Run lengths of control charts for correlated output of feedback processes

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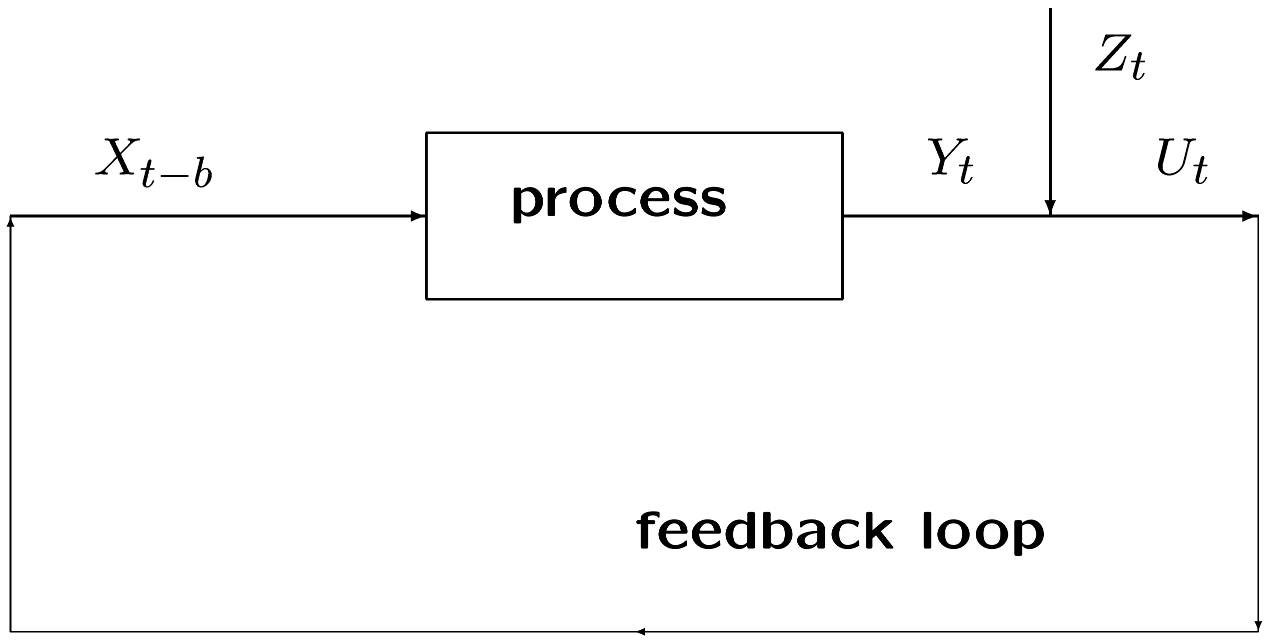
Outline

- Practical problem
- Statistical model
- Run lengths of Shewhart control charts
- Run lengths of CUSUM control charts
- Topics under investigation

Sputter process for optical discs



$Y_t =$ ideal thickness tellurium layer

$Z_t =$ disturbance

$U_t =$ actual thickness tellurium layer

$X_t =$ feedback (MMSE controller)

Simple statistical model

$$
\begin{aligned}
Y_t &= \mu + g\,X_{t-b} \\
U_t &= \mu + g\,X_{t-b} + Z_t \\
Z_t &= \varphi\,Z_{t-1} + a_t,
\end{aligned}
$$

where

- $a_t \sim \mathrm{NID}(0, \sigma_a^2)$.
- X_t is MMSE feedback controller
- g is the gain (known)
- b is the delay time (known)
- U_t is observed

