EMI CKP Asymptotische Technieken*

Theory, assignments and exercises - 2WAK0

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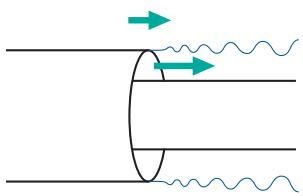
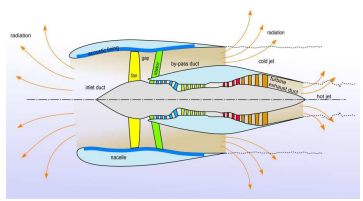
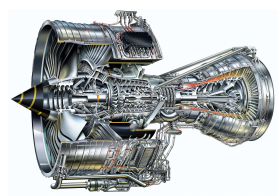
Chapter 1

Modelling and Perturbation Methods

Mathematical modelling is an art. It is the art of portraying a real, often physical, problem mathematically, by sorting out the whole spectrum of effects that play or may play a role, and then making a judicious selection by including what is relevant and excluding what is too small. This selection is what we call a *model* or *theory*. Models and theories, applicable in a certain situation, are not *isolated islands of knowledge* provided with a logical flag, labelling it *valid* or *invalid*. A model is never unique, because it depends on the type, quality and accuracy of answers we are aiming for, and of course the means (time, money, numerical power, mathematical skills) that we have available.







Concept of hierarchy (turbofan engine)

Normally, when the problem is rich enough, this spectrum of effects does not simply consist of two classes *important* and *unimportant*, but is a smoothly distributed hierarchy varying from *essential* effects via *relevant* and *rather relevant* to *unimportant* and *absolutely irrelevant* effects. As a result, in practically any model there will be effects that are small but not small enough to be excluded. We can ignore their smallness, and just assume that all effects that constitute our model are equally important. This is the usual approach when the problem is simple enough for analysis or a brute force numerical simulation.

$$x^2 = 4 + 10^{-6}x^5$$
$$x^2 = 4$$

Comparing "exact" and approximate models.

There are situations, however, where it could be wise to utilise the smallness of these small but important effects, but in such a way, that we simplify the problem without reducing the quality of the model. Usually, an otherwise intractable problem becomes solvable and (most importantly) we gain great insight in the problem.

Perturbation methods do this in a systematic manner by using the sharp fillet knife of mathematics in general, and asymptotic analysis in particular. From this perspective, perturbation methods are the continuation of modelling by other means and are therefore much more important for the understanding and analysis of practical problems than they're usually credited with. David Crighton[1] called "*Asymptotics - an indispensable complement to thought, computation and experiment in applied mathematical modelling*".

Examples are numerous: simplified geometries reducing the spatial dimension, small amplitudes allowing linearization, low velocities and long time scales allowing incompressible description, small relative viscosity allowing inviscid models, zero or infinite lengths rather than finite lengths, etc.

The question is: how can we use this gradual transition between models of different level. Of course, when a certain aspect or effect, previously absent from our model, is included in our model, the change is abrupt and big: usually the corresponding equations are more complex and more difficult to solve. This is, however, only true if we are merely interested in exact or numerically *exact* solutions. But an exact solution of an approximate model is not better than an approximate solution of an exact model. So there is absolutely no reason to demand the solution to be more exact than the corresponding model. If we accept approximate solutions, based on the inherent small or large modelling parameters, we do have the possibilities to gradually increase the complexity of a model, and study small but significant effects in the most efficient way.

The methods utilizing systematically this approach are called *perturbations methods*. Usually, a distinction is made between regular and singular perturbations. A (loose definition of a) regular perturbation is one in which the solutions of perturbed and unperturbed problem are everywhere close to each other.

We will find many applications of this philosophy in continuous mechanics (fluid mechanics, elasticity), and indeed many methods arose as a natural tool to understand certain underlying physical phenomena. We will consider here four methods relevant in continuous mechanics: (1) the method of slow variation and (2) the method of Lindstedt-Poincaré as examples of regular perturbation methods; then (3) the method of matched asymptotic expansions and (4) the method of multiple scales (with as a special case the WKB method) as examples of singular perturbation methods. In (1) the typical length scale in one direction is much greater than in the others, while in (2) the relevant time scale is unknown and part of the problem. In (3) several approximations, coupled but valid in spatially distinct regions, are solved in parallel. Method (4) relates to problems in which several length scales act in the same direction, for example a wave propagating through a slowly varying environment.

In order to quantify the used small effect in the model, we will always introduce a small positive dimensionless parameter ε. Its physical meaning depends on the problem, but it is always the ratio between two inherent length scales, time scales, or other characteristic problem quantities.

[1] D.G. Crighton. "Asymptotics – an indispensable complement to thought, computation and experiment in applied mathematical modelling." In *Proceedings of the Seventh European Conference on Mathematics in Industry, March 2-6, 1993, Montecatini*, A. Fasano and M. Primicerio, editors, volume ECMI 9, pages 3–19, Stuttgart, 1994. European Consortium for Mathematics in Industry, B.G.Teubner.

Chapter 2

Modelling and Scaling

2.1 Theory

2.1.1 What is a model? Some philosophical considerations.

Mathematics has, historically, its major sources of inspiration in applications. It is just the unexpected question from practice that forces one to go off the beaten track. Also it is usually easier to portray properties of a mathematical abstraction with a concrete example at hand. Therefore, it is safe to say that most mathematics is applied, applicable or emerges from applications.

Before mathematics can be applied to a real problem, the problem must be described mathematically. We need a mathematical representation of its primitive elements and their relations, and the problem must be formulated in equations and formulas, to render it amenable to formal manipulation and to clarify the inherent structure. This is called mathematical modelling. An informal definition could be:

> *Describing a real-world problem in a mathematical way by what is called a* model, *such that it becomes possible to deploy mathematical tools for its solution. The model should be based on first principles and elementary relations and it should be accurate enough, such that it has reasonable claims to predict both quantitative and qualitative aspects of the original problem. The accuracy of the description should be limited, in order to make the model not unnecessary complex.*

This is evidently a very loose definition. Apart from the question what is meant with: a problem being described in a mathematical way, there is the confusing paradox that we only know the precision of our model, if we can compare it with a better model, but this better model is exactly what we try to avoid as it is usually unnecessarily complex! In general we do not know a problem and its accompanying model well enough to be absolutely sure that the sought description is both consistent, complete and sufficiently accurate for the purpose, ànd not too formidable for any treatment. A model is, therefore, to a certain extent a vague concept. Nevertheless, modelling plays a key rôle in applied mathematics, since mathematics cannot be applied to any real world problem without the intermediate steps of modelling. Therefore, a more structured approach is necessary, which is the aim of the present chapter.

Some people define *modelling* as the process of translating a real-world problem into mathematical terms. We will not do so, as this definition is too wide to include the subtle aspects of "limited precision" (to be discussed below). Therefore we will introduce the word *mathematising*, defined as the process of translating a real-world problem into mathematical terms. It is a translation in the sense that we translate from the inaccurate, verbose "everyday" language to the language of mathematics. For example, the geometrical presence and evolution of objects in space and time may be described parametrically in a suitable coordinate system. Any properties or fields that are expected to play a rôle may be formulated by functions in time and space, explicitly or implicitly, for example as a differential equation.

Mathematising is an elementary but not trivial step. In fact, it forms probably the single most important step in the progress of science. It requires the distinction, naming, and exact specification of the essential relevant elementary objects and their interrelations, where mathematics acts as a language in which the problem is described. If theory is available for the mathematical problem obtained this way, the problem considered may be subjected to the strict logic of mathematics, and reasoning in this language will transcend over the limited and inaccurate ordinary language. Mathematising is therefore, apart from providing the link between the mathematical world and the real world, also important for science in general.

A very important point to note is the fact that such a mathematised formulation is *always* at some level simplified. The earth can be modelled by a point or a sphere in astronomical applications, or by an infinite half-space or modelled not at all in problems of human scale. Based on the level of simplification, sophistication or accuracy, we can associate an inherent hierarchy to the set of possible descriptions. A model may be too crude, but also it may be too refined. It is too crude if it just doesn't describe the problem considered, or if the numbers it produces are not accurate enough to be acceptable. It is too refined if it includes irrelevant effects that make the problem untreatable, or make the model so complicated that important relations or trends remain hidden.

The ultimate goal for mathematising a problem is a deeper understanding and a more profound analysis and solution of the problem. Usually, a more refined problem translation is more accurate but also more complicated and more difficult – if not impossible! – to analyse and solve than a simpler one. Therefore, not every mathematical translation is a good one. We will call a good mathematical translation a *model* or *mathematical model* if it is *lean* or *thrifty* in the sense, that it describes our problem quantitatively or qualitatively in a suitable or required accuracy with a *minimal* number of essentially different parameters and variables. (We say "essentially different", in view of a reduction that is always possible by writing the problem in dimensionless form. See Buckingham's Theorem below.) Again, this definition is rather subjective, as it greatly depends on the context of the problem considered and our knowledge and resources. So there will rarely be one "best" model. At the same time, it shows that modelling, even if relying significantly on intuition, is part of the mathematical analysis.

2.1.2 Types of models

We will distinguish the following three classes of models.

- **Systematic models.**

  Other possible names are *asymptotic models* or *reducing models*, and it is the most important type for us here. The starting point is to use available complete models, which are adequate, but over-complete in so far that effects are included which are irrelevant, uninteresting, or negligibly small, making the mathematical problem unnecessarily complex. By using available additional information (order of magnitude of the parameters) assumptions can be made which minimize in a systematic way the *over-complete* model into a *good* model by taking a parameter that is already large or small to its asymptotic limit: small parameters are taken zero, large parameters become infinite, an almost symmetry becomes a full symmetry.

  Examples of systematic models are found in particular in the well-established fields of continuum physics (fluid mechanics, elasticity). An ordinary flow is usually described by a model which is reduced from the full, *i.e.* compressible and viscous, Navier-Stokes equations.

  An example is the convection-diffusion problem described by the "complete" model

  $$\frac{\partial T}{\partial t} + \boldsymbol{v} \cdot \nabla T = \alpha \nabla^2 T,$$

  which is difficult to solve, but may be reduced to the much simpler

  $$\frac{\partial T}{\partial t} + \boldsymbol{v} \cdot \nabla T = 0$$

  if we have reasons to believe that diffusion term $\alpha \nabla^2 T$ is small compared to convection.

  Another example is the (again difficult) nonlinear pendulum equation

  $$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\frac{g}{L}\sin\theta,$$

  which may be reduced to the much simpler linear equation

  $$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\frac{g}{L}\theta,$$

  if we know or conjecture that angle θ is small and $\sin\theta \simeq \theta$.

- **Constructing models**

  Another possible name is *building block models*. Here we build our problem description step by step from low to high, from simple to more complex, by adding effects and elements lumped together in building blocks, until the required accuracy or adequacy is obtained. This type of model is usually the first if a new scientific discipline is explored.

  An example is the 1D Euler-Bernoulli model of a flexible bar with small displacements and where the bending moment is assumed to be a linear function of the radius of curvature.

  $$EI\frac{\partial^4 y}{\partial x^4} - T\frac{\partial^2 y}{\partial x^2} + Q + m_0\frac{\partial^2 y}{\partial t^2} = 0.$$

- **Canonical models.**

  Another possible name is *characteristic models* or *quintessential models*. Here an existing model is further reduced to describe only the essence of a certain aspect of the problem. These models are particularly important if the mathematical analysis of a model from one of the other categories is lacking available theory. The development of such theory is usually hindered by too much irrelevant details. These models are useful for the understanding, but usually far away from the original full problem setting and therefore not suitable for direct industrial application.

  An example is Burgers' equation, originally formulated as an "unphysically" reduced version of the Navier-Stokes equations in order to study certain fundamental effects,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}.$$

Note that an asymptotic model may start as a building-block model, which is only found at a later stage to be too comprehensive. Similarly, a canonical model may reduce from an asymptotic model if the latter appears to contain a particular, not yet understood effect, which should be investigated in isolation before any progress with the original model can be made.

The type of model which is most relevant in the context of asymptotic techniques, is the *asymptotic* or *systematic* model. In the following we will explain this further.

2.1.3 Perturbation methods: the continuation of modelling by other means

We have seen above that a real-world problem described by a *systematic* model, is essentially described by a hierarchy of systematic models, where a higher level model is more comprehensive and more accurate than one from a lower level. Now suppose that we have a fairly good model, describing the dominating phenomena in good order of magnitude. And suppose that we are interested in improving on this model by adding some previously ignored aspects or effects. In general, this implies a very abrupt change in our model. The equations are more complex and more difficult to solve. As an illustration, consider the simple "model" $x^2 = a^2$, and the more complete "model" $x^2+\varepsilon x^5 = a^2$. The first one can be solved easily analytically, the second one with much more effort only numerically. So it seems that the relation between solution and model is not continuous in the problem parameters. Whatever small ε we take, from a transparent and exact solution of the simple model at $\varepsilon = 0$, we abruptly face a far more complicated solution of a model that is just a little bit better. This is a pity, because certain type of useful information (parametric dependencies, trends) become increasingly more difficult to dig out of the more complicated solution of the complex model. This discontinuity of models in the parameter ε may therefore be an argument to retain the simpler model.

The (complexity of the) model is, however, only discontinuous if we are merely interested in exact or numerically "exact" solutions (for example for reasons of benchmarking or validation of solution methods). This is not always the case. As far as our modelling objectives are concerned, we have to keep in mind that also the improved model is only a next step in the modelling hierarchy and not exact in any absolute sense. So there is no reason to require the solution to be more exact than the corresponding model, as **an exact solution of an approximate model is not better than an approximate solution of an exact model**. Moreover, the *type* of information that analytical solutions may provide (functional relationships, *etc.*) is sometimes so important that numerical accuracy may be worthwhile to sacrifice.

Let us go back to our "fairly good", improved model. The effects we added are relatively small. Otherwise, the previous lower level model was not fairly good as we assumed, but just completely wrong. Usually, this smallness is quantified by small dimensionless parameters (see below) occurring in the equations and (or) boundary conditions. This is the generic situation. The transition from a lower-level to a higher-level theory is characterized by the appearance of one or more modelling parameters, which are (when made dimensionless) small or large, and yield in the limit a simpler description. Examples are infinitely large or small geometries with circular or spherical symmetry that reduce the number of spatial dimensions, small amplitudes allowing linearization, low velocities and long time scales in flow problems allowing incompressible description, small relative viscosity allowing inviscid models, *etc.* In fact, in any practical problem it is really the rule rather than the exception that dimensionless numbers are either small or large.

If we accept approximate solutions, where the approximation is based on the inherently small or large modelling parameters, we do have the possibility to gradually increase the complexity of a model, and study small but significant effects in the most efficient way. The methods utilizing this approach systematically are called "perturbation methods". The approximation constructed is almost always an asymptotic approximation, *i.e.* where the error reduces with the small or large parameter.

Usually, a distinction is made between *regular* and *singular* perturbations. A (loose definition of a) regular perturbation problem is where the approximate problem is everywhere close to the unperturbed problem. This, however, depends of course on the domain of interest and, as we will see, on the choice of coordinates. If a problem is regular without any need for other than trivial reformulations, the construction of an asymptotic solution is straightforward. In fact, it forms the usual strategy in modelling when terms are linearised or effects are neglected. The more interesting perturbation problems are those where this straightforward approach fails.

We will consider here four methods relevant in the presented modelling problems. The first two are examples of regular perturbation methods, but only after a suitable coordinate transformation. The first is called the method of slow variation, where the typical axial length scale is much greater than the transverse length scale. The second is the Lindstedt-Poincaré method or the method of strained coordinates, for periodic processes. Here, the intrinsic time scale ($\sim$ the period of the solution) is unknown and has to be found. The other two methods are of singular perturbation type, because there is no coordinate transformation possible that renders the problem into one of regular type. The third one is the method of matched asymptotic expansions (MAE). To render the problem into one of regular type, different scalings are necessary in spatially distinct regions (boundary layers). The fourth singular perturbation method considered here is the method of multiple scales and may be considered as a combination of the method of slow variation and the method of strained coordinates, as now several (long, short, shorter) length scales occur in parallel. This cannot be repaired by a single coordinate transformation. Therefore, the problem is temporarily reformulated into a higher dimensional problem by taking the various length scales apart. Then the problem is regular again, and can be solved. A refinement of this method is the WKB method, where the coordinate transformation of the fast variable becomes itself slowly varying.

2.1.4 Energy consumption of a car

Consider a car of mass m in position $x(t)$ and velocity $v(t) = x'(t)$ at time t, moving along a straight horizontal road of length L in time T, subject to acceleration force mv', air drag $\frac{1}{2}\rho A C_D v|v|$ (where ρ is the density of air, A is the car's frontal area, and C_D is its drag coefficient), and engine thrust $F(t)$, such that (with always positive velocity)

$$mv' + bv^2 = F(t), \quad b = \tfrac{1}{2}\rho C_D A.$$

1. Suppose that the force F is constant (like gravity in free fall $F = mg$). What is the final velocity in steady state?
2. Suppose that the power $P = Fv$ is constant (like a[1] typical engine). What is the final velocity in steady state?

