ANSWERS

1. a) True. b) True. c) False. d) True.
2.

$$\left(\frac{\sqrt{2}-\mathrm{i}\sqrt{5}}{\sqrt{5}+\mathrm{i}\sqrt{2}}\right)^{11} = (-\mathrm{i})^{11} = \mathrm{i}, \qquad \left(1+\mathrm{i}\sqrt{3}\right)^9 = \left(2\,\mathrm{e}^{\frac{1}{3}\pi \mathrm{i}}\right)^9 = -512.$$

3. Valid between the nearest singularities: $0 < |z| < 2$.

$$\frac{1}{\mathrm{e}^{\pi z} - 1} = \frac{1}{1 + \pi z + \frac{1}{2}\pi^2 z^2 + \cdots - 1} = \frac{1}{\pi z(1 + \frac{1}{2}\pi z + \ldots)} =$$

$$\frac{1}{\pi z}\left(1 - \tfrac{1}{2}\pi z + \ldots\right) = \frac{1}{\pi z} - \frac{1}{2} + \ldots$$

4. We can write $f(z) = h(z)/z(z-1)^2$, with $h(z)$ an entire function. Since $zf(z) = h(z)/(z-1)^2 \to 1$ it follows with a corollary of Liouville that $h(z)$ is a 2nd order polynomial $h(z) = z^2 + bz + c$. Due to the symmetric zeros, $b = 0$. Since the residue of $f(z)/z$ in $z = 0$ is equal to $2c$, the given integral yields $c = 2$. So together we have

$$f(z) = \frac{z^2+2}{z(z-1)^2} = \frac{2}{z} + \frac{3}{(z-1)^2} - \frac{1}{z-1}.$$

5.

$$\int_0^{2\pi} \frac{\cos(50\theta)}{5+4\cos(\theta)}\,\mathrm{d}\theta = \mathrm{Re}\left[\int_0^{2\pi} \frac{\mathrm{e}^{50\theta \mathrm{i}}}{5+4\cos(\theta)}\,\mathrm{d}\theta\right] = \mathrm{Re}\left[\int_{|z|=1} \frac{\frac{1}{2}z^{50}}{\frac{5}{2}+z+z^{-1}}\,\frac{\mathrm{d}z}{\mathrm{i}z}\right] =$$

$$\mathrm{Re}\left[\frac{1}{2\mathrm{i}}\int_{|z|=1} \frac{z^{50}}{(z+2)(z+\frac{1}{2})}\,\mathrm{d}z\right] = \tfrac{2}{3}\pi\left(\tfrac{1}{2}\right)^{50}$$

6. a) w is negative real along the positive imaginary axis, therefore negative real along the upper side of the branch cut and hence positive real along the lower side.
b) w is everywhere analytic except along the branch cut $[-1,1]$. Let the contour $|z| = R$ shrink to the branch cut, and parametrise $z = t$ with $t = -1\ldots 1$ down under the cut where $w(z) = \sqrt{1-t^2} > 0$, and $z = -t$ with $t = -1\ldots 1$ above the cut where $w(z) = -\sqrt{1-t^2} < 0$. We obtain, using the transformation $t = \sin\theta$,

$$\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}}\,\mathrm{d}t + \int_{-1}^{1} \frac{1}{-\sqrt{1-t^2}}\cdot(-1)\,\mathrm{d}t = 2\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}}\,\mathrm{d}t = 2\int_{-\pi/2}^{\pi/2} \mathrm{d}\theta = 2\pi$$

Alternative: one may expand $1/w$ into a Laurent series in $|z| > 1$, starting with $(\mathrm{i}z)^{-1} + \ldots$.

7. In the usual way (close the contour along the upper/lower half for $\mathrm{e}^{\mathrm{i}z}/\mathrm{e}^{-\mathrm{i}z}$) we find

$$\int_{-\infty}^{\infty} \frac{\cos z}{z-t-\mathrm{i}\varepsilon}\,\mathrm{d}z = \pi\mathrm{i}\,\mathrm{e}^{\mathrm{i}t}\,\mathrm{e}^{-\varepsilon}, \qquad \int_{-\infty}^{\infty} \frac{\cos z}{z-t+\mathrm{i}\varepsilon}\,\mathrm{d}z = -\pi\mathrm{i}\,\mathrm{e}^{-\mathrm{i}t}\,\mathrm{e}^{-\varepsilon},$$

so

$$\lim_{\varepsilon\downarrow 0} \frac{1}{2\pi}\left(\pi\mathrm{i}\,\mathrm{e}^{\mathrm{i}t}\,\mathrm{e}^{-\varepsilon} - \pi\mathrm{i}\,\mathrm{e}^{-\mathrm{i}t}\,\mathrm{e}^{-\varepsilon}\right) = -\sin t.$$

Alternative: take the integrals together and write the result as

$$\mathrm{Re}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2(z-t)\,\mathrm{e}^{\mathrm{i}z}}{(z-t)^2+\varepsilon^2}\,\mathrm{d}z\right].$$