Typical values of model parameters

- $b = 1$

- $X_0 = 1.82$

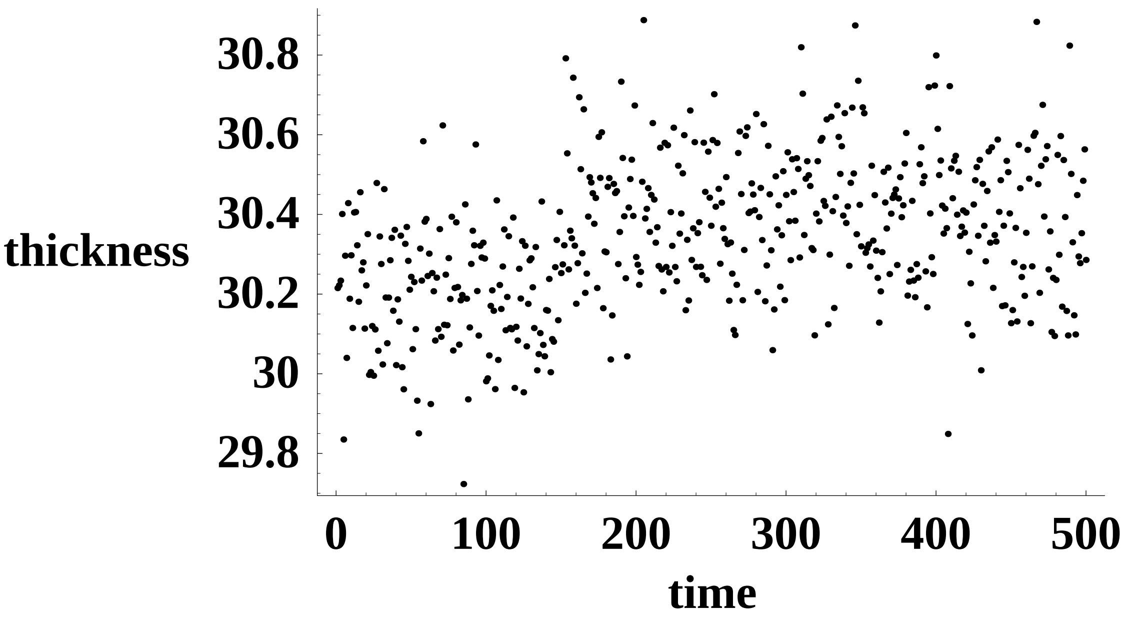
- $\varphi = -.25$

- $g = 16.6$

- $\mu = 30.2$; shifts of size 0.17

- $\sigma_a = 0.17$

Simulation



SPC and APC

Two different approaches to process control:

SPC (Statistical Process Control):

- originated in parts industry
- developed by statisticians
- process monitoring by hypothesis testing

APC (Automatic Process Control):

- originated in process industry
- developed by control engineers
- process adjustment by estimation

Integration of SPC and APC

Advantage SPC: detects special causes

Disadvantage SPC: not all special causes can be eliminated

Advantage APC: direct compensation instead of interrupting process

Disadvantage APC: does not give insight in process / overcompensation

Box and Kramer (1992) suggest to combine SPC and APC:

use control charts on feedback system

Statistical problem: control charts for *correlated* data

General statistical model

$$\begin{aligned} Y_t &= \mu + g\,X_{t-b} \\ U_t &= \mu + g\,X_{t-b} + Z_t \\ \Phi(B)\,Z_t &= \Theta(B)\,a_t, \end{aligned}$$

where

$$\Phi(B) = 1 - \varphi_1\,B - \varphi_2\,B^2 - \ldots \varphi_p\,B^p$$

and

$$\Theta(B) = 1 - \theta_1\,B - \theta_2\,B^2 - \ldots \theta_q\,B^q$$

(B is backward shift operator: $B^k\,a_t := a_{t-k}$).

Assumption: roots of Φ and Θ outside unit circle. Hence,

$$Z_t = \Gamma(B)\,a_t = \left(1 - \sum_{i=1}^{\infty} \gamma_i\,B^i\right) a_t,$$

and

$$a_t = \Xi(B)\,Z_t = \left(1 - \sum_{i=1}^{\infty} \xi_i\,B^i\right) Z_t,$$

Controller X_t

Causal linear filter X_t on error terms a_t:

$$X_t = \sum_{j=0}^{\infty} c_j \, a_{t-j} = \left(\sum_{j=0}^{\infty} c_j \, B^j \right) a_t$$

Note: a_t are not observed: rewrite in terms of observed U_t:

$$X_t = \frac{\sum_{j=0}^{\infty} c_j \, B^j}{1 - \sum_{j=1}^{b-1} \gamma_j \, B^j + \sum_{j=b}^{\infty} \left(g \, c_{j-b} - \gamma_j \right) B^j} U_t$$

Special case: $c_j = \gamma_{b+j}/g$ leads to MMSE control

Influence of shift in mean

Assumption: persistent change of mean of disturbance Z_t occurs at time c and has size δ

$$U_t = \left(1 - \sum_{j=1}^{b-1} \gamma_j \, B^j + \sum_{j=b}^{\infty} \left(g \, c_{j-b} - \gamma_j\right) B^j\right) \cdot \\ (\Xi(B) \, \delta \, I[t \geq c] + a_t)$$