From here on we will not assume a given F, but assume a resulting v for which the required F is given by the equation.

We are interested in the extra energy consumption due to velocity fluctuations. So we compare the energy consumption in case of a steady velocity $v_0(t) = V_0 = L/T$ and in case of a velocity that fluctuates around the average of V_0, for example like

$$v_a(t) = V_0 + a\sin(\omega t).$$

Let's assume for convenience that T corresponds to an integer multiple of $2\pi/\omega$ such that

$$\sin(\omega T) = 0, \quad \int_0^T \sin(\omega t)\,\mathrm{d}t = 0, \quad \int_0^T \sin^2(\omega t)\,\mathrm{d}t = \tfrac{1}{2}T, \quad \int_0^T \sin^3(\omega t)\,\mathrm{d}t = 0.$$

Since the energy is the time-integral of the power $F(t)v(t)$, we find

$$E_0 = \int_0^T F_0 v_0\,\mathrm{d}t = \int_0^T bV_0^3\,\mathrm{d}t = bV_0^3 T = bV_0^2 L$$

and

$$\begin{aligned} E_a = \int_0^T F_a v_a \mathrm{d}t &= \int_0^T (mv_a' v_a + bv_a^3)\,\mathrm{d}t = \tfrac{1}{2}m(v_a^2(T) - v_a^2(0)) \\ &+ b\int_0^T V_0^3 + 3V_0^2 a\sin(\omega t) + 3V_0 a^2\sin^2(\omega t) + a^3\sin^3(\omega t)\,\mathrm{d}t = bL(V_0^2 + \tfrac{3}{2}a^2). \end{aligned}$$

The extra energy consumption is then

$$E_a - E_0 = \tfrac{3}{2}ba^2 L.$$

This is a rather neat result. Note, for example, that it is independent of mass m. Can we generalise this result? Assume

$$v_a(t) = V_0 + v_1(t), \quad v_1(0) = v_1(T) = 0, \quad \int_0^T v_1(t)\,\mathrm{d}t = 0.$$

[1] ... crude approximation of a ...

Then

$$E_a = \int_0^T F_a v_a \mathrm{d}t = \int_0^T (m v_a' v_a + b v_a^3)\,\mathrm{d}t = \tfrac{1}{2} m (v_a^2(T) - v_a^2(0))$$
$$+ b \int_0^T (V_0^3 + 3V_0^2 v_1(t) + 3V_0 v_1^2(t) + v_1^3(t))\,\mathrm{d}t = bLV_0^2 + 3bV_0 \int_0^T v_1^2(t)\,\mathrm{d}t + b\int_0^T v_1^3(t)\,\mathrm{d}t$$

and so

$$E_a - E_0 = 3bV_0 \int_0^T v_1^2(t)\,\mathrm{d}t + b\int_0^T v_1^3(t)\,\mathrm{d}t$$

This is clearly not such a nice result, and more difficult to grasp.

We obtain more structure if we use the fact that v_1 is small compared to V_0. But how do we do that? We write

$$v_1(t) = \varepsilon V_0 u(t)$$

where u and ε are dimensionless, ε is small, and u is scaled such that it is of order 1. Then we obtain

$$E_a - E_0 = 3b\varepsilon^2 V_0^3 \int_0^T u^2(t)\,\mathrm{d}t + b\varepsilon^3 V_0^3 \int_0^T u^3(t)\,\mathrm{d}t.$$

Since ε is small, we can approximate

$$E_a - E_0 \simeq 3b\varepsilon^2 V_0^3 \int_0^T u^2(t)\,\mathrm{d}t$$

which is almost of the same form as the former result if we realise that $a = \varepsilon V_0$. The main unusual term is

$$\int_0^T u^2(t)\,\mathrm{d}t.$$

This, however, is of dimension time and being the integral of a function of $O(1)$ over an interval T, it must be a number of $O(T)$. So if we divide by T it is dimensionless and of $O(1)$. Moreover, we are free to scale u (only the product εu is relevant) such that this quadratic average is unity

$$\frac{1}{T}\int_0^T u^2(t)\,\mathrm{d}t = 1.$$

Then we have in the end again a transparent result

$$E_a - E_0 \simeq 3b\varepsilon^2 V_0^2 L.$$

So scaling and non-dimensionalisation is useful for clear results. We scaled and non-dimensionalised v_1, but not time t and the other variables. Furthermore, we made only dimensionless afterwards, while it is more systematic to do it right from the start with the differential equation.

A time scale of the problem is obviously T. (Is this the most natural time scale? Are there other possible?) Let's introduce

$$t = T\tau$$

and redefine

$$v(t) = V_0 U(\tau), \quad F(t) = bV_0^2 f(\tau), \quad E = bV_0^2 L\epsilon.$$

(Is this the natural scaling for F and E? Are there other possible?) Then we have[2]

$$\frac{mV_0}{T}U' + bV_0^2U^2 = bV_0^2 f,$$

which is then

$$\alpha U' + U^2 = f, \quad \alpha = \frac{m}{bL}.$$

The problem has apparently one dimensionless parameter, α. The energy consumed is now

$$\epsilon = \int_0^1 fU \,\mathrm{d}\tau = \tfrac{1}{2}\alpha(U^2(1) - U^2(0)) + \int_0^1 U^3 \,\mathrm{d}\tau$$

We can introduce fluctuations of a constant mean velocity as follows

$$U = 1 + \varepsilon u, \quad \int_0^1 u \,\mathrm{d}\tau = 0, \quad u(0) = u(1) = 0,$$

with the assumption that $u = O(1)$, but otherwise we can normalise it as we may find convenient, like

$$\int_0^1 u^2 \,\mathrm{d}t = 1.$$

Altogether we find

$$\epsilon_a - \epsilon_0 = \int_0^1 1 + 3\varepsilon u + 3\varepsilon^2 u^2 + \varepsilon^3 u^3 \,\mathrm{d}\tau - 1 = 3\varepsilon^2 + O(\varepsilon^3) \simeq 3\varepsilon^2.$$

A much cleaner result!

[2] If $v(t) = V_0 U(\tau) = V_0 U\left(\frac{t}{T}\right)$, then with the chain rule

$$v'(t) = \frac{\mathrm{d}}{\mathrm{d}t} v(t) = V_0 \frac{\mathrm{d}}{\mathrm{d}t} U\left(\frac{t}{T}\right) = \frac{V_0}{T}\frac{\mathrm{d}}{\mathrm{d}\tau} U(\tau) = \frac{V_0}{T} U'(\tau).$$

2.1.5 Nondimensionalisation

Buckingham's Π-Theorem:

If a physical problem is described by n variables and parameters in r dimensions, the number of dimensionless groups is at least $n-r$. Exactly $n-r$ if all r dimensions play a role. More than $n-r$ if some dimensions are redundant, or occur in the same combination. In that case r is effectively smaller.

Note: mol, rad or dB do *not* count, because they are dimensionless units.

A way to see this theorem intuitively is as follows.
From the problem variables, parameters, and their combinations we can construct time, length, etc. scales. They follow from the problem and are therefore called inherent (length, time) scales. For example, from a velocity V and a length L we have a time L/V. These new scales can be used for measuring, instead of meters or seconds. In this way we can replace the original r dimensions by r new dimensions from (combinations of) r variables. These r variables, when measured in the new dimensions, are by definition equal to unity, and play no visible role anymore. The remaining $n-r$ variables, on the other hand, may be expressed in the new dimensions to constitute the essential (and nondimensional) problem parameters.

Example. A problem with the 4 variables force F, length L, velocity V and viscosity η are expressed in 3 dimensions kg, m and s by $[F] = \mathrm{kg\,m/s^2}$, $[L] = \mathrm{m}$, $[V] = \mathrm{m/s}$ and $[\eta] = \mathrm{kg/m\,s}$.

With the inherent unit of length L, inherent unit of time L/V and inherent unit of mass $\eta L^2/V$, the variables L, V and η become simply 1 (times L, V and η, respectively). Only force F becomes some (dimensionless) number $\mathcal{F}$ times the new units as follows:

$$F = \mathcal{F} \cdot \frac{\dfrac{\eta L^2}{V} \cdot L}{\left(\dfrac{L}{V}\right)^2} = \mathcal{F} \cdot LV\eta, \qquad \text{in other words} \qquad \mathcal{F} = \frac{F}{LV\eta}.$$

A more formal way to obtain this is by utilizing a bit linear algebra. We have for any dimensionless quantity G the condition that it should satisfy for some combination of $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$

$$[G] = [F^{\alpha_1} L^{\alpha_2} V^{\alpha_3} \eta^{\alpha_4}] = \mathrm{m}^{\alpha_1+\alpha_2+\alpha_3-\alpha_4}\,\mathrm{kg}^{\alpha_1+\alpha_4}\,\mathrm{s}^{-2\alpha_1-\alpha_3-\alpha_4} = \mathrm{m}^0\,\mathrm{kg}^0\,\mathrm{s}^0 = 1.$$

In other words we have $r = 3$ equations for $n = 4$ unknowns

$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 0 & 0 & 1 \\ -2 & 0 & -1 & -1 \end{pmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since all equations are independent, this system has rank $= 3$, the number of equations r, and so $4-3=1$ linearly independent solutions. Therefore, there is one dimensionless variable G. (If some rows are dependent, the rank would have been less than r and the number of independent solutions

more than $n - r$.) Solving this system yields the solution $\boldsymbol{\alpha} = (1, -1, -1, -1)$, or any multiple of it. The corresponding dimensionless number is then

$$G = \frac{F}{LV\eta},$$

which confirms the above result with $G = \mathcal{F}$. Note that other forms, like $G^2, \sqrt{G}, 1/G$ etc. are equally possible dimensionless numbers, equivalent to G.

Weber's Law.

Normally, we have in the problems studied several variables and parameters of the same unit (dimension), which act as each others reference to compare with. The opposite situation, when there is *no* reference available, is also meaningful.

When a variable is perceived for which there is no reference quantity available to compare with, c.q. to scale on, the actual value of the variable itself will be the reference. The resulting logarithmic relation (see below) is known as *Weber's Law*[3].

Take for example the perceived loudness of sound. Since the range of our human audible sensitivity is incredibly large (10^{14} in energy), the loudest and quietest levels are practically infinitely far away. Therefore, we have no reference or scaling level to compare with, other than the actually perceived sound itself.

As a result, variations in sound loudness $\mathrm{d}L$ are perceived proportional to *relative* variations of the physical sound intensity $\mathrm{d}I/I$:

$$\mathrm{d}L = K\frac{\mathrm{d}I}{I},$$

for a suitably chosen constant K. After integration we obtain that L varies logarithmically in I.

$$L = L_0 + K \log I$$

with L_0 a conveniently chosen reference level.

As the intensity (the time-averaged energy flux) I is, for a single tone, proportional to the mean squared acoustic pressure p_{rms}^2, we have the relation $L = K \log(p_{\mathrm{rms}}^2) + L_0$. If

$$L = 2\log_{10}(p_{\mathrm{rms}}/p_0)$$

for a reference value $p_0 = 2 \cdot 10^{-5}$ Pascal is taken, we call L the Sound Pressure Level in Bells. The usual unit is one tenth of it, the decibel.

[3]Ernst Heinrich Weber, 1834

2.2 Exercises

2.2.1 A car with viscous friction and hills

Repeat the analysis of section (2.1.4) for

a) an engine with linear (viscous) friction, leading to the model: $mv' + bv^2 + cv = F(t)$.
b) a road varying in height $h(x)$, leading to the model: $mv' + bv^2 + mgh'(x) = F(t)$.

Think about the various possible scalings and nondimensionalisations.

2.2.2 Membrane resonance

The resonance frequency ω of a freely suspended membrane (like a frame drum, a skin stretched over a frame without a resonance cavity) is determined by the membrane tension T, membrane surface density σ, membrane diameter a, air density ρ_a and sound speed c_a. In other words, there is a relation

$$\omega = f(T, \sigma, a, \rho_a, c_a).$$

According to Buckingham, this relation can be reduced to a relation between three dimensionless groups:

frequency	ω ,	dimension	1/s	Buckingham: $6 - 3 = 3$ dimensionless groups G
memb. tension	T ,	”	kg/s^2	
memb. density	σ,	”	kg/m^2	
memb. diameter	a ,	”	m	
air density	ρ_a,	”	kg/m^3	
air soundspeed	c_a ,	”	m/s	

$$G = \omega^{\alpha_1} T^{\alpha_2} \sigma^{\alpha_3} a^{\alpha_4} \rho_a^{\alpha_5} c_a^{\alpha_6}$$

$$\begin{aligned}[G] &= \left(\frac{1}{\mathrm{s}}\right)^{\alpha_1} \left(\frac{\mathrm{kg}}{\mathrm{s}^2}\right)^{\alpha_2} \left(\frac{\mathrm{kg}}{\mathrm{m}^2}\right)^{\alpha_3} \mathrm{m}^{\alpha_4} \left(\frac{\mathrm{kg}}{\mathrm{m}^3}\right)^{\alpha_5} \left(\frac{\mathrm{m}}{\mathrm{s}}\right)^{\alpha_6} \\ &= \mathrm{m}^{-2\alpha_3+\alpha_4-3\alpha_5+\alpha_6} \mathrm{s}^{-\alpha_1-2\alpha_2-\alpha_6} \mathrm{kg}^{\alpha_2+\alpha_3+\alpha_5} = \mathrm{m}^0 \mathrm{s}^0 \mathrm{kg}^0\end{aligned}$$

a) Give (mutually independent) examples of the 3 possible dimensionless numbers G.
b) Show that it is possible to write the functional dependence between the frequency and the other parameters as

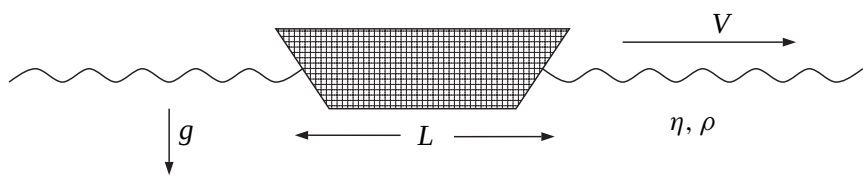
$$G_\omega = F(G_1, G_2)$$

where G_ω is the only parameter that depends on ω. You may introduce for convenience $c_M = (T/\sigma)^{\frac{1}{2}}$, the propagation speed of transversal waves in the membrane in the absence of air loading.

2.2.3 Ship drag: wave and viscosity effects

A ship of typical size L, moving with velocity V in deep water of density ρ and viscosity η, feels a drag D due to gravity waves and due to viscous friction, apart from density, velocity and geometry effects. Symbolically, we have

$$D = f(g, \eta, \rho, V, L).$$



According to Buckingham, this relation can be reduced to a relation between three dimensionless groups:

drag	D ,	dimension	$\mathrm{kg\,m/s^2}$	Buckingham: $6-3=3$ dimensionless groups G
length	L ,	”	m	
velocity	V ,	”	m/s	
viscosity	η,	”	kg/m s	
gravity	g ,	”	$\mathrm{m/s^2}$	
water density	ρ,	”	$\mathrm{kg/m^3}$	

$$G = D^{\alpha_1} L^{\alpha_2} V^{\alpha_3} \eta^{\alpha_4} g^{\alpha_5} \rho^{\alpha_6}$$

$$\begin{aligned}[G] &= \left(\frac{\mathrm{kg\,m}}{\mathrm{s}^2}\right)^{\alpha_1} \mathrm{m}^{\alpha_2} \left(\frac{\mathrm{m}}{\mathrm{s}}\right)^{\alpha_3} \left(\frac{\mathrm{kg}}{\mathrm{m\,s}}\right)^{\alpha_4} \left(\frac{\mathrm{m}}{\mathrm{s}^2}\right)^{\alpha_5} \left(\frac{\mathrm{kg}}{\mathrm{m}^3}\right)^{\alpha_6} \\ &= \mathrm{m}^{\alpha_1+\alpha_2+\alpha_3-\alpha_4+\alpha_5-3\alpha_6} \mathrm{s}^{-2\alpha_1-\alpha_3-\alpha_4-2\alpha_5} \mathrm{kg}^{\alpha_1+\alpha_4+\alpha_6} = \mathrm{m}^0\mathrm{s}^0\mathrm{kg}^0\end{aligned}$$

a) Give (mutually independent) examples of the 3 possible dimensionless numbers G.

