{R}$.

c) There exist $\eta \in {}^*\mathbb{R}$ such that $\eta > 0$ and $\eta <^* x$ for all positive $x \in \mathbb{R}$.

d) The set $\{{}^*r : r \in \mathbb{R}\}$ is an external subset of ${}^*\mathbb{R}$.

Note that b) implies that $^*\{r : r \in \mathbb{R}\} \neq {}^*\mathbb{R}$.

Proof: a) This follows from applying the Transfer Principle to the corresponding statements for $\mathbb{R}$.
b) The sets $^*\{x \in \mathbb{R} : x > n\}$ with $n \in \mathbb{N}$ are internal and have the finite intersection property. Hence, by the κ -Saturation Principle, $A := \bigcap_{n\in\mathbb{N}} {}^*\{x \in \mathbb{R} : x > n\} \neq \emptyset$. Any $\omega \in A$ suffices.
c) Apply the κ-Saturation Principle to the internal sets $^*\{x \in \mathbb{R} : 0 < x < 1/n\}$ or consider $1/\omega$, where ω is as in the proof of b).
d) For convenience we will identify $\{^*r : r \in \mathbb{R}\}$ with $\mathbb{R}$. Suppose there exists a subset A of $\mathbb{R}$ such that $^*A =^* \mathbb{R}$. We distinguish two possibilities: A is bounded by above by some $x \in \mathbb{R}$. Then by the Transfer Principle, $^*a \in^* A \Rightarrow^* a \leq x$, hence $^*A \neq \mathbb{R}$. If A is not bounded above, then $\forall x \in \mathbb{R}\, \exists a \in A : a > x$. By the Transfer Principle, $\forall^* x \in^* \mathbb{R}\, \exists^* a \in^* A : {}^*a > {}^*x$. It follows from b) that $^*A \neq \mathbb{R}$. □

Definition 5

- *A number* $s \in^* \mathbb{R}$ *is* ***infinite*** *if* $n < |s|$ *for all* $n \in \mathbb{N}$.
- *A number* $s \in^* \mathbb{R}$ *is* ***finite*** *if* $|s| < n$ *for some* $n \in \mathbb{N}$.
- *A number* $s \in^* \mathbb{R}$ *is* ***infinitesimal*** *if* $|s| < 1/n$ *for all* $n \in \mathbb{N}$.

Infinite integers are sometimes called **hyperfinite integers**. If s is infinitesimal we will write $s \approx 0$. If $s, t \in^* \mathbb{R}$ are such that $s - t \approx 0$, then we also write $s \approx t$.

Proposition 6 *If* $\rho \in^* \mathbb{R}$ *is finite, then there exists a unique* $x \in \mathbb{R}$ *such that* $x \approx \rho$ *(*x *is called the* ***standard part*** *of* ρ *, notation* $x = st(\rho)$ *).*

Proof: Define subsets A and B of $\mathbb{R}$ by $A := \{t \in \mathbb{R} : \rho \leq t\}$ and $B := \{t \in \mathbb{R} : t < \rho\}$. Since ρ is finite, B is non-empty and bounded above in $\mathbb{R}$. Let x be the supremum of B. For each $n \in \mathbb{N}$, $x + 1/n \in A$ and $x - 1/n \in B$, so $x - 1/n < \rho \leq x + 1/n$. Hence, $|x - \rho| < 1/n$ for all $n \in \mathbb{N}$. Uniqueness is clear. □

Proposition 7 *Let* $^*\mathbb{R}$ *be as in Theorem 3 with* $\kappa > card(\mathbb{N})$*. A function* $f : [0, 1] \to \mathbb{R}$ *is continuous in* x *if and only if for all* $y \in^* [0, 1]$*,* $y -^* x \approx 0 \Rightarrow {}^*f(^*x) - {}^*f(y) \approx 0$.

Proof: exercise (the Transfer Principle implies that $^*f$ is a function).

We will apply Proposition 7 to give a nonstandard proof of the following well-known theorem:

Theorem 8 *Let* $f : [0, 1] \to \mathbb{R}$ *be continuous. Then* f *attains a maximum on [0,1].*

Proof: Let ω be a hyperfinite natural number and consider the set $S := \{^*f(k/\omega) : k \in {}^*\mathbb{N}$ and $0 \leq k \leq \omega\}$. Since every finite subset of $\mathbb{R}$ has a largest element, the Transfer Principle yields that S has a largest element, say $^*f(n/\omega)$. It is easy to see that $\mathrm{st}(n/\omega) \in [0, 1]$. We claim that $f(x) \leq f(\mathrm{st}(n/\omega))$ for all $x \in [0, 1]$. Let $x \in [0, 1]$ be arbitrary. Choose a $k \in {}^*\mathbb{N}$ such that $k/\omega - {}^*x \approx 0$. Recall that by the Extension Principle, $f(x) = {}^*f(^*x)$. By Proposition 7, $^*f(^*x) \approx {}^*f(k/\omega) \leq {}^*f(n/\omega) \approx {}^*f(^*\mathrm{st}(n/\omega))$. We conclude that either $f(x) < f(\mathrm{st}(n/\omega))$ or $^*f(^*x) \approx {}^*f(^*\mathrm{st}(n/\omega))$. Using the elementary fact that $x, y \in \mathbb{R}$ and $^*x \approx {}^*y$ implies $x = y$, we see that $f(x) \leq f(\mathrm{st}(n/\omega))$. □

There is another form of Nonstandard Analysis due to Nelson (see [5]). The approach by Nelson is axiomatic. See also [4].

Exercises

a) Let ${}^*\mathbb{R}$ be as in Theorem 3 with $\kappa > \mathrm{card}(\mathbb{N})$. Prove that for every real sequence $(a_n)_{n\in\mathbb{N}}$: $\lim_{n\to\infty} a_n = a$ if and only if ${}^*a_\omega \approx a$ for all $\omega \in {}^*\mathbb{N} \setminus \mathbb{N}$.

b) Let $W \in V(\mathbb{R})$ be an infinite dimensional vector space and let ${}^*\mathbb{R}$ be as in Theorem 3 with $\kappa > \mathrm{card}(W)$. Prove that ${}^*W$ is also a vector space and that there exists a linear subspace F of ${}^*W$ such that $\dim(F) \in {}^*\mathbb{N} \setminus \mathbb{N}$ (hint: use saturation).

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