Recall: $a_t = \Xi(B) \, Z_t$

Special case: $c_j = \gamma_{b+j}/g$ leads to MMSE control, hence

$$U_t = \left(1 - \sum_{j=1}^{b-1} \gamma_j \, B^j\right) (\Xi(B) \, \delta \, I[t \geq c] + a_t)$$

Vulnerability of MMSE

Special case white noise disturbance: $\Xi(B) = 1$, hence $X_t = 0$

In general: $Z_t = \Gamma(B)\, a_t$ implies mean of Y_t equals

$$\left(1 + \Xi(B) \sum_{j=0}^{\infty} \gamma_{b+j}\, B^{b+j}\right) \delta\, I[t \geq c]$$

Mean converges to

$$(1 - \xi_1 - \xi_2 - \cdots)\,(1 - \gamma_1 - \cdots - \gamma_{b-1})\,\delta$$

Roots $\Xi(B)$, hence mean disturbance unless $\gamma_1 + \cdots + \gamma_{b-1} = 1$

Conclusion: in general, MMSE control cannot compensate for persistent change of mean

Lag $b = 1$ *versus* $b > 1$

If $b = 1$, then $U_t = a_t$, hence uncorrelated

If $b > 1$, then U_t correlated.

Solution:

When no shift: $e_{U_t}(1) = Z_t - \widehat{Z}_t(t-1) = a_t$.

When shift: $e_{U_t}(1) = \Xi(B)\,\delta\,I[t \geq c] + a_t$.

This is identical to expression for U_t when $b = 1$.

Chart e_t instead of U_t when $b > 1$

Control charts

Use control chart to signal if feedback cannot cope with disturbance.

In-control run length: false alarm rate

Out-of-control run length: power to detect process disturbances

Several types of control charts available:

- Shewhart charts
- CUSUM charts
- ...

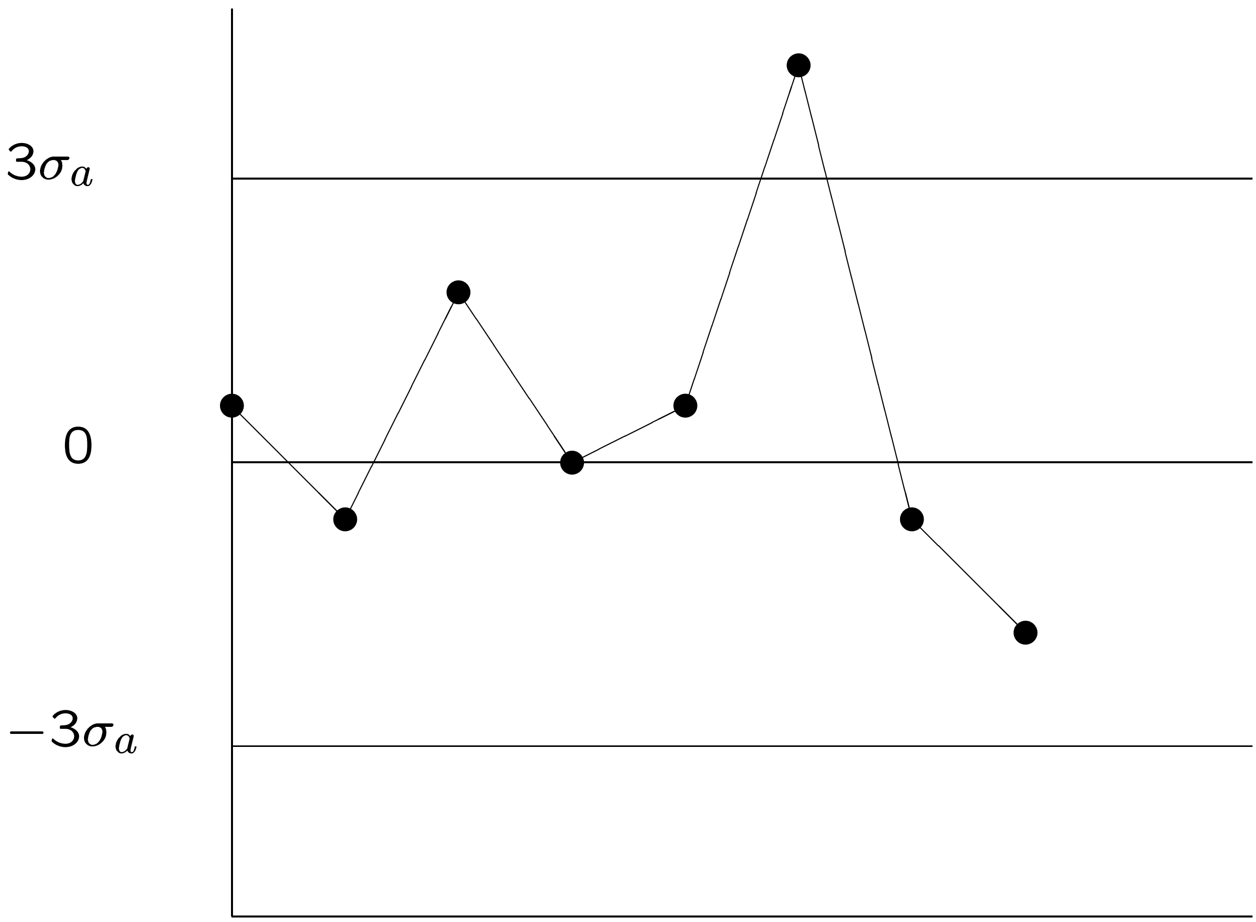
Problem: how do control charts perform when applied to correlated data?