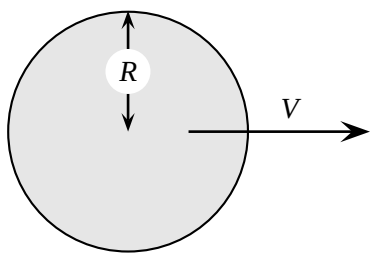
b) Show that it is possible to write the functional dependence between the drag and the other parameters as

$$G_D = F(G_g, G_\eta)$$

where G_D is a parameter that depends on D but not on g or η, G_g depends on g but not on D or η, and G_η depends on η but not on D or g.

2.2.4 Sphere in viscous flow

Work out in detail – using Buckingham's theorem – scaling and non-dimensionalisation of the problem of the viscous air resistance (drag D, velocity V) of a sphere (radius R) in a fluid (density ρ, viscosity η). What would be a suitable scaling if viscosity dominates the resistance? And what if pressure difference dominates?



Sphere in viscous fluid

length	R,	dimension	m
velocity	V,	”	m/s
drag	D,	”	kg m/s^2
viscosity	η,	”	kg/m s
density	ρ,	”	kg/m^3

$$\left.\right\} \quad D = f(\rho, V, \eta, R)$$

2.2.5 Cooling of a cup of tea

The total amount of thermal energy in a cup of tea of volume V, water density ρ, specific heat c and temperature T at time t is $E(t) = \rho c V T(t)$. According to Newton's cooling law, the heat flux through the surface A is $q = -hA(T - T_\infty)$ with heat transfer coefficient h. What is the dimension of h? Make the problem dimensionless and determine the characteristic time scale of the problem.

Confirm this by solving the equation for the decaying temperature $T(t)$

$$\frac{\mathrm{d}E}{\mathrm{d}t} = q, \qquad T(0) = T_0.$$

2.2.6 The velocity of a rowing boat.

Determine the functional dependence of the velocity v of a rowing boat on the number n of rowers by using the following modelling assumptions.

The size of the boat scales with the number of rowers (*i.e.* their volume) but has otherwise the same shape. So if the volume per rower is G, the volume of the boat is $V = nG$. Furthermore, the volume of the boat can be written as a length L times a cross section $A = \ell^2$ and $L = \lambda\ell$ for a shape factor λ.

The drag only depends on the water pressure distribution and is for high enough Reynolds numbers given by $D = \frac{1}{2}\rho v^2 A C_D$, where ρ is the water density and C_D the drag coefficient, which is a constant as it depends only on the shape of the boat.

The required thrust is therefore $F = D$, while the necessary power to maintain the velocity v is then $P = \frac{\mathrm{d}}{\mathrm{d}t}\int^x F\,\mathrm{d}x' = Fv$. The available power per rower is a fixed p.

2.2.7 Travel time in cities

A simple model for the travel time by car between two addresses in a big city is: the time T in minutes is equal to the distance L in kilometers plus the number N of traffic lights passed,

$$T = L + N.$$

a) What is this formula if time is measured in hours and distance in miles?
b) Generalise the formula for arbitrary units of time and length.
c) Make this last version dimensionless in a suitable way.

2.2.8 A sessile drop with surface tension.

The height h of a drop of liquid at rest on a horizontal surface with the effect of gravity being balanced by surface tension is a function of liquid density ρ, volume L^3, acceleration of gravity g, surface tension γ and contact angle θ. As $[h] = \text{m}$, $[\rho] = \text{kg/m}^3$, $[L] = \text{m}$, $[g] = \text{m/s}^2$, $[\gamma] = \text{kg/s}^2$, and $[\theta] = 1$, we have $6 - 3 = 3$ dimensionless numbers. One is of course the already dimensionless θ. The second dimensionless number is the Bond number, known to control this kind of problems, and is given by

$$B = \frac{\rho g L^2}{\gamma}.$$

The third is a dimensionless number containing h, leading to a functional relationship given by

$$h = \ell F(B, \theta),$$

where F is dimensionless and ℓ is an inherent length scale. We have practically two useful choices for ℓ. One is suitable when B is small (high relative surface tension) and the drop becomes spherical. The other is the proper scaling when B is large (low relative surface tension), such that the drop will spread out, flat as a pancake, and $h \ll L$. In particular, $h/L = O(B^{-1/2})$

Find these two (mutually independent) possible ℓ_1 and ℓ_2.

2.2.9 The drag of a plate sliding along a thin layer of lubricant

A flat plate of length L and width W, slipping with constant velocity V along a thin layer of lubricant of thickness h and viscosity η, experiences a drag D. Assume that the drag is linearly proportional to the wetted surface $L \times W$. Find a functional relation between D and the other problem parameters.

length	L,	dimension	m
width	W,	”	m
velocity	V,	”	m/s
viscosity	η,	”	kg/m s
thickness	h,	”	m

2.2.10 The stiffened catenary

A cable, suspended between the points $X = 0$, $Y = 0$ and $X = D$, $Y = 0$, is described as a linear elastic, geometrically non-linear inextensible bar[4] of bending stiffness EI and weight Q per unit length.

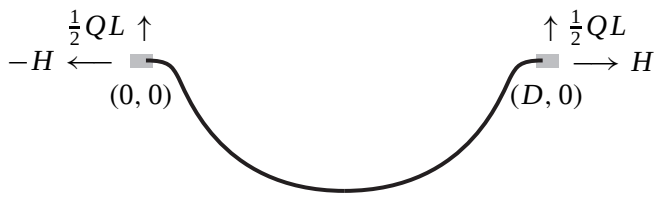


Figure 2.1: A suspended cable

At the suspension points the cable is horizontally clamped such that the cable hangs in the vertical plane through the suspension points. The total length L of the cable is larger than D, so the cable is not stretched.

In order to keep the cable in position, the suspension points apply a reaction force, with horizontal component $-H$ resp. H, and a vertical component V, resp. $QL - V$. From symmetry we already have $V = QL - V$ so $V = \frac{1}{2}QL$ is known. On the other hand, H, the force that keeps the cable ends apart, is unknown.

Let s be the arc length along the cable, and $\psi(s)$ the tangent angle with the horizon. Then the cartesian co-ordinates $(X(s), Y(s))$ of a point on the cable are given by

$$X(s) = \int_0^s \cos\psi(s')\,\mathrm{d}s', \quad Y(s) = \int_0^s \sin\psi(s')\,\mathrm{d}s'.$$

The shape of the cable $\psi(s)$ and the necessary force H, are determined by the following differential equation and boundary conditions

$$EI\frac{\mathrm{d}^2\psi}{\mathrm{d}s^2} = H\sin\psi - (Qs - V)\cos\psi$$

$$\psi(0) = 0, \quad \psi(L) = 0, \quad X(L) = D, \quad Y(L) = 0.$$

a. Make the equations and boundary conditions dimensionless by scaling all lengths on L.
   How many (and which) dimensionless problem parameters do we have?
b. Under what conditions can we approximate the equation by

   $$0 = H\sin\psi - (Qs - V)\cos\psi.$$

   Can we keep all the boundary conditions? Which would you keep? Can you solve the remaining equation?
c. Under what conditions can we approximate the equation by

   $$EI\frac{\mathrm{d}^2\psi}{\mathrm{d}s^2} = H\psi - (Qs - V).$$

   Can we keep all the boundary conditions? Do we have to adapt any to bring it in line with the used approximation? Can you solve the remaining equation (up to a numerical evaluation)?

[4] A so called Euler-Bernoulli bar.

2.2.11 Electrically heated metal

A piece of metal Ω of size L is heated, from an initial state $T(\boldsymbol{x}, t) \equiv 0$, to a temperature distribution T by applying at $t = 0$ an electric field with potential ψ and typical voltage V (Fig. 2.2). This heat

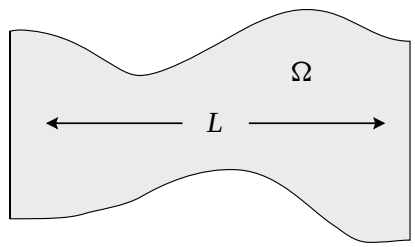


Figure 2.2: A piece of metal heated by an electric field.

source, the energy dissipation of the electric field, is given by the inhomogeneous term $\sigma|\nabla\psi|^2$ in the following inhomogeneous heat equation

$$C\frac{\partial T}{\partial t} = \kappa\nabla^2 T + \sigma|\nabla\psi|^2.$$

The edges are kept at $T = 0$, yielding a dissipation of thermal energy. As time proceeds, the temperature distribution will converge to a steady state corresponding to an equilibrium of heat production by the source and heat loss via the edges. We are interested in the typical time this takes and the typical final temperature.

If we introduce the formal scaling $T = T_0 u$, $t = t_0\tau$, $\boldsymbol{x} = L\boldsymbol{\xi}$, and $\psi = V\Psi$, then we get

$$\frac{CT_0}{t_0}\frac{\partial u}{\partial \tau} = \frac{\kappa T_0}{L^2}\nabla_{\xi}^2 u + \frac{\sigma V^2}{L^2}|\nabla_{\xi}\Psi|^2.$$

a. If we take the final (steady state) situation as reference, what would then be our choice for T_0?
b. What is then the choice for the time t_0?
c. Note that the boundary conditions are rather important. If the edges were thermally isolated, we would, at least initially, have no temperature gradients scaling on L, and the diffusion term $\kappa\nabla^2 T$ would be negligible. Only the storage term $C\frac{\partial}{\partial t}T$ would balance the source term, and there would be no other temperature to scale on than $\sigma V^2 t_0/CL^2$. In other words, the temperature would rise approximately linearly in time.

2.2.12 The Korteweg-de Vries equation

A version of the Korteweg-de Vries equation (an equation for certain types of water waves) is given by

$$A\zeta_t + B\zeta_{xxx} + C\zeta\zeta_x = 0$$

Rescale the $\zeta = \lambda\sigma$, $x = \alpha z$ and $t = \beta\tau$, such that the remaining equation has only coefficients equal to 1.

2.2.13 Traffic waves

A simple (but nonlinear) one-dimensional wave equation, used (for example) to model traffic flow density ρ at position x and time t, is

$$\frac{\partial \rho}{\partial t} + C(\rho)\frac{\partial \rho}{\partial x} = 0, \qquad \rho(x,0) = F(x).$$

Since dimensional quantities must include an inherent scale, we can write (with dimensionless shape functions g and f)

$$C(u) = C_0 g\left(\frac{\rho}{D}\right), \qquad F(x) = \rho_0 f\left(\frac{x}{L}\right).$$

a. Make the problem dimensionless in a sensible way. What is the remaining dimensionless parameter?
b. Show that the solution ρ is implicitly given by

$$\rho = F(x - C(\rho)t).$$

It is sufficient to consider the original equation. The dimensionless solution is similar.

2.2.14 The pendulum

Consider a pendulum consisting of a bob of mass m, suspended from a fixed, massless support of length L. The acceleration of gravity is g. Depending on time variable t, the pendulum angular displacement $\phi(t)$ swings between angle $-\alpha$ and α.

angle	ϕ,	dimension	-
angle	α,	”	-
time	t,	”	s
mass	m,	”	kg
length	L,	”	m
gravity	g,	”	m/s^2

a) What is the inherent time scale of the problem?
b) The motion is given by the equation

$$mL\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} + mg\sin\phi = 0.$$

Using a), make this equation dimensionless.
c) Under what condition can we approximate the dimensionless equation by

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} + \phi - \tfrac{1}{6}\phi^3 = 0$$

2.2.15 An equation

x satisfies the following equation

$$ax^2 + bf\left(\frac{x}{L}\right) = 0$$

with parameters a, b and L, and dimensionless function f with dimensionless argument, while $[x] =$ meters and $[b] =$ seconds.

a) What are the dimensions of a and L?

b) Find, by scaling $x = \lambda X$ for some suitable λ and collecting parameters in dimensionless groups R, equivalent equations of the form

$$X^2 + Rf(X) = 0, \quad X^2 + f(RX) = 0.$$

c) Under what conditions can the original equation be approximated by

$$f\left(\frac{x}{L}\right) = 0$$

2.2.16 Heat convection and diffusion

Consider a steady flow field $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x})$ of air of uniform density ρ and specific heat capacity c, and temperature $T = T(\boldsymbol{x}, t)$ at position $\boldsymbol{x}$ and time t. The heat is convected by the flow and diffused by Fourier's law for heat conduction, leading to the equations

$$\rho c\left(\frac{\partial T}{\partial t} + \boldsymbol{v}\cdot\nabla T\right) = -\nabla\cdot\boldsymbol{q}, \quad \boldsymbol{q} = -\kappa\nabla T,$$

where $\boldsymbol{q}$ is the heat flux density and κ is the coefficient of conductivity.

Assume that the typical velocity of the velocity field is U_0, and the length scale of the variation of both the flow field and the temperature field is L. Neglecting transient effects we have thus a typical time scale of L/U_0.

temperature	T,	dimension	K
length scale	L,	”	m
velocity	U_0,	”	m/s
density	ρ,	”	kg/m^3
heat flux density	$\boldsymbol{q}$,	”	W/m^2
specific heat capacity	c ,	”	J/kgK
conductivity	κ,	”	W/mK

a) Under what conditions (*i.e.* for which small parameter) can the diffusion be neglected, such that we obtain the simplified equation

$$\frac{\partial T}{\partial t} + \boldsymbol{v}\cdot\nabla T = 0$$

b) Show that (under these conditions) the temperature is constant along any streamline $\boldsymbol{x} = \boldsymbol{\xi}(t)$, given by

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t}.$$

2.2.17 Falling through the center of the earth.

Although it is unlikely that such a tunnel will ever be excavated in the near future, we assume a vacuum straight tunnel right through the center of the earth. It connects two opposite points on the earth's surface, separated by the earth's diameter $2R$. If the earth's mass density ρ is uniform, then according to Newton's law of gravitation any object in the tunnel at radial position r is attracted only by the part of the earth's mass that is inside the concentric sphere of radius r. The proportionality constant is the universal gravitation constant G.

At time $t = 0$ at position $r = R$ we drop a stone of negligible mass (compared to the mass of the earth) with zero initial speed. We wait until the stone returns at time $t = T$ (about 84 minutes).

The problem parameters and variables, according to our model, are

radius	R,	dimension	m
position	r,	”	m
time	t,	”	s
return time	T,	”	s
density	ρ,	”	kg/m^3
gravity constant	G,	”	m^3/s^{2}kg

Show by dimensional arguments that T depends only on ρ and G, and not on R. In other words, at whatever depth we release the stone, the return time is the same.

2.2.18 Heat conduction in a long bar

A semi-infinite isolated metal bar, given by $0 \leqslant x < \infty$, is heated by a uniform heat source of constant flux density Q at $x = 0$, starting from $t = 0$. Assume that the initial temperature $T = 0$, such that T is linearly proportional to Q. The bar metal has a specific heat capacity c and conductivity κ. Due to the uniform source and the isolation, the temperature along a cross section is uniform.

temperature	T,	dimension	K
length	x,	”	m
time	t,	”	s
density	ρ,	”	kg/m^3
specific heat capacity	c ,	”	J/kgK
conductivity	κ,	”	W/mK
heat source	Q,	”	W/m^2

a) According to Buckingham's Pi theorem, there are $6 - 4 = 2$ dimensionless groups possible (note that $T \propto Q$, so T/Q is to be considered as one variable). Give examples of such groups.
b) Show, by using a), that the most general form for $T(x, t)$ is[5]

$$T(x, t) = \frac{Qx}{\kappa} F\left(\sqrt{\frac{x^2 \rho c}{\kappa t}}\right).$$

[5]The seemingly different $T(x, t) = (Qt/\rho c x) G\left(\sqrt{x^2 \rho c/\kappa t}\right)$ is in reality of the same form. Write $F(\eta) = \eta^{-2} G(\eta)$.

c) Assume that T satisfies the equation

$$\rho c \frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2},$$

and define the *similarity variable* $\eta = \sqrt{x^2 \rho c / \kappa t}$. Derive the (ordinary) differential equation in the variable η for function $F(\eta)$ of b). Use the chain rule carefully when differentiating T to x and t. Make sure that the final equation *only* depends on η and contains no x or t dependence anymore.

The solution of this equation is not standard but can be found (for example) by Mathematica or Wolfram Alpha.

2.2.19 A Simple Balloon

A balloon rises in the atmosphere of density ρ_a such that it is at height $h(t)$ at time t. The balloon of mass m, fixed volume V and cross sectional surface A is subject to inertia $-mh''$, Archimedean (buoyancy) force $g\rho_a V$, weight $-mg$ and air drag $-\frac{1}{2}\rho_a C_d A (h')^2$, where $g = 9.8$ m/s^2 is the acceleration of gravity, and drag coefficient C_d depends on the geometry but is for a sphere (and high enough Reynolds number) in the order of 0.5.

Together these forces cancel out each other, so altogether we have the following equation for the dynamics of the balloon

$$m \frac{\mathrm{d}^2 h}{\mathrm{d}t^2} = g\rho_a V - gm - \frac{1}{2}\rho_a \left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2 C_d A.$$

Assume that $h(0) = 0$ and $h'(0) = 0$. The atmospheric air density will vary (in the troposphere, *i.e.* for $0 \leqslant h \leqslant 11$ km) with the height according to

$$\rho_a(h) = \rho_0 \left(1 - \frac{h}{L}\right)^{\alpha} \text{ kg/m}^3, \quad \text{with } \rho_0 = 1.225\,\text{kg/m}^3, \quad L = 44.33\,\text{km}, \quad \alpha = 4.256.$$

In practice a flexible balloon will grow in size with the decreasing atmospheric pressure, but we will ignore this and assume that the material is very stiff. What is the maximum attainable height, *i.e.* where $h' = h'' = 0$?

Make the equation dimensionless on the inherent length and time scales. There are two natural length scales in the problem (the atmospheric variation L and the diameter of the balloon $\sim V^{3/2}, \sim A^{1/2}$). What seems to be the most sensible one? Try both if you hesitate. The suitable time scale can be found by assuming that the dynamics is dominated by the balance between the buoyancy and the drag. When is this possibly not the case?

Introduce convenient dimensionless parameters and (in the case of ρ_a) shape function. Can you interpret these parameters? For what conditions can we neglect the inertia term? Is this reasonable for a balloon of $m = 1$ kg, $V = 2$ m^3 and $A = 1.9$ m^2. What about the initial conditions? Can you solve the remaining equation, at least implicitly?

2.2.20 A pulsating sphere

The radially symmetric sound field of a pulsating sphere $r = a_0 + a(t)$ (with a small) in a medium with mean density ρ_0 and sound speed c_0 is described by the following (linearised) equations for pressure perturbation p, density perturbation ρ and velocity perturbation v.

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \rho_0 \left(\frac{\partial v}{\partial r} + 2\frac{v}{r} \right) &= 0, \\ \rho_0 \frac{\partial v}{\partial t} + \frac{\partial p}{\partial r} &= 0, \\ p - c_0^2 \rho &= 0. \end{aligned}$$

while

$$v = \frac{\partial a}{\partial t} \quad \text{at } r = a_0.$$

If the sphere pulsates harmonically with frequency ω, we write for convenience

$$a = \mathrm{Re}(\hat{a}\, \mathrm{e}^{\mathrm{i}\omega t}), \;\; p = \mathrm{Re}(\hat{p}\, \mathrm{e}^{\mathrm{i}\omega t}), \;\; v = \mathrm{Re}(\hat{v}\, \mathrm{e}^{\mathrm{i}\omega t}), \;\; \rho = \mathrm{Re}(\hat{\rho}\, \mathrm{e}^{\mathrm{i}\omega t}).$$

leading to the equations (we eliminate ρ)

$$\begin{aligned} \mathrm{i}\omega \hat{p} + \rho_0 c_0^2 \left(\frac{\partial \hat{v}}{\partial r} + 2\frac{\hat{v}}{r} \right) &= 0, \\ \mathrm{i}\omega \rho_0 \hat{v} + \frac{\partial \hat{p}}{\partial r} &= 0. \end{aligned}$$

with

$$\hat{v} = \mathrm{i}\omega \hat{a} \quad \text{at} \quad r = a_0.$$

The proper solution of the equations can be shown to be

$$\begin{aligned} \hat{p} &= \frac{A}{r}\, \mathrm{e}^{-\mathrm{i}kr} \\ \hat{v} &= \frac{1}{\rho_0 c_0} \frac{A}{r} \left(1 + \frac{1}{\mathrm{i}kr} \right) \mathrm{e}^{-\mathrm{i}kr} \end{aligned}$$

with constant A to be determined, and the acoustic wavenumber

$$k = \frac{\omega}{c_0} = \frac{2\pi}{\lambda}$$

where λ is the free field wavelength.

a. Determine A by applying the boundary condition at $r = a_0$.
b. Since the perturbations depend linearly on the pulsation amplitude, we scale $\hat{p}$ and $\hat{v}$ on $\hat{a}/a_0$. Furthermore, we make dimensionless: $\hat{p}$ on ρc_0^2 and $\hat{v}$ on c_0. Altogether we rewrite therefore $\hat{p} = \rho c_0^2 (\hat{a}/a_0) p$ and $\hat{v} = c_0 (\hat{a}/a_0) v$ (where p and v are not the dimensional quantities of the original problem, but we left the notation just for convenience). Lengths can be scaled on a_0 and on $1/k$. Do both. Their ratio, dimensionless number $\varepsilon = ka_0$, is called Helmholtz number.
c. Simplify the formulas for small source size (known as a *compact source*), *i.e.* $\varepsilon = ka_0 \ll 1$. What do you get in each case of scaling? Can you interpret the results?

Chapter 3

Asymptotic Analysis

3.1 Basic Definitions and Theorems

We will be interested to analyse the behaviour of a function near a particular point y_0, say $y \to y_0$, especially when this point is a singularity of some kind. Usually, we have to distinguish between the behaviour on the right side (*i.e.* $y \downarrow y_0$) and on the left side (*i.e.* $y \uparrow y_0$). On the other hand, y_0 can always be assumed to be 0. If it is finite, it can be identified to 0 by the transformation $\varepsilon = y - y_0$ or $\varepsilon = y_0 - y$. If it is $\pm\infty$, it can be identified to 0 by the transformation $\varepsilon = 1/y$ or $\varepsilon = -1/y$. In general, we consider therefore $f(\varepsilon)$ or $f(x, \varepsilon)$ for $\varepsilon \downarrow 0$.

1. $\boldsymbol{O}$ (Big O)
   $f(\varepsilon) = O(\varphi(\varepsilon))$ as $\varepsilon \to 0$ if there are positive constants K and ε_1 (both independent of ε) such that
   $$|f(\varepsilon)| \leqslant K|\varphi(\varepsilon)| \text{ for } 0 < \varepsilon < \varepsilon_1.$$
   Intuitive interpretation: f can be embraced completely by $|\varphi|$ (up to a multiplicative constant) in a neighbourhood of 0. A crude estimate (for example $\sin\varepsilon = O(1/\varepsilon)$) is not incorrect, but a sharp estimate is more informative.

   Examples: $\sin\varepsilon = O(\varepsilon)$, $(1-\varepsilon)^{-1} = O(1)$, $\sin(1/\varepsilon) = O(1)$, $(\varepsilon + \varepsilon^2)^{-1} = O(\varepsilon^{-1})$, $\ln((1+\varepsilon)/\varepsilon) = O(\ln\varepsilon)$.

2. $\boldsymbol{o}$ (small o)
   $f(\varepsilon) = o(\varphi(\varepsilon))$ as $\varepsilon \to 0$ if for every $\delta > 0$, there is an ε_1 (independent of ε) such that
   $$|f(\varepsilon)| \leqslant \delta|\varphi(\varepsilon)| \text{ for } 0 < \varepsilon < \varepsilon_1.$$
   Intuitive interpretation: f is always smaller than any multiple (however small) of $|\varphi|$ in a neighbourhood of 0. Again, a crude estimate is not incorrect, but a sharp estimate is more informative.

   Examples: $\sin(2\varepsilon) = o(1)$, $\cos\varepsilon = o(\varepsilon^{-1})$, $\mathrm{e}^{-a/\varepsilon} = o(\varepsilon^n)$ for any $a > 0$ and any n.

3. $\boldsymbol{O_s}$ (sharp O)
   $f(\varepsilon) = O_s(\varphi(\varepsilon))$ as $\varepsilon \to 0$ if $f(\varepsilon) = O(\varphi(\varepsilon))$ and $f(\varepsilon) \neq o(\varphi(\varepsilon))$.

   Intuitive interpretation: f is behaves exactly the same (up to a multiplicative constant) as φ in a neighbourhood of 0.

   Examples: $2\sin\varepsilon = O_s(\varepsilon)$, $3\cos\varepsilon = O_s(1)$, but there is *no* n such that $\ln\varepsilon = O_s(\varepsilon^n)$.

4. **Similar behaviour.** Implications (ii), (iii) and (iv) often serve as the definition of O, o and O_s:

   (i) If $f = o(\varphi)$ then $f = O(\varphi)$.

   (ii) If $\lim_{\varepsilon \downarrow 0} \left| \frac{f(\varepsilon)}{\varphi(\varepsilon)} \right| = L \in [0, \infty)$ then $f = O(\varphi)$.

   (iii) If $\lim_{\varepsilon \downarrow 0} \left| \frac{f(\varepsilon)}{\varphi(\varepsilon)} \right| = 0$ then $f = o(\varphi)$.

   (iv) If $\lim_{\varepsilon \downarrow 0} \left| \frac{f(\varepsilon)}{\varphi(\varepsilon)} \right| = L \in (0, \infty)$ then $f = O_s(\varphi)$.

   (v) If $f = O(\varphi)$ and $\varphi = O(f)$ then $f = O_s(\varphi)$.

5. **Asymptotic approximation.**
   $\varphi(\varepsilon)$ is an asymptotic approximation to $f(\varepsilon)$ as $\varepsilon \to 0$, denoted by $f \sim \varphi$, if

$$f(\varepsilon) = \varphi(\varepsilon) + o(\varphi(\varepsilon)) \quad \text{as } \varepsilon \to 0,$$

   Intuitive interpretation: If $\lim_{\varepsilon \to 0} f/\varphi = 1$ then $f \sim \varphi$. *Note:* $f \sim 0$ is only possible if $f \equiv 0$.
   Examples: $\sin \varepsilon \sim \varepsilon$, $(\varepsilon + \varepsilon^2)^{-1} \sim 1/\varepsilon$, $\ln(a\varepsilon) \sim \ln \varepsilon$ for any $a > 0$.

6. **Pointwise asymptotic approximation.**
   $\varphi(x, \varepsilon)$ is a pointwise asymptotic approximation to $f(x, \varepsilon)$ as $\varepsilon \to 0$ if

$$f(x, \varepsilon) \sim \varphi(x, \varepsilon) \quad \text{for fixed } x.$$

   Intuitive interpretation: $f(x, \varepsilon)$ is approximated asymptotically better and better by $\varphi(x, \varepsilon)$ for $\varepsilon \to 0$ and x *fixed*. We don't know anything yet if we allow x to become small or large (within the domain).

   Examples: $\sin(x + \varepsilon) \sim \sin x$ and $\sin x \neq 0$, $1/(\varepsilon + x) \sim 1/x$ and $x \neq 0$. Note that in the last example the approximation fails if we would scale $x = \varepsilon^n t$ for any $n \geqslant 1$.

7. **Uniform asymptotic approximation.**
   The continuous function $\varphi(x, \varepsilon)$ is a uniform asymptotic approximation to the continuous function $f(x, \varepsilon)$ for $x \in \mathcal{D}$ as $\varepsilon \to 0$, if the way φ approaches f is the same for all x.
   More precisely: if for any positive number δ there is an ε_1 (independent of x and ε) such that

$$|f(x, \varepsilon) - \varphi(x, \varepsilon)| \leqslant \delta |\varphi(x, \varepsilon)| \quad \text{for } x \in \mathcal{D} \quad \text{and } 0 < \varepsilon < \varepsilon_1.$$

   Intuitive interpretation:
   $f(x, \varepsilon)$ is approximated uniformly by $\varphi(x, \varepsilon)$, if the approximation is preserved with any scaling of $x = a(\varepsilon) + b(\varepsilon)t$, valid in the domain of f. In formulas (with a scaling $x = \varepsilon t \in [0, K]$ as an example):

$$\text{if } f(x, \varepsilon) \sim \varphi(x, \varepsilon) \text{ and } \varphi(\varepsilon t, \varepsilon) \sim g(t, \varepsilon), \text{ then also } f(\varepsilon t, \varepsilon) \sim g(t, \varepsilon).$$

Examples:

$\cos(\varepsilon) + \mathrm{e}^{-x/\varepsilon} \sim 1$	only pointwise for $x \in (0, \infty)$. Not uniform: take $x = \varepsilon t$.
$\cos(\varepsilon) + \mathrm{e}^{-x/\varepsilon} \sim 1$	pointwise and uniformly for $x \in [a, \infty)$, $a > 0$.
$\cos(\varepsilon) + \mathrm{e}^{-t} \sim 1 + \mathrm{e}^{-t}$	uniformly for $t \in [0, \infty)$.
$\sin(\varepsilon x + \varepsilon) \sim \varepsilon(x + 1)$	only pointwise for $x \in (-\infty, \infty)$. Take $x = t/\varepsilon$.
$\sin(\varepsilon x + \varepsilon) \sim \varepsilon(x + 1)$	uniformly for $x \in [-a, a]$, $0 < a < \infty$.
$2 + \sin(t + \varepsilon) \sim 2 + \sin(t)$	uniformly for $t \in (-\infty, \infty)$.
$2 + \mathrm{e}^{\varepsilon} \sin(t + \varepsilon t) \sim 2 + \sin(t)$	only pointwise for $t \in \mathbb{R}$. Note that $\sin(t + \varepsilon t) = \sin t + O(\varepsilon t)$.
$2 + \mathrm{e}^{\varepsilon} \sin(\tau) \sim 2 + \sin(\tau)$	uniform for $\tau \in \mathbb{R}$. Note that we rescaled $\tau = (1 + \varepsilon)t$.

Uniform implies *pointwise*, but the reverse is not necessarily true. See the above examples.

8. If f and φ are absolutely integrable, and $f(x, \varepsilon) \sim \varphi(x, \varepsilon)$ uniformly on a domain $\mathcal{D}$, while $\int_{\mathcal{D}} |\varphi| \mathrm{d}x = O(\int_{\mathcal{D}} \varphi \mathrm{d}x)$, then $\int_{\mathcal{D}} f(x, \varepsilon)\, \mathrm{d}x \sim \int_{\mathcal{D}} \varphi(x, \varepsilon)\, \mathrm{d}x$.

9. **Asymptotic sequence.**
The sequence $\{\mu_n(\varepsilon)\}$ is called an asymptotic sequence, if $\mu_{n+1} = o(\mu_n)$ as $\varepsilon \to 0$ for each $n = 0, 1, 2, \cdots$. This is denoted symbolically

$$\mu_0 \gg \mu_1 \gg \mu_2 \gg \cdots \gg \mu_n \gg \ldots$$

Common examples are $\mu_n = \varepsilon^n$, or more generally $\mu_n = \delta(\varepsilon)^n$ if $\delta(\varepsilon) = o(1)$. Combinations of ε and $\ln(\varepsilon)$ yield the sequence $\mu_{n,k} = \varepsilon^n \ln(\varepsilon)^k$, where $k = n, \cdots, 0$ and

$$\ln \varepsilon \gg 1 \gg \varepsilon \ln(\varepsilon) \gg \varepsilon \gg \varepsilon^2 \ln(\varepsilon)^2 \gg \varepsilon^2 \ln(\varepsilon) \gg \varepsilon^2 \gg \ldots$$

10. **Asymptotic expansion.**
If $\{\mu_n(\varepsilon)\}$ is an asymptotic sequence, then $f(\varepsilon)$ has an asymptotic expansion of $N + 1$ terms with respect to this sequence, denoted by

$$f(\varepsilon) \sim \sum_{n=0}^{N} a_n \mu_n(\varepsilon),$$

where the coefficients a_n are independent of ε, if for each $M = 0, \ldots, N$

$$f(\varepsilon) - \sum_{n=0}^{M} a_n \mu_n(\varepsilon) = o(\mu_M(\varepsilon)) \quad \text{as } \varepsilon \to 0.$$

$\mu_n(\varepsilon)$ is called a **gauge function** or **order function**.

If $\mu_n(\varepsilon) = \varepsilon^n$, we call the expansion an asymptotic power series. Any Taylor series in ε around $\varepsilon = 0$ is also an asymptotic power series.