Shewhart chart

Upper control limit : $E\,U_t + 3\sqrt{\mathrm{Var}\,U_t} = 3\,\sigma_a$

Centre line : $E\,U_t = 0$

Lower control limit : $E\,U_t - 3\sqrt{\mathrm{Var}\,U_t} = -3\,\sigma_a$



Run length of Shewhart chart for $b = 1$

$$p_{k+1} = P(|U_{c+k}| > 3\,\sigma_a)$$

$$P(\mathsf{RL}_{\mathsf{out}} \geq j) = \prod_{i=1}^{j-1} (1 - p_i) \text{ for } j = 2, \ldots$$

$$P(z) := \sum_{j=1}^{\infty} P(\mathsf{RL}_{\mathsf{out}} = j)\, z^j$$

$$\widetilde{P}(z) := \sum_{j=1}^{\infty} P(\mathsf{RL}_{\mathsf{out}} \geq j)\, z^j$$

Then

$$P(z) = 1 + \frac{z-1}{z}\,\widetilde{P}(z)$$

$$EX = P'(1) = \widetilde{P}(1)$$

$$\mathrm{Var}(X) = 2\widetilde{P}'(1) - \widetilde{P}(1) - (\widetilde{P}(1))^2.$$

Run length of Shewhart chart: $b = 1$ and AR(p)-disturbances

$$p_{k+1} = 1 - F\left(3 - (1 - \xi_1 - \cdots - \xi_k)\frac{\delta}{\sigma_a}\right) + F\left(-3 - (1 - \xi_1 - \cdots - \xi_k)\frac{\delta}{\sigma_a}\right).$$

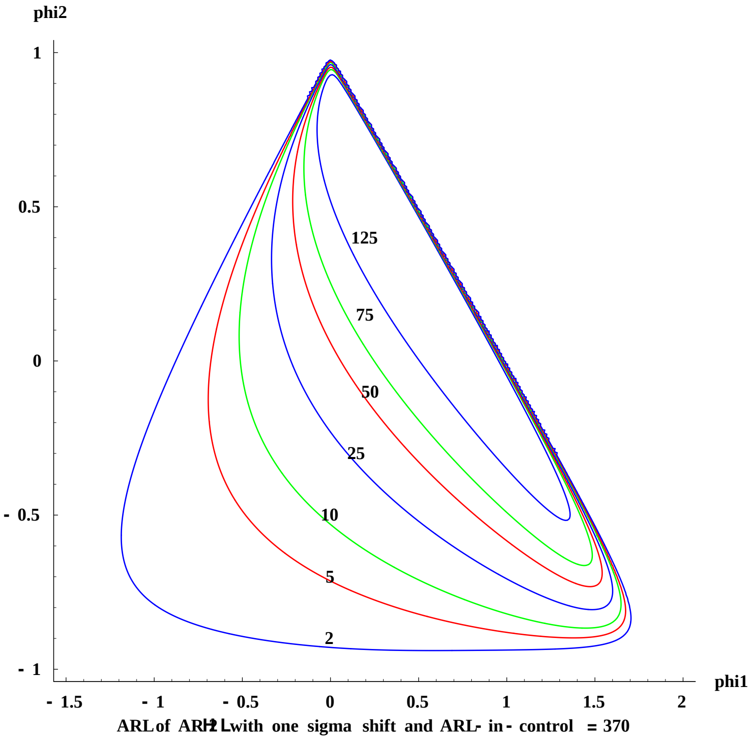
For AR(p) processes (i.e., $\Phi(B)Z_t = a_t$), Ξ is polynomial $\rightarrow p_k = p_{p+1}$ for $k \geq p + 1$.