Asymptotic expansions, based on Taylor expansions in ε^n, of elementary functions:

$$\begin{aligned}
\mathrm{e}^{\varepsilon} &= 1 + \varepsilon + \tfrac{1}{2}\varepsilon^2 + \ldots \\
\sin(\varepsilon) &= \varepsilon - \tfrac{1}{6}\varepsilon^3 + \ldots \\
\cos(\varepsilon) &= 1 - \tfrac{1}{2}\varepsilon^2 + \ldots \\
\frac{1}{1-\varepsilon} &= 1 + \varepsilon + \varepsilon^2 + \ldots \\
\ln(1-\varepsilon) &= -\varepsilon - \tfrac{1}{2}\varepsilon^2 - \tfrac{1}{3}\varepsilon^3 - \ldots \\
\ln(1+\varepsilon) &= \varepsilon - \tfrac{1}{2}\varepsilon^2 + \tfrac{1}{3}\varepsilon^3 - \ldots \\
(1+\varepsilon)^{\alpha} &= 1 + \alpha\varepsilon + \tfrac{1}{2}\alpha(\alpha-1)\varepsilon^2 + \ldots
\end{aligned}$$

Examples of combinations (which are sometimes not Taylor expansions in ε^n)

$$\begin{aligned}
\varepsilon^{\varepsilon} &= \mathrm{e}^{\varepsilon \ln \varepsilon} = 1 + \varepsilon \ln \varepsilon + \tfrac{1}{2}\varepsilon^2 (\ln \varepsilon)^2 + \ldots \\
\ln(\sin \varepsilon) &= \ln \varepsilon - \tfrac{1}{6}\varepsilon^2 + \ldots \\
\ln(\cos \varepsilon) &= -\tfrac{1}{2}\varepsilon^2 - \tfrac{1}{12}\varepsilon^4 + \ldots \\
\frac{1}{1-f(\varepsilon)} &= 1 + f(\varepsilon) + f(\varepsilon)^2 + \ldots, \qquad \text{if } \; f(\varepsilon) = o(1).
\end{aligned}$$

11. **How to determine the coefficients.**
The coefficients a_n of an asymptotic expansion can be determined uniquely (for given $\mu_n(\varepsilon)$) by the following recursive procedure

$$a_0 = \lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{\mu_0(\varepsilon)}, \quad a_1 = \lim_{\varepsilon \to 0} \frac{f(\varepsilon) - a_0\mu_0(\varepsilon)}{\mu_1(\varepsilon)}, \quad \ldots \quad a_N = \lim_{\varepsilon \to 0} \frac{f(\varepsilon) - \sum_{n=0}^{N-1} a_n \mu_n(\varepsilon)}{\mu_N(\varepsilon)},$$

provided μ_n are nonzero for ε near 0 and each of the limits exist.

12. **Convergent and asymptotic.**
Let $\{\mu_n(\varepsilon)\}$ be an asymptotic sequence, with $\mu_0 = 1$ and $\varepsilon > 0$, and let

$$f(\varepsilon) = \sum_{n=0}^{N} a_n \mu_n(\varepsilon) + R_N(\varepsilon).$$

If the series converges for $N \to \infty$, then $\lim_{N\to\infty} R_N(\varepsilon) = 0$. If the series is an asymptotic expansion for $\varepsilon \to 0$, then $\lim_{\varepsilon\to 0} R_N(\varepsilon) = 0$. A convergent power series (like a Taylor series) is also an asymptotic expansion. An asymptotic expansion is not necessarily convergent.

13. **Asymptotically equal**
Two functions f and g are asymptotically equal up to N terms, with respect to the asymptotic sequence $\{\mu_n\}$, if $f - g = o(\mu_N)$.

14. **The fundamental theorem of asymptotic expansions [10]**
An asymptotic expansion vanishes only if the coefficients vanish, i.e.

$$\big\{a_0\mu_0(\varepsilon) + a_1\mu_1(\varepsilon) + a_2\mu_2(\varepsilon) + \ldots = 0 \quad (\varepsilon \to 0)\big\} \;\Leftrightarrow\; \big\{a_0 = a_1 = a_2 = \ldots = 0\big\}.$$

15. **Poincaré expansion.**
Let $\{\mu_n(\varepsilon)\}$ be an asymptotic sequence of **order functions**. If $f(x,\varepsilon)$ has an asymptotic expansion with respect to this sequence, given by

$$f(x,\varepsilon) \sim \sum_{n=0}^{N} a_n(x)\mu_n(\varepsilon),$$

where the **shape functions** $a_n(x)$ are **independent** of ε, then this expansion is called a Poincaré expansion. *Note*: a Poincaré expansion is *never* Poincaré anymore after (nontrivial) rescaling x.

16. **Regular and singular expansion.**
If a Poincaré expansion is uniform in x on a given domain $\mathcal{D}$ this expansion is called a regular expansion. Else, the expansion is called a singular expansion.

Note: A typical indication for non-uniformity is a scaling, such that the asymptotic ordering of the terms is violated. In other words, a scaled $x = x(\varepsilon)$ with $a_1(x)\mu_1(\varepsilon) \not\ll a_0(x)\mu_0(\varepsilon)$, etc.

17. **Role of scaling.**
A Poincaré expansion and its region of uniformity depends (among other things) on the chosen scaling $x = x_0 + \delta(\varepsilon)\xi$ and the domain $\mathcal{D}$.

For example, $\mathrm{e}^{-x/\varepsilon} + \sin(x+\varepsilon) = \sin(x) + O(\varepsilon)$ is regular on any positive interval $[a,b]$ with $a, b = O(1)$ but is singular on $(0,b]$, while $\mathrm{e}^{-t} + \sin(\varepsilon t + \varepsilon) = \mathrm{e}^{-t} + \varepsilon(t+1) + O(\varepsilon^3)$ is regular on any finite fixed interval.

18. **Manipulations of asymptotic expansions.**
Let $f(x,\varepsilon)$ and $g(x,\varepsilon)$ have Poincaré expansions on $\mathcal{D}$ with asymptotic sequence $\{\mu_n(\varepsilon)\}$

$$f(x,\varepsilon) = \mu_0(\varepsilon)a_0(x) + \mu_1(\varepsilon)a_1(x) + \cdots$$
$$g(x,\varepsilon) = \mu_0(\varepsilon)b_0(x) + \mu_1(\varepsilon)b_1(x) + \cdots$$

<u>*Addition*</u>. Then the sum has the following asymptotic expansion

$$f + g = \mu_0(a_0 + b_0) + \mu_1(a_1 + b_1) + \cdots$$

<u>*Multiplication.*</u> If $\{\mu_k\mu_n\}$ can be asymptotically ordered to the asymptotic sequence $\{\gamma_n\}$, with $\gamma_0 = \mu_0^2$, $\gamma_1 = \mu_0\mu_1$, $\gamma_2 = O(\mu_0\mu_2 + \mu_1^2)$, etc., then the product has the asymptotic expansion

$$fg = (\mu_0 a_0 + \mu_1 a_1 + \cdots)(\mu_0 b_0 + \mu_1 b_1 + \cdots) = \gamma_0 a_0 b_0 + \gamma_1(a_0 b_1 + a_1 b_0) + \gamma_2(\cdots) + \cdots$$

<u>*Integration.*</u> If the approximation is uniform, f, a_0, a_1, etc. are absolute-integrable on $\mathcal{D}$, while $\int_{\mathcal{D}} a_n \,\mathrm{d}x \neq 0$, then we can integrate term by term and obtain the asymptotic expansion

$$\int_{\mathcal{D}} f(x,\varepsilon)\,\mathrm{d}x = \mu_0 \int_{\mathcal{D}} a_0(x)\,\mathrm{d}x + \mu_1 \int_{\mathcal{D}} a_1(x)\,\mathrm{d}x + \cdots$$

<u>*Differentiation.*</u> This is the least obvious. If both f and f' have asymptotic expansions on $\mathcal{D}$

$$f(x,\varepsilon) = \mu_0(\varepsilon)a_0(x) + \mu_1(\varepsilon)a_1(x) + \cdots, \quad f'(x,\varepsilon) = \mu_0(\varepsilon)q_0(x) + \mu_1(\varepsilon)q_1(x) + \cdots$$

then the derivative of the expansion of f is the expansion of derivative f', and satisfy

$$q_0 = a_0', \quad q_1 = a_1', \quad \text{etc.}$$

Counterexample: $f(x,\varepsilon) = \frac{1}{2}x^2 + \varepsilon\cos(\frac{x}{\varepsilon}) = \frac{1}{2}x^2 + O(\varepsilon)$, but $f'(x,\varepsilon) \neq x + O(\varepsilon)$.

3.2 Asymptotic expansions applied

3.2.1 General procedure for algebraic equations

The existence of an asymptotic expansion yields a class of methods to solve problems that depend on a parameter which is typically small in the range of interest. Such methods are called *perturbation methods*. The importance of these methods are two-fold. They provide analytic solutions to otherwise intractable problems, and the asymptotic structure of the solution provides instant insight into the dominating qualities.

If $x(\varepsilon)$ is implicitly given as the solution of an algebraic equation

$$\mathcal{F}(x, \varepsilon) = 0 \tag{3.1}$$

we may solve this asymptotically for $\varepsilon \to 0$ in the following steps.

(i) First we prove, make plausible, or check in one way or another that a solution *exists*, and try to find out if this solution is unique or there are more. This is not really an asymptotic question, but important because the approximations involved later in the solution process may fool us: the approximated equation may have no solutions while the original has, or the other way round. Sometimes the existence of solutions is obvious straightaway, but sometimes global arguments should be invoked.

(ii) Then we have to find the order of magnitude of the sought solution, say $x(\varepsilon) = \gamma(\varepsilon)X(\varepsilon)$ with $X = O_s(1)$. Unless we have scaled the problem already correctly, the solution is not necessarily $O(1)$. Often, we cannot decide with certainty, and we have to make a suitable assumption that is consistent with all the information we have, and proceed to construct successfully a solution or until we encounter a contradiction.

Another point of concern is the fact that there may be more solutions with different scalings.

The scaling function $\gamma(\varepsilon)$ is found such that it yields a meaningful $X = O_s(1)$ in the limit $\varepsilon \to 0$. This is called a *distinguished limit*, while the reduced equation for $X(0)$, *i.e.* $\mathcal{F}_0(X) = 0$, is called a *significant degeneration* (there may be more than one.) We can rescale $\mathcal{F}$ and x such $\mathcal{F}(x, \varepsilon) = 0$ becomes $\mathcal{G}(X, \varepsilon) = 0$ while $\mathcal{G}(X, 0) = O(1)$.

(iii) The final stage is to make an assumption about an asymptotic expansion of the solution X for small ε

$$X(\varepsilon) = X_0 + \mu_1(\varepsilon)X_1 + \mu_2(\varepsilon)X_2 + \dots$$

This is *only* an assumption, based on a successful and consistent construction later. If we encounter a contradiction, we have to go back and correct or alter the assumed expansion.

If both $X(\varepsilon)$ and $\mathcal{G}(X, \varepsilon)$ have an asymptotic series expansion with the same gauge functions, $X(\varepsilon)$ may be determined asymptotically by the following perturbation method. We expand X, substitute this expansion in $\mathcal{G}$, and expand $\mathcal{G}$ to obtain

$$\mathcal{G}(X, \varepsilon) = \mathcal{G}_0(X_0) + \mu_1(\varepsilon)\mathcal{G}_1(X_1, X_0) + \mu_2(\varepsilon)\mathcal{G}_2(X_2, X_1, X_0) + \dots = 0.$$

From the Fundamental Theorem of asymptotic expansions (3.14) it follows that each term $\mathcal{G}_n$ vanishes, and the sequence of coefficients (X_n) can be determined by induction:

$$\mathcal{G}_0(X_0) = 0, \qquad \mathcal{G}_1(X_1, X_0) = 0, \qquad \mathcal{G}_2(X_2, X_1, X_0) = 0, \qquad \textit{etc.} \tag{3.2}$$

It should be noted that finding the sequence of gauge functions (μ_n) is of particular importance. This is in general done iteratively, but sometimes a good guess also works. For example, if $\mathcal{G}$ is a smooth function of ε, in particular in $\varepsilon = 0$, then in most cases an asymptotic power series will work, *i.e.* $\mu_n(\varepsilon) = \varepsilon^n$.

We have to realise that a successful construction is not a *proof* for its correctness. Strictly mathematical proofs are usually very difficult, and in the context of modelling not common. Successfully finding a consistent solution is normally the strongest indication for its correctness we can obtain.

3.2.2 Example: roots of a polynomial

We illustrate this procedure by the following example. Consider the roots for $\varepsilon \to 0$ of the equation

$$x^3 - \varepsilon x^2 + 2\varepsilon^3 x + 2\varepsilon^6 = 0.$$

Since the polynomial is of 3^{d} order, and is negative for $x = -1$ (and ε small), positive in $x = \varepsilon^2$, negative in $x = -\frac{1}{2}\varepsilon$, and positive in $x = 1$, there are exactly 3 real solutions $x^{(1)}, x^{(2)}, x^{(3)}$.

From the structure of the equation it seems reasonable to assume that the order of magnitude of the solutions scale like a power of ε. We write

$$x = \varepsilon^n X(\varepsilon), \quad X = O_s(1)$$

We have to determine exponent n first. This is done by balancing terms, and then seek such n that produce a non-trivial limit under the limit $\varepsilon \to 0$: the *distinguished limits* of step (ii) above.
We compare asymptotically the coefficients in the equation that remain after scaling

$$\varepsilon^{3n} X^3 - \varepsilon^{1+2n} X^2 + 2\varepsilon^{3+n} X + 2\varepsilon^6 = 0.$$

Consider now the order of magnitude of the coefficients:

$$\varepsilon^{3n}, \quad \varepsilon^{1+2n}, \quad \varepsilon^{3+n}, \quad \varepsilon^6.$$

By dividing by the biggest coefficient (this depends on n), we can always make sure that one coefficient is 1 and the others are smaller. For example, if $n = 0$ we have

$$1, \quad \varepsilon, \quad \varepsilon^3, \quad \varepsilon^6.$$

If $n = 2$ we have

$$\varepsilon, \quad 1, \quad 1, \quad \varepsilon.$$

If $n = 4$ we have

$$\varepsilon^6, \quad \varepsilon^3, \quad \varepsilon, \quad 1.$$

If none balance (like for $n = 0$ and $n = 4$), the asymptotically biggest, with coefficient 1, would be zero on its own, which thus implies to leading order that $X = 0$. However, this is *not* $O_s(1)$ and therefore not a valid scaling. So at least two should be of the same order of magnitude *and* dominate (like with $n = 2$).

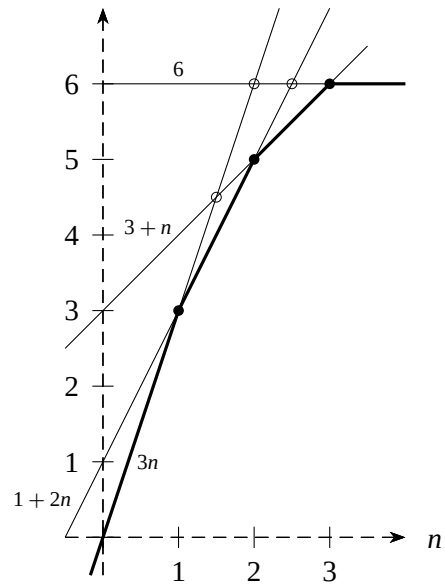


Figure 3.1: Analysis of distinguished limits.

In other words: in order to have a meaningful (or "significant") degenerate solution $X(0) = O_s(1)$, at least two terms of the equation should be asymptotically equivalent, and at the same time of leading order when $\varepsilon \to 0$.

So this leaves us with the task to compare the exponents $3n$, $1+2n$, $3+n$, 6 as a function of n. Consider the Figure 3.1. The solid lines denote the exponents of the powers of ε, that occur in the coefficients of the equation considered. At the intersections of these lines, denoted by the open and closed circles, we find the candidates of distinguished limits, *i.e.* the points where at least two coefficients are asymptotically equivalent. Finally, only the closed circles are the distinguished limits, because these are located along the lower envelope (thick solid line) and therefore correspond to leading order terms when $\varepsilon \to 0$. We have now three cases.

$n = 1$.

$$\varepsilon^3 X^3 - \varepsilon^3 X^2 + 2\varepsilon^4 X + 2\varepsilon^6 = 0, \quad \text{or} \quad X^3 - X^2 + 2\varepsilon X + 2\varepsilon^3 = 0.$$

From the structure of the equation it seems reasonable to assume that X has an asymptotic expansion in powers of ε. If we assume the expansion $X = X_0 + \varepsilon X_1 + \ldots$, we finally have

$$X_0^3 - X_0^2 = 0, \quad 3X_0^2 X_1 - 2X_0 X_1 + 2X_0 = 0, \quad \textit{etc.}$$

and so $X_0 = 1$, and $X_1 = -2$, *etc.* leading to $x(\varepsilon) = \varepsilon - 2\varepsilon^2 + \ldots$ Note that solution $X_0 = 0$ is excluded because that would change the order of the scaling!

$n = 2$.

$$\varepsilon^6 X^3 - \varepsilon^5 X^2 + 2\varepsilon^5 X + 2\varepsilon^6 = 0, \quad \text{or} \quad \varepsilon X^3 - X^2 + 2X + 2\varepsilon = 0.$$

From the structure of the equation it seems reasonable to assume that X has an asymptotic expansion in powers of ε. If we assume the expansion $X = X_0 + \varepsilon X_1 + \ldots$, we finally have

$$-X_0^2 + 2X_0 = 0, \quad X_0^3 - 2X_0 X_1 + 2X_1 + 2 = 0, \quad \textit{etc.}$$

and so $X_0 = 2$, $X_1 = 5$, *etc.* , leading to $x(\varepsilon) = 2\varepsilon^2 + 5\varepsilon^3 + \ldots$

$n = 3$.

$$\varepsilon^9 X^3 - \varepsilon^7 X^2 + 2\varepsilon^6 X + 2\varepsilon^6 = 0, \quad \text{or} \quad \varepsilon^3 X^3 - \varepsilon X^2 + 2X + 2 = 0.$$

From the structure of the equation it seems reasonable to assume that X has an asymptotic expansion in powers of ε. If we assume the expansion $X = X_0 + \varepsilon X_1 + \ldots$, we finally have

$$2X_0 + 2 = 0, \quad -X_0^3 + 2X_1 = 0, \quad \textit{etc.}$$

and so $X_0 = -1$, $X_1 = -\frac{1}{2}$, *etc.* , leading to $x(\varepsilon) = -\varepsilon^3 - \frac{1}{2}\varepsilon^4 + \ldots$ □

Epilogue

The presented method of finding the distinguished limits depends greatly on the power-law scaling function ε^n, inherent in polynomial equations. This will not be possible for more general algebraic equations where it will not always be so easy to guess the general form of the scaling function or the gauge functions of the asymptotic expansion. In general, the scaling function or functions will have to found by careful ad-hoc balancing arguments, while the terms of the expansion will have to be estimated iteratively by a similar process of balancing. See the exercises.

3.3 Exercises

3.3.1 Asymptotic expansions in ε

3.3.1.1 What values of α, if any, yield (i) $f = O(\varepsilon^\alpha)$, (ii) $f = o(\varepsilon^\alpha)$, (iii) $f = O_s(\varepsilon^\alpha)$ as $\varepsilon \to 0$?

(a) $f = \sqrt{1+\varepsilon^2}$

(b) $f = \varepsilon \sin(\varepsilon)$

(c) $f = (1 - \mathrm{e}^\varepsilon)^{-1}$

(d) $f = \ln(1+\varepsilon)$

(e) $f = \varepsilon \ln(\varepsilon)$

(f) $f = \sin(1/\varepsilon)$

(g) $f = \sqrt{x+\varepsilon}$, where $0 \leqslant x \leqslant 1$

(h) $f = \mathrm{e}^{-x/\varepsilon}$, where $x \geqslant 0$

3.3.1.2 Determine asymptotic expansions for $\varepsilon \to 0$ of

(a) $\varepsilon / \tan \varepsilon$,

(b) $\varepsilon/(1-\varepsilon^\varepsilon)$,

(c) $1/(\log(\varepsilon) - \varepsilon)$,

(d) $\log(\sin \varepsilon)$.

(e) $(1 - \varepsilon + \varepsilon^2 \ln \varepsilon)/(1 - \varepsilon \ln \varepsilon - \varepsilon + \varepsilon^2 \ln \varepsilon)$.

3.3.1.3 Assuming $f \sim a\varepsilon^\alpha + b\varepsilon^\beta + \ldots$, find α, β (with $\alpha < \beta$) and nonzero a, b for the following functions:

(a) $f = 1/(1 - \mathrm{e}^\varepsilon)$

(b) $f = \sinh(\sqrt{1+\varepsilon x})$ for $0 < x < \infty$.

(c) $f = \int_0^\varepsilon \sin(x + \varepsilon x^2)\,\mathrm{d}x$

3.3.2 Asymptotic sequences

3.3.2.1 Are the following sequences *asymptotic sequences* for $\varepsilon \to 0$. If not, arrange them so that they are or explain why it is not possible to do so.

(a) $\phi_n = (1 - \mathrm{e}^{-\varepsilon})^n$ for $n = 0, 1, 2, 3, \ldots$

(b) $\phi_n = [2\sinh(\varepsilon/2)]^{n/2}$ for $n = 0, 1, 2, 3, \ldots$

(c) $\phi_n = 1/\varepsilon^{1/n}$ for $n = 1, 2, 3, \ldots$

(d) $\phi_1 = 1, \phi_2 = \varepsilon, \phi_3 = \varepsilon^2, \phi_4 = \varepsilon\ln(\varepsilon), \phi_5 = \varepsilon^2\ln(\varepsilon), \phi_6 = \varepsilon\ln^2(\varepsilon), \phi_7 = \varepsilon^2\ln^2(\varepsilon)$.

(e) $\phi_n = \varepsilon^{n\varepsilon}$ for $n = 0, 1, 2, 3, \ldots$

(f) $\phi_n = \varepsilon^{n/\varepsilon}$ for $n = 0, 1, 2, 3, \ldots$

3.3.3 Asymptotic expansions in x and ε

3.3.3.1 Find a one-term asymptotic approximation, for $\varepsilon \to 0$, of the form $f(x, \varepsilon) \sim \phi(x)$ that holds for $-1 < x < 1$. Sketch $f(x, \varepsilon)$ and ϕ, and then explain why the approximation is not uniform for $-1 < x < 1$.

(a) $f(x, \varepsilon) = x + \exp\big((x^2 - 1)/\varepsilon\big)$

(b) $f(x, \varepsilon) = x + \tanh(x/\varepsilon)$

(c) $f(x, \varepsilon) = x + 1/\cosh(x/\varepsilon)$

3.3.3.2 Determine, if possible, regular expansions (*i.e.* uniform Poincaré expansions) for $\varepsilon \to 0$ and $x \in [0, 1]$ of

(a) $\sin(\varepsilon x)$,

(b) $1/(\varepsilon + x)$,

(c) $x\log(\varepsilon x)$,

(d) $\mathrm{e}^{-\sin(x)\varepsilon}$,

(e) $\mathrm{e}^{-\sin(x)/\varepsilon}$.

(f) $2\log(1 + x)/(x^2 + \varepsilon^2)$.

3.3.4 Solving algebraic equations asymptotically

3.3.4.1 Find a two-term asymptotic expansion, for $\varepsilon \to 0$, of each solution x of the following equations.

(a) $\varepsilon x^3 - 3x + 1 = 0$,

(b) $\varepsilon x^3 - x + 2 = 0$,

(c) $x^{2+\varepsilon} = 1/(x + 2\varepsilon)$, $(x > 0)$.

(d) $x^2 - 1 + \varepsilon \tanh(x/\varepsilon) = 0$

(e) $x = a + \varepsilon x^k$ for $x > 0$. Consider $0 < k < 1$ and $k > 1$.

(f) $1 - 2x + x^2 - \varepsilon x^3 = 0$.

3.3.4.2 Derive step by step, by iteratively scaling $x(\varepsilon) = \mu_0(\varepsilon)x_0 + \mu_1(\varepsilon)x_1 + \mu_2(\varepsilon)x_2 + \dots$ and balancing, that a third order asymptotic solution (for $\varepsilon \to 0$) of the equation

$$\ln(\varepsilon x) + x = a,$$

is given by

$$x(\varepsilon) = \ln \varepsilon^{-1} - \ln\big(\ln \varepsilon^{-1}\big) + a + o(1).$$

Find a more efficient expansion based on an alternative asymptotic sequence of gauge functions by combining $\mathrm{e}^{-a}\,\varepsilon$.

3.3.4.3 Analyse asymptotically for $\varepsilon \to 0$ the zeros of $\mathrm{e}^{-x/\varepsilon^2} + x - \varepsilon$.

3.3.4.4 Solve the n-th solution $x = x_n$ of

$$x = \tan x$$

asymptotically for large n.
<u>Hint</u>: for large n and $x_n > 0$, $x_n = \tan(x_n)$ is large, and so x_n must be near (in fact: just before) a pole of tan. *If we count the trivial first solution as $x_0 = 0$, then $x_n \simeq (n + \frac{1}{2})\pi$. Write $\varepsilon^{-1} = (n + \frac{1}{2})\pi$, and $x_n = \varepsilon^{-1} - y(\varepsilon)$ with $0 < y < \frac{1}{2}\pi$ such that* $\tan(x) = \cot(y)$. *Solve asymptotically for small ε.*
Generalise this result to the solutions of

$$x = \alpha \tan x$$

for $\alpha > 0$. Note the slight difference between $\alpha > 1$ and $\alpha < 1$.

3.3.4.5 Find an asymptotic approximation, for $\varepsilon \to 0$, of each solution of

$$y^2 + (1 + \varepsilon + x)y + x = 0, \quad \text{for } 0 < x < 1,$$

and determine if it is uniform in x over the indicated interval.

3.3.5 Solving differential equations asymptotically

3.3.5.1 Find a two-term asymptotic expansion, for $\varepsilon \to 0$, of the solution of the following problems.

(a) $y'' + \varepsilon y' - y = 1$, where $y(0) = y(1) = 1$.

(b) $y'' + y + y^3 = 0$, where $y(0) = 0$ and $y(\frac{1}{2}\pi) = \varepsilon$.

3.3.6 A quadratic equation

x satisfies the following equation

$$ax^2 + bx + c = 0$$

with parameters a, b and c, while $[x] =$ meters and $[c] =$ seconds.

a) What are the dimensions of a and b?
b) Find, by scaling $x = LX$ for some suitable L and collecting parameters in dimensionless groups R, equivalent equations of the form

$$X^2 + X + R = 0, \quad X^2 + RX + 1 = 0, \quad RX^2 + X + 1 = 0.$$

c) Under what conditions can the original equation be approximated by

$$ax^2 + bx = 0, \quad ax^2 + c = 0, \quad bx + c = 0.$$

Is it systematic to allow the solution $x = 0$ of the first equation?

3.3.7 A cubic equation

x satisfies the following equation

$$qx^3 + ax^2 + bx + c = 0$$

with parameters q, a, b and c, while $[x] =$ seconds and $[q] =$ meters.

a) What are the dimensions of a, b and c?
b) Find, by scaling $x = TX$ for some suitable T and collecting parameters in dimensionless groups R and ρ, equivalent equations of the form

$$RX^3 + \rho X^2 + X + 1 = 0, \quad RX^3 + X^2 + \rho X + 1 = 0, \quad RX^3 + X^2 + X + \rho = 0.$$

c) Under what conditions can the original equation be approximated by

$$ax^2 + bx + c = 0.$$

3.3.8 The catenary

Consider the problem of the catenary. An inextensible chain of length L, and mass-per-length m, is suspended between the points $x = 0$, $y = 0$ and $x = D$, $y = 0$. To keep the chain in position, at each end a force is applied of which the vertical component is equal to half the weight of the chain ($\frac{1}{2}mgL$), and the horizontal component equal to H. This H is unknown if D is given, but for the present problem, we assume for simplicity that H is given, while D is a result.

Parametrise the chain position $x = X(s)$, $y = Y(s)$ and the tension $T(s)$ and tangent angle $\psi = \psi(s)$ by the arclength s, leading to

$$X(s) = \int_0^s \cos\psi(t)\,\mathrm{d}t, \quad Y(s) = \int_0^s \sin\psi(t)\,\mathrm{d}t.$$

Formulate from the balance of forces, applied to a small section $\mathrm{d}s$, a set of equations for T, ψ, X and Y:

$$\begin{aligned}(T + \mathrm{d}T)\cos(\psi + \mathrm{d}\psi) - T\cos\psi &= 0\\ (T + \mathrm{d}T)\sin(\psi + \mathrm{d}\psi) - T\sin\psi &= mg\mathrm{d}s\end{aligned}$$

From symmetry, the tangent angle $\psi(\frac{1}{2}L)$ and the vertical tension component $T(\frac{1}{2}L)\sin\psi(\frac{1}{2}L)$ are half way the chain equal to zero, while at each end the horizontal tension component is given by: $T(0)\cos\psi(0) = T(L)\cos\psi(L) = H$.

Taking all together, we have finally

$$T = \sqrt{m^2g^2(s - \tfrac{1}{2}L)^2 + H^2}, \quad \psi = \arctan\Big(\frac{mg(s - \frac{1}{2}L)}{H}\Big)$$

Introduce the dimensionless parameter $\varepsilon = mgL/H$. Find an approximate solution (in T, ψ, X and Y) for small ε.

Can you interpret the approximation physically?

3.3.9 A water-bubbles mixture

A mixture of water and air (in the form of bubbles) with volume fraction α air and volume fraction $1 - \alpha$ water, has a mean density ρ and sound speed c given by

$$\rho = \alpha\rho_a + (1 - \alpha)\rho_w, \qquad \frac{1}{\rho c^2} = \frac{\alpha}{\rho_a c_a^2} + \frac{1 - \alpha}{\rho_w c_w^2}.$$

Typical values are $\rho_w = 1000$ kg/m^3, $\rho_a = 1.2$ kg/m^3, $c_w = 1470$ m/s, $c_a = 340$ m/s. Develop strategies to approximate c for values of α, based on an inherent small problem parameter. When is c minimal? What is the effect of even a very small fraction of air (common in the wake of a ship's propeller, or in a fresh central heating system)?

3.3.10 A car changing lanes

A car rides along a double lane straight road given by $-\infty < x < \infty$, $-2b \leqslant y \leqslant 2b$. The position of the car at time t is given by

$$x = \xi(t), \quad y = \eta(t).$$

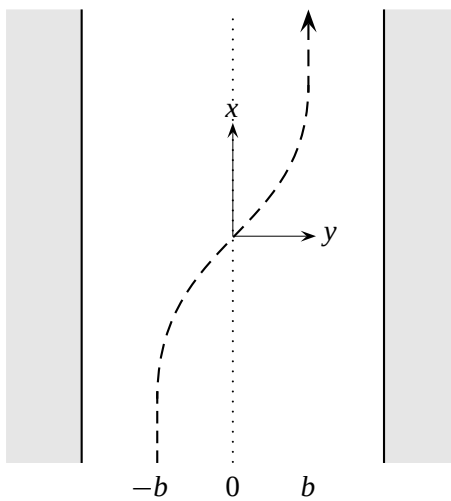


Figure 3.2: The trajectory of a car that changes lane

For $x \to -\infty$, the car is at $y = -b$, but near $x = 0$ it changes lane and shifts smoothly to $y = b$ according to a trajectory given by

$$\eta(t) = F(\xi(t)),$$

where F is given and $\xi = \xi(t)$ is to be found under the condition that all along the trajectory, the car travels with the same speed V, so

$$\dot{\xi}(t)^2 + \dot{\eta}(t)^2 = V^2, \quad \text{and so} \quad \dot{\xi}(t)^2 + F'(\xi)^2\,\dot{\xi}(t)^2 = V^2.$$

Note that both F and its argument x have dimension "length", so if F describes a changes of the order of b over a distance of the order of L, we should be able to write F as

$$F(x) = b f(x/L)$$

for $f = O(1)$. Take for definiteness $\xi(0) = 0$, and

$$f(z) = \tanh(z) \quad \text{where} \quad f'(z) = 1 - f(z)^2.$$

We assume that the change of lane happens gradually, such that

$$\varepsilon = \frac{b}{L} \ll 1.$$

a. Make the problem dimensionless by the inherent length scale b and corresponding time scale b/V. Write $\xi = bX$. Note the appearance of the small parameter ε. Do you see the appearance of a term of the form $f'(\varepsilon X)$? If we expand this for small ε we obtain something like

$$f'(\varepsilon X) = f'(0) + \varepsilon X f''(0) + \dots$$

which is already incorrect for $X = O(1/\varepsilon)$, the order of magnitude we are interested in! Therefore this choice is NOT clever. Indeed, b is not the typical length scale for ξ.

b. Make the problem dimensionless by the inherent length scale L and corresponding time scale L/V. Write $\xi = LX$ and $t = (L/V)\tau$.

c. By separation of variables we can write τ as a function, in the form of an integral, of X. Otherwise, it is impossible to find an explicit expression for X. Therefore, we will try to find an asymptotic expansion for small ε by assuming the Poincaré expansion

$$X(\tau, \varepsilon) = X_0(\tau) + \varepsilon^2 X_1(\tau) + O(\varepsilon^2),$$

and substitute this in the equation, and expand the equation also asymptotically. Find the first two terms. Do you see why we can expand in powers of ε^2 rather than (for example) ε?

Hint: note that for small δ: $\sqrt{1+\delta} = 1 + \frac{1}{2}\delta + \dots, \quad (1+\delta)^{-1} = 1 - \delta + \dots,$ and use

$$\int 1 - \tanh(x)^2 \, \mathrm{d}x = \tanh(x), \qquad \int \left(1 - \tanh(x)^2\right)^2 \mathrm{d}x = \tanh(x) - \tfrac{1}{3}\tanh(x)^3$$

3.3.11 A chemical reaction-diffusion problem (regular limit)

A catalytic reaction is a chemical reaction between reactants, of which one – the catalyst – returns after the reaction to its original state. Its rôle is entirely to enable the reaction to happen. An example of a catalyst is platinum. The primary reactant is usually a liquid or a gas. As the catalyst and the reactant are immiscible, the reaction occurs at the catalyst surface, which is therefore made as large as possible. A way to achieve this is by applying the catalyst in the pores of porous pellets in a so-called fixed bed catalytic reactor. The reactant diffuses from the surface to the inside of the pellet. Meanwhile, being in contact with the catalyst, the reactant is converted to the final product.