$$\mathrm{ARL}_{\mathrm{out}}(\mathrm{AR}(p)) = 1 + \sum_{j=1}^{p} \prod_{i=1}^{j} (1 - p_i) + \frac{\prod_{i=1}^{p+1}(1 - p_i)}{p_{p+1}}$$

Similar but more complicated expression for variance of run length.

This simplifies approach of Longnecker and Ryan (1992).

Contour plot for AR(2) processes



Contour plot of standard deviation of run length almost identical!

Run length of Shewhart chart: $b = 1$ and MMSE feedback

$$\text{ARL}_{\text{out}} = 1 + \sum_{j=1}^{\infty} \prod_{i=1}^{j} (1 - p_i)$$

Fix n and assume $p_{\text{low}} \leq p_k \leq p_{\text{up}}$ for $k > n$.

Substitute inequality in general formula for ARL_{out}:

$$1 + \sum_{j=1}^{n-1} \prod_{i=1}^{j} (1 - p_i) + \frac{\prod_{i=1}^{n} (1 - p_i)}{p_{\text{up}}} \leq$$

$$\text{ARL}_{\text{out}} \leq 1 + \sum_{j=1}^{n-1} \prod_{i=1}^{j} (1 - p_i) + \frac{\prod_{i=1}^{n} (1 - p_i)}{p_{\text{low}}}$$

Similar but more complicated formula for variance of run length.

Bounds converge to true ARL, but how to find p_{low} and p_{up}?

Bounds for MA(1)-model

$$p_{k+1} = 1 - F\left(3 - (1 - \xi_1 - \cdots - \xi_k)\,\frac{\delta}{\sigma_a}\right)$$
$$+ F\left(-3 - (1 - \xi_1 - \cdots - \xi_k)\,\frac{\delta}{\sigma_a}\right).$$

$$Z_t = (1 - \theta\, B)\, a_t$$

$$\sigma_z^2 = (1 + \theta^2)\, \sigma_a^2.$$

$$\xi_k = -\theta^k$$

$$\frac{1 - |\theta|^{k+1}}{1 - \theta} \leq 1 + \theta + \cdots + \theta^k \leq \frac{1 + |\theta|^{k+1}}{1 - \theta}.$$

Bounds are positive and monotone in k. Hence, we can bound p_k.

Bounds for ARMA(1,1)-model

$$(1 - \varphi\, B)\, Z_t = (1 - \theta\, B)\, a_t$$

$$\sigma_z^2 = \frac{1 + \theta^2 - 2\varphi\theta}{1 - \varphi^2}\, \sigma_a^2.$$

$$\xi_k = (\varphi - \theta)\, \theta^{k-1}$$

$$\begin{aligned} \frac{1 - \varphi - |\varphi - \theta|\, |\theta|^k}{1 - \theta} &\leq \\ 1 - (\varphi - \theta)\left(1 + \theta + \cdots + \theta^{k-1}\right) &\leq \\ \frac{1 - \varphi + |\varphi - \theta|\, |\theta|^k}{1 - \theta}. & \end{aligned}$$

Lower bound is positive for large k.

Bounds are positive and monotone in k. Hence, we can bound p_k.

CUSUM charts

$$\begin{aligned}
\mathsf{S_H}(t) &= \max(0, U_t - K + \mathsf{S_H}(t-1)) \\
\mathsf{S_L}(t) &= \max(0, -U_t - K + \mathsf{S_L}(t-1)) \\
\mathsf{S_H}(0) &= \mathsf{S_L}(0) = 0,
\end{aligned}$$

$K = \delta/2$ for uncorrelated data

Suggestion (cf. Runger et al. (1995)):

$$K = \frac{\delta}{2}\left(1 - \sum_{i=1}^{\infty} \xi_i\right)$$

Run lengths can be expressed as (numerical)solutions of integral equations.

Topics under investigation

- non-stationary disturbances (ARIMA etc.)

- run lengths when parameters are estimated

- other control charts, like EWMA charts, cuscore, ...

- other feedback controllers

- wavelets to decorrelate data

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