Assume reactant A reacts to product B at the pellet pores surface via an nth-order, irreversible reaction

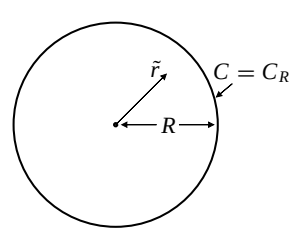
$$A \xrightarrow{k} B$$

with concentration inside the pellet $C = [A]$ mol/m^3, production rate kC^n mol/m^{3}s and rate constant k. This reaction acts as a sink term for A. Under the additional assumption of a well stirred fluid in order to maintain a constant concentration $C = C_R$ at the outer surface of spherically shaped pellets, we obtain the following instationary reaction-diffusion equation:

$$\frac{\partial C}{\partial t} - \nabla \cdot (D \nabla C) = -kC^n, \quad 0 < \tilde{r} < R, t > 0,$$

$$C(r, 0) = 0, \quad 0 < \tilde{r} < R,$$

$$C(R, t) = C_R, \ \frac{\partial}{\partial \tilde{r}} C(0, t) = 0, \quad t > 0,$$



where D is the diffusion coefficient of C inside the pellet. After sufficiently long time the concentration C attains a steady state distribution within the pellet. Assuming spherical symmetry and a constant diffusion coefficient D, we have the stationary reaction-diffusion equation

$$D\frac{1}{\tilde{r}^2}\frac{\mathrm{d}}{\mathrm{d}\tilde{r}}\left(\tilde{r}^2\frac{\mathrm{d}C}{\mathrm{d}\tilde{r}}\right) = kC^n, \quad 0 < \tilde{r} < R,$$
$$C(R) = C_R, \ \frac{\mathrm{d}}{\mathrm{d}\tilde{r}}C(0) = 0.$$

The net mass flux into the pellet, an important final result, is given by $4\pi R^2 D\frac{\mathrm{d}}{\mathrm{d}r}C(R)$ (Fick's law).

We make the problem dimensionless as follows:

$$c = \frac{C}{C_R}, \quad r = \frac{\tilde{r}}{R}, \quad \phi^2 = \frac{kR^2C_R^{n-1}}{D},$$

such that

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}c}{\mathrm{d}r}\right) = \phi^2c^n, \quad 0 < r < 1,$$
$$c(1) = 1, \ \ c'(0) = 0,$$

where the prime ($'$) denotes differentiation with respect to r, ϕ is called the Thiele modulus, and reaction order $n = 1, 2, 3, \ldots$.

We are interested in the asymptotic behaviour of c for $\varepsilon = \phi^2 \to 0$. Assume a regular Poincaré expansion of c in powers of ε and find the first three terms. *Hint.* Introduce $y = rc$.

3.3.12 The pivoted barrier

Consider a horizontal barrier of length L, free on one end and pivoted at the other end, such that it can swivel horizontally around a vertical pivot. The hinge is constructed in such a way that the barrier is fixed perpendicularly to the upper end of a vertical hollow cylinder of diameter B and length H. This upper end is closed, the other end is open. With this open end the cylinder is placed over a vertical axis which is firmly anchored in the ground. Of course, the length of the axis is more than H and the diameter of the axis, b, is less than B.

Depending on the clearance between cylinder and axis, and the length of the cylinder, the free end of the barrier (which is otherwise perfectly stiff) will lean down from the exactly horizontal position by an angle α of the barrier with the horizon. The question is: how much will this be.

You may assume that the construction is reasonable. In other words, the clearance will be small but not very small, and the length of the cylinder is ample.

Show by elementary geometry that

$$B = H\tan\alpha + b\cos\alpha.$$

We can immediately see that if α happens to be small, such that $\tan\alpha \approx \alpha$ and $\cos\alpha \approx 1$, then

$$\alpha \approx \frac{B-b}{H}.$$

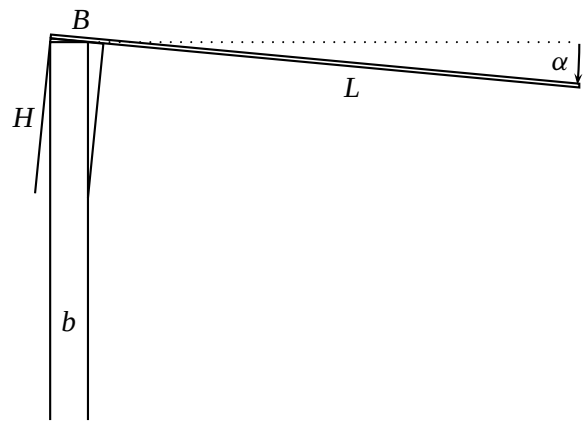


Figure 3.3: Slightly tilted barrier

For many practical applications this may be good enough, but the approach is very ad-hoc and not systematic. For example, how would we make higher order corrections? We have two dimensionless parameters (for example B/H and b/H, or $(B-b)/H$ and b/H, etc.) and it is of interest to see by what process and which assumptions the approximation appears. We will consider two limits: relatively small $B-b$, and relatively small b.

Reformulate the problem as one of finding a zero of a 4-th order polynomial equation in $X = \sin\alpha$:

$$b^2X^4 - 2HbX^3 + (H^2 + B^2 - 2b^2)X^2 + 2HbX + b^2 - B^2 = 0.$$

A formal solution is theoretically possible (Gerolamo Cardano, 1545) but difficult and clumsy. We will now try to make reasonable approximations to construct an adequate and transparent approximate solution.

a) Divide the equation by b^2. We assume that B is slightly larger than b, while H is of the same order of magnitude as b. This is made explicit by writing

$$\lambda = \frac{H}{b}, \quad B^2 = b^2(1+\varepsilon) \quad \text{where } 0 < \varepsilon \ll 1\ .$$

(Using B^2 and b^2 is for convenience later.) Assess the order of magnitude of the various terms. How big is X in terms of ε? Rescale X and find an approximate solution X for small ε.

b) Divide the equation by H^2. We assume that b is much smaller than H, while B is of the same order of magnitude as b. This is made explicit by writing

$$\mu = \frac{B}{b}, \quad \varepsilon = \frac{b}{H}.$$

Assess the order of magnitude of the various terms. How big is X in terms of ε? Rescale X and find an approximate solution X for small ε.

c) How does solution (a) compare with solution (b)? Show that they are the same if $B \approx b$.

Chapter 4

Method of Slow Variation

4.1 Theory

4.1.1 General Procedure

Suppose we have a function $\varphi(\boldsymbol{x}, \varepsilon)$ of spatial coordinates $\boldsymbol{x}$ and a small parameter ε, such that the typical variation in one direction, say x, is of the order of length scale ε^{-1}. We can express this behaviour most conveniently by writing $\varphi(x, y, z, \varepsilon) = \Phi(\varepsilon x, y, z, \varepsilon)$. Now if we were to expand Φ for small ε, we might, for example, get something like

$$\Phi(\varepsilon x, y, z, \varepsilon) = \Phi(0, y, z; 0) + \varepsilon(x\Phi_x(0, y, z; 0) + \Phi_\varepsilon(0, y, z; 0)) + \ldots,$$

which is only uniform in x on an interval $[0, L]$ if $L = O(1)$, and the inherent slow variation on the longer scale of $x = O(\varepsilon^{-1})$ would be masked. It is clearly much better to introduce the scaled variable $X = \varepsilon x$, and a (assumed) regular expansion of $\Phi(X, y, z, \varepsilon)$

$$\Phi(X, y, z, \varepsilon) = \mu_0(\varepsilon)\varphi_0(X, y, z) + \ldots \tag{4.1}$$

now retains the slow variation in X in the shape functions of the expansion. In other words, the scaled variable X in combination with order function μ_0 yields with $\lim_{\varepsilon\to 0} \mu_0^{-1}\Phi$ the *distinguished limit* or *significant degeneration* of φ.

This situation frequently happens when the geometry involved is slender. The theory of one dimensional gas dynamics, lubrication flow, or sound propagation in horns (Webster's horn equation) are important examples, although they are usually derived not systematically, without explicit reference to the slender geometry. We will illustrate the method both for heat flow in a varying bar, quasi 1-D gas flow and the shallow water problem.

4.1.2 Example: heat flow in a bar

Consider the stationary problem of the temperature distribution T in a long heat-conducting bar with outward surface normal $\boldsymbol{n}$ and slowly varying cross section $\mathcal{A}$. The bar is kept at a temperature difference such that a given heat flux is maintained, but is otherwise isolated. As there is no leakage of

heat, the flux is constant. With spatial coordinates made dimensionless on a typical bar cross section, we have the following equations and boundary conditions

$$\nabla^2 T = 0, \qquad \nabla T \cdot \boldsymbol{n} = 0, \qquad \iint_{\mathcal{A}} \frac{\partial T}{\partial x} \mathrm{d}S = -Q.$$

After integrating $\nabla^2 T$ over a slice $x_1 \leqslant x \leqslant x_2$, and applying Gauss's theorem[1], we find that the axial flux Q is indeed independent of x.

We will assume here the cross section and the temperature field circular symmetric, but that is not a necessary simplification for a manageable analysis. As a result we have in cylindrical coordinates (x, r, θ)

$$\frac{\partial^2 T}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = 0, \qquad \frac{\partial T}{\partial x} n_x + \frac{\partial T}{\partial r} n_r = 0, \qquad 2\pi \int_0^R r \frac{\partial T}{\partial x}\, \mathrm{d}r = -Q.$$

The typical length scale of diameter variation is assumed to be much larger than a diameter. We introduce the ratio between a typical diameter and this length scale as the small parameter ε, and write for the bar surface

$$S(X, r) = r - R(X) = 0, \quad X = \varepsilon x,$$

where (x, r, θ) form a cylindrical coordinate system (see Figure 4.1). By writing R as a continuous

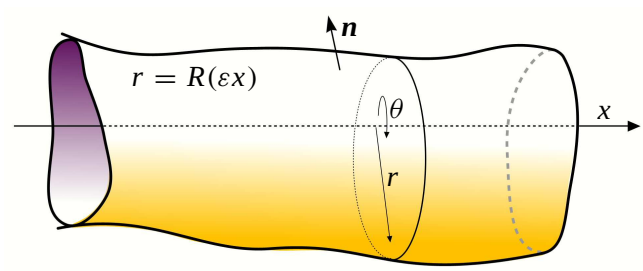


Figure 4.1: Slowly varying bar.

function of slow variable X, rather than x, we have made our formal assumption of slow variation explicit in a convenient and simple way, since $R_x = \varepsilon R_X = O(\varepsilon)$. From calculus (section 8.3) we know, that ∇S is a normal of the surface $S = 0$. So we can write

$$\boldsymbol{n} \sim \nabla S, \quad \text{or} \quad n_x \boldsymbol{e}_x + n_r \boldsymbol{e}_r \sim -\varepsilon R_X \boldsymbol{e}_x + \boldsymbol{e}_r.$$

The crucial step will now be the assumption that the temperature is *only* affected by the geometric variation induced by R. Any initial or entrance effects are ignored or have disappeared. As a result, in the limit of small ε,

$$\text{the temperature field } T(x, r, \varepsilon) = \tilde{T}(X, r, \varepsilon) \text{ is a function of } X,$$

rather than x – in other words: $\tilde{T}$ yields the distinguished limit of T – and

$$\text{its axial gradient scales on } \varepsilon, \text{ as } \frac{\partial T}{\partial x} = \varepsilon \frac{\partial \tilde{T}}{\partial X} = O(\varepsilon).$$

[1]The integral of divergence $\nabla \cdot \vec{\boldsymbol{v}}$ over a volume Ω is equal to the outward flux integral of $\vec{\boldsymbol{v}}$ across its surface $\partial\Omega$: $\int_\Omega \nabla \cdot \vec{\boldsymbol{v}}\, \mathrm{d}\boldsymbol{x} = \oint_{\partial\Omega} \vec{\boldsymbol{v}} \cdot \boldsymbol{n}\, \mathrm{d}S$.

For simplicity we will write in the following T, instead of $\tilde{T}$. If we rewrite the equations from x into X, we obtain the rescaled heat equation

$$\varepsilon^2 T_{XX} + \frac{1}{r}\big(rT_r\big)_r = 0. \tag{$*$}$$

At the wall $r = R(X)$ the boundary condition of vanishing heat flux is

$$-\varepsilon^2 T_X R_X + T_r = 0. \tag{$\dagger$}$$

The flux condition, for lucidity rewritten with $Q = 2\pi\varepsilon q$, is given by

$$\int_0^{R(X)} r\frac{\partial T}{\partial X}\,\mathrm{d}r = -q.$$

This problem is too difficult in general, so we try to utilize the small parameter ε in a systematic manner. From the flux condition, it seems that $T = O(1)$. Since the perturbation terms are $O(\varepsilon^2)$, we assume the asymptotic expansion of Poincaré-type (note that all terms are independent of ε!)

$$T(X, r, \varepsilon) = T_0(X, r) + \varepsilon^2 T_1(X, r) + O(\varepsilon^4).$$

After substitution in equation ($*$) and boundary condition ($\dagger$), further expansion in powers of ε^2 and equating like powers of ε, we obtain to leading order the following equation in r

$$(rT_{0,r})_r = 0 \qquad \text{with} \quad T_{0,r} = 0 \quad \text{at } r = R(X) \text{ and regular at } r = 0.$$

An obvious solution is $T_0(X, r)$ is constant. Since X is present as parameter we have thus

$$T_0 = T_0(X).$$

We can substitute this directly in the flux condition, to find

$$\frac{1}{2}R^2(X)\frac{\mathrm{d}T_0}{\mathrm{d}X} = -q$$

and therefore

$$T_0(X) = T_{\text{in}} - \int_0^X \frac{q}{\frac{1}{2}R^2(\xi)}\,\mathrm{d}\xi.$$

We can go on to find the next term T_1, but this leading order solution contains already most of the physical information.

In summary: we assumed that the slowly varying bar induces a slowly varying temperature distribution. This is not always true, but depends on the type of physical phenomenon. Then we rescaled the equations such that we used this slow variation. After assuming an asymptotic expansion of the solution we obtained a simplified sequence of problems. The original partial differential equations simplified to ordinary differential equations, which are far easier to solve.

It should be noted that we did not include in our analysis any boundary conditions at the ends of the bar. It is true that the present method fails here. The found solution is uniformly valid on $\mathbb{R}$ (since $R(X)$ is assumed continuous and independent of ε), but only as long as we stay away from any end. Near the ends the boundary conditions induce x-gradients of $O(1)$ which makes the prevailing length scale again x, rather than X. This region is asymptotically of boundary layer type, and should be treated differently (see below).

4.2 Exercises

4.2.1 Heat flux in 2D

The same problem as before, but now strictly in 2D.

Consider the stationary two-dimensional problem of the temperature T in a long heat-conducting bar of typical height H, slowly varying (*i.e.* on a length scale $L \gg H$) in diameter. The bar is thermally isolated, but axially a temperature gradient is maintained, for example by applying a temperature difference at the ends. However, we will not consider the neighbourhood of the ends, and we will not explicitly apply boundary conditions at the ends. Instead, we will assume a given axial heat flux.

In *dimensional* form, we have

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \quad \text{on } -\infty < x < \infty, \quad 0 \leqslant y \leqslant Hh\Big(\frac{x}{L}\Big), \quad \varepsilon = \frac{H}{L}$$

with ε small, shape function $h(\xi) = O(1)$, and a given cross sectional heat flux (per m) equal to

$$\int_0^{Hh} \vec{\boldsymbol{\phi}} \cdot \boldsymbol{e}_x \,\mathrm{d}y = \int_0^{Hh} -k\frac{\partial}{\partial x} T \,\mathrm{d}y = Q,$$

where $\vec{\boldsymbol{\phi}} = -k\nabla T$ is the heat flux density vector. The dimensions are $[T] = \mathrm{K}$, $[H] = [L] = [x] = [y] = \mathrm{m}$, $[\vec{\boldsymbol{\phi}}] = \mathrm{Wm}^{-2}$, $[k] = \mathrm{Wm}^{-1}\mathrm{K}^{-1}$, $[Q] = \mathrm{Wm}^{-1}$.

a) Write the boundary condition of thermal isolation, $\vec{\boldsymbol{\phi}} \cdot \vec{\boldsymbol{n}} = 0$, explicitly in terms of h.

b) Make the problem dimensionless by scaling lengths on typical bar height H, and the temperature on the intrinsic temperature of the problem. (Hint: make sure that the cross sectional heat flux becomes unity.)

c) Apparently, the essential co-ordinate in (dimensionless) x-direction is εx, and significant changes in x-direction are felt only on a length scale $x = O(\varepsilon^{-1})$, so we rewrite $X = \varepsilon x$.

   Assume that the field varies slowly in x (*i.e.* essentially in X, while any end-effects are local and irrelevant for the x's considered).

   Solve the problem to leading order of an assumed asymptotic expansion of (dimensionless) temperature $T = T(X, y, \varepsilon)$ in powers of ε.

4.2.2 Lubrication flow

Lubrication theory deals with an incompressible viscous flow of not-large Reynolds number, through a narrow channel of slowly varying cross section.

Consider the Navier-Stokes equations in a two-dimensional narrow channel, with prescribed mass flux. In practice this flux is created by a pressure difference or pressure gradient, but by using the flux here, we can estimate the typical flow velocity and thus the Reynolds number. We have

$$\rho\Big(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\Big)+\frac{\partial p}{\partial x}=\mu\Big(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\Big),$$
$$\rho\Big(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\Big)+\frac{\partial p}{\partial y}=\mu\Big(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\Big),$$
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

for the velocity $\vec{\boldsymbol{v}}=(u,v)$, pressure p, density ρ, and viscosity μ in the channel

$$-\infty<x<\infty,\qquad Hg\Big(\frac{x}{L}\Big)\leqslant y\leqslant Hh\Big(\frac{x}{L}\Big).$$

(Neither end conditions in x nor origin $y=0$ are important. If convenient you may assume $y=0$ somewhere inside or near the channel.) Boundary conditions are a vanishing velocity at the walls:

$$u=v=0\quad\text{at } y=Hg\Big(\frac{x}{L}\Big),\ \text{and } y=Hh\Big(\frac{x}{L}\Big)$$

and, due to an assumed pressure difference, a fixed mass flux

$$\int_{Hg}^{Hh}\rho u(x,y)\,\mathrm{d}y=F.$$

The dimensionless channel opening $0<h-g=O(1)$. We assume the typical length scale L of the channel variations large compared to the typical opening H, so $\varepsilon=H/L$ is small. The mass flux F, density ρ and viscosity μ are such, that the resulting typical velocity U corresponds to a Reynolds number $\mathrm{Re}=\rho UH/\mu$ of order 1 or less.

a) Make the problem dimensionless by scaling lengths on typical channel height H, the velocity on the intrinsic velocity of the problem, and the pressure on the inherent pressure required to have flow at all. (Hint: make sure that the cross sectional mass flux becomes unity, and that the pressure gradient balances the viscous terms.)

b) Apparently, the essential co-ordinate in (dimensionless) x-direction is εx, and significant changes in x-direction are felt only on a length scale $x=O(\varepsilon^{-1})$, so we rewrite $X=\varepsilon x$.

   Assume that the field varies slowly in x (*i.e.* essentially in X, while any end-effects are local and irrelevant for the x's considered).

   Solve the problem to leading order of an assumed asymptotic expansion of (dimensionless) velocity $\vec{\boldsymbol{v}}=\vec{\boldsymbol{v}}(X,y,\varepsilon)$ and pressure $p=P(X,y,\varepsilon)$ in powers of ε. (Hint: by construction is $u=O(1)$, but v and p will require rescaling in ε).

   As is common in incompressible flow problems, the pressure is only determined up to a constant.

4.2.3 Quasi 1D gas dynamics

Consider a compressible, subsonic inviscid irrotational flow through a slowly varying cylindrical duct, given by $r = HR(x/L)$, with $H \ll L$. The flow is assumed nearly uniform. Because of symmetry, it is assumed to be independent of the circumferential co-ordinate θ. As the flow is irrotational, we can assume a potential ϕ for the velocity $\vec{\boldsymbol{v}} = \nabla\phi$, leading to a Bernoulli integral of the momentum equations. Due to the absence of friction and heat conduction we may assume adiabatic changes, leading to a pressure p of the form $p = p_0(\rho/\rho_0)^\gamma$, where p_0 and ρ_0 are reference values and $\gamma = 1.4$ is a dimensionless gas constant[2]. A related quantity is the sound speed given by $c^2 = \gamma p/\rho$.

Inside the duct $0 \leqslant r \leqslant HR(x/L)$ we have the equation of mass conservation

$$\nabla \cdot (\rho\vec{\boldsymbol{v}}) = \frac{\partial}{\partial x}(\rho u) + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v) = 0, \text{ where } \vec{\boldsymbol{v}} = \nabla\phi = u\boldsymbol{e}_x + v\boldsymbol{e}_r$$

and Bernoulli's equation

$$\tfrac{1}{2}|\vec{\boldsymbol{v}}|^2 + \frac{c^2}{\gamma - 1} = E, \quad c^2 = \gamma\frac{p}{\rho}, \quad \frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^\gamma,$$

where E is an integration constant, fixing the thermodynamical properties of the problem. The duct walls, with normal vector $\boldsymbol{n}$, are impermeable, so

$$\vec{\boldsymbol{v}} \cdot \boldsymbol{n} = \nabla\phi \cdot \boldsymbol{n} = 0 \text{ at } r = HR(x/L),$$

while a mass flux F is given by

$$2\pi \int_0^{HR} \rho u r \, \mathrm{d}r = F.$$

a) Make the problem dimensionless by scaling lengths on typical duct radius H, pressure and density on the reference values p_0 and ρ_0, and the velocity on the corresponding reference sound speed $c_0 = \sqrt{\gamma p_0/\rho_0}$. In order to have an interesting problem, we assume that the resulting dimensionless quantities p/p_0, ρ/ρ_0, c/c_0, $\vec{\boldsymbol{v}}/c_0$, E/c_0^2, $F/H^2\rho_0 c_0^2 = O(1)$.

b) Apparently, the essential co-ordinate in (dimensionless) x-direction is εx, and significant changes in x-direction are felt only on a length scale $x = O(\varepsilon^{-1})$, so we rewrite $X = \varepsilon x$.

   Assume that the field varies slowly in x (*i.e.* essentially in X, while any end-effects are local and irrelevant for the x's considered).

   Solve the problem to leading order of an assumed asymptotic expansion of (dimensionless) potential $\phi = \phi(X, r, \varepsilon)$ and density $\rho = \rho(X, y, \varepsilon)$ in powers of ε. (Hint: by construction is $u = O(1)$, but ϕ will require rescaling in ε).

   Note: the very last equation is an algebraic equation that has to be solved numerically.

[2]Note, that this type of flow is called: 1D gas dynamics. A better name would be: quasi-1D gas dynamics.

4.2.4 Webster's horn

Consider acoustic waves of fixed frequency ω through a slowly varying horn (duct). The typical wave length λ is long, *i.e.* of the same order of magnitude as the typical length scale L of the duct diameter variations. For simplicity we consider a two-dimensional horn, with a constant lower wall given by $y = 0$ and an upper wall given by $y = Hh(x/L)$, where $h = O(1)$ is dimensionless and $H \ll L$.

The sound field is given by the velocity potential ϕ, where velocity is $\vec{\boldsymbol{v}} = \nabla\phi$ (and pressure $p = -\mathrm{i}\omega\rho_0\phi$ but this is here unimportant), obeying the reduced wave equation (Helmholtz equation)

$$\nabla^2\phi + k^2\phi = 0, \quad \text{in } -\infty < x < \infty, \;\; 0 \leqslant y \leq Hh(x/L),$$

where $k = \omega/c$ is the free field wave number, which is equal to $2\pi/\lambda$.

The wall (with normal vectors $\boldsymbol{e}_y$ and $\boldsymbol{n}$) are impermeable, so we have the boundary conditions

$$\vec{\boldsymbol{v}} \cdot \boldsymbol{e}_y = 0 \quad \text{at } y = 0, \qquad \vec{\boldsymbol{v}} \cdot \boldsymbol{n} = 0 \quad \text{at } y = Hh(x/L).$$

Assume that there is a sound field (the problem is linear, so it's enough to assume that $\phi \not\equiv 0$).

a) Make the lengths in the problem dimensionless on the typical duct height H and ϕ on an (unimportant) reference value Φ. Verify that the equations remain the same. Introduce the small parameter $\varepsilon = H/L$.

b) Apparently, the x variations scale on L, and so the essential co-ordinate in (dimensionless) x-direction is εx. Significant changes in x-direction are felt only on a length scale $x = O(\varepsilon^{-1})$, and so we rewrite $X = \varepsilon x$.

   Note that the dimensionless $k = O(\varepsilon)$, so we scale $k = \varepsilon\kappa$.

   Assume that the field varies slowly in X (any end-effects are local and irrelevant here).

   Assume in scaled coordinates for ϕ an obvious asymptotic expansion in ε, and derive the equation for (leading order) ϕ_0. This equation is called "Webster's equation".

c) Solve this equation for $h(z) = \mathrm{e}^{2\alpha z}$.

4.2.5 Shallow water waves along a varying bottom

Consider the following inviscid incompressible irrotational 2D water flow in (x, z)-domain along a slowly varying bottom. The bottom is given by $z = b(x/L)$, where L is a typical length scale along which bottom variations occur. The water level is given by $z = h$.

The velocity vector $\boldsymbol{v}$ can be given by a potential ϕ

$$\boldsymbol{v} = \nabla\phi.$$

Conservation of mass requires

$$\nabla^2\phi = 0 \text{ for } -\infty < x < \infty, \quad b < z < h.$$

Because of the assumptions we can integrate the momentum equation to Bernoulli's equation and obtain for pressure p

$$\tfrac{1}{2}|\nabla\phi|^2 + \frac{p}{\rho_0} + gz = C,$$

where ρ_0 denotes the water density, g the acceleration of gravity, and C is a constant, related to the chosen reference pressure level.

At the impermeable bottom we have a vanishing normal component of the velocity, yielding the boundary condition

$$\nabla\phi \cdot \nabla(b - z) = \frac{\partial\phi}{\partial x}\frac{\partial b}{\partial x} - \frac{\partial\phi}{\partial z} = 0 \text{ at } z = b.$$

Since the water surface $z = h$ is a streamline, it follows that for a particle moving along $(x(t), z(t))$ we have $\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\partial h}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}$, leading to

$$\frac{\partial\phi}{\partial z} = \frac{\partial h}{\partial x}\frac{\partial\phi}{\partial x} \text{ at } z = h.$$

Furthermore, the water surface takes the pressure of the air above the water, say $p = p_a$, so

$$\tfrac{1}{2}|\nabla\phi|^2 + gh = C - \frac{p_a}{\rho_0} \text{ at } z = h.$$

The water flow is defined by means of a prescribed volume flux F, which is the same for all positions x.

$$\int_b^h \frac{\partial\phi}{\partial x}\,\mathrm{d}z = F.$$

By assuming far upstream a constant bottom level $b = b_\infty$, a constant water level $h = h_\infty = b_\infty + D_\infty$ and a uniform flow with velocity $U_\infty = F/D_\infty$, we can determine the Bernoulli constant in physical terms

$$C = \frac{p_a}{\rho_0} + \tfrac{1}{2}U_\infty^2 + gh_\infty.$$

Introduce $\varepsilon = D_\infty/L$ where ε is small.

a) Make the problem dimensionless. Scale lengths on D_∞, velocities on U_∞. Assume that the inverse-squared Froude number (or Richardson number) $\gamma = gD_\infty/U_\infty^2 = O(1)$.

b) Solve the problem to leading order for small ε by application of the method of slow variation. Note that both ϕ and h are unknowns, and have to be expanded in ε. Bottom variation b and constants F and $C - p_a/\rho_0$, on the other hand, are given.

   Note. The very last equation cannot be integrated explicitly.

4.2.6 A laterally heated bar

A 2-dimensional slowly varying heat conducting bar is described by $-\infty < x < \infty, \quad 0 \leqslant y \leqslant Hh(x/L)$, where the geometry h is a smooth function of its argument. The bar is kept along the lower side (at $y = 0$) at fixed temperature $T(x, 0) = \theta_0$, and along the upper side (at $y = Hh(x/L)$) at fixed temperature $T(x, Hh) = \theta_1$. This constitutes a stationary temperature distribution $T(x, y)$, which satisfies the heat equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0.$$

a) Make the problem dimensionless. Scale lengths on H and temperature by $T = \theta_0 + (\theta_1 - \theta_0)\tilde{T}$. Introduce the geometric ratio $\varepsilon = H/L$. Assume that ε is small. As the notation suggests, $h(z)$ does not depend on ε and $0 < h(z) = O(1)$.

b) Assuming that $\tilde{T}$ is slowly varying with geometry h in x (no end effects), solve the problem asymptotically for small ε to first and second order by application of the Method of Slow Variation.

Chapter 5

Method of Lindstedt-Poincaré

5.1 Theory

5.1.1 General Procedure

When we have a function y, depending on a small parameter ε, and periodic in t with fundamental frequency $\omega(\varepsilon)$, we can write y as a Fourier series

$$y(t,\varepsilon) = \sum_{n=-\infty}^{\infty} A_n(\varepsilon)\,\mathrm{e}^{\mathrm{i}n\omega(\varepsilon)t} \tag{5.1}$$

If amplitudes and frequency have an asymptotic expansion for small ε, say

$$A_n(\varepsilon) = A_{n,0} + \varepsilon A_{n,1} + \ldots, \qquad \omega(\varepsilon) = \omega_0 + \varepsilon\omega_1 + \ldots, \tag{5.2}$$

we have a natural asymptotic series expansion for y of the form

$$y(t,\varepsilon) = \sum_{n=-\infty}^{\infty} A_{n,0}\,\mathrm{e}^{\mathrm{i}n\omega_0 t} + \varepsilon \sum_{n=-\infty}^{\infty} \Big(A_{n,1} + \mathrm{i}n\omega_1 t A_{n,0}\Big)\,\mathrm{e}^{\mathrm{i}n\omega_0 t} + \ldots \tag{5.3}$$

This expansion, however, is only uniform in t on an interval $[0, T]$, where $T = o(\varepsilon^{-1})$. On a larger interval, for example $[0, \varepsilon^{-1}]$, the asymptotic hierarchy in the expansion becomes invalid, because $\varepsilon t = O(1)$. This happens because of the occurrence of algebraically growing oscillatory terms, called "secular terms". *Secular* = occurring once in a century, and *saeculum* = generation, referring to their astronomical origin.

Definition. The terms proportional to $t^m \sin(n\omega_0 t)$, $t^m \cos(n\omega_0 t)$ are called "*secular terms*". More generally, the name refers to any algebraically growing terms that limit the region of validity of an asymptotic expansion.

It is therefore far better to apply first a coordinate transformation $\tau = \omega(\varepsilon)t$, introduce $Y(\tau, \varepsilon) = y(t, \varepsilon)$, and expand Y, rather than y, asymptotically. We get

$$Y(\tau, \varepsilon) = \sum_{n=-\infty}^{\infty} A_n(\varepsilon)\, \mathrm{e}^{\mathrm{i}n\tau} = \sum_{n=-\infty}^{\infty} A_{n,0}\, \mathrm{e}^{\mathrm{i}n\tau} + \varepsilon \sum_{n=-\infty}^{\infty} A_{n,1}\, \mathrm{e}^{\mathrm{i}n\tau} + \dots \tag{5.4}$$

which is now, in variable τ, a *uniformly valid* approximation!

The method is called the *Lindstedt-Poincaré method* or the *method of strained coordinates*. In practical situations, the function $y(t, \varepsilon)$ is implicitly given, often by a differential equation, and to be found. A typical, but certainly not the only example [38] is a weakly nonlinear harmonic equation of the form

$$y'' + \varepsilon h(y, y') + \alpha^2 y = 0,$$

where h is assumed to allow the existence of one or more periodic solutions for $y = O(1)$ with frequency $\omega(\varepsilon) \approx \alpha$ for $\varepsilon \to 0$. In view of the above, it makes sense to construct an asymptotic approximation like $Y = Y_0 + \varepsilon Y_1 + \varepsilon^2 Y_2 + \cdots$ with a rescaled variable $\tau = \omega t$. However, except for trivial situations, the frequency ω is unknown, and has to be found too. Therefore, when constructing the solution we have to allow for an unknown coordinate transformation. In order to construct the unknown $\omega(\varepsilon)$ we expand this in a similar way, for example like

$$\tau = (\omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \dots)t \tag{5.5}$$

but details depend on the problem. Note that the only purpose of the scaling is to render the asymptotic expansion of Y regular, so it is no restriction to assume for ω_0 something convenient, like $\omega_0 = \alpha$. The other coefficients $\omega_1, \omega_2, \dots$ are determined from the additional condition that the asymptotic hierarchy should be respected as long as possible. In other words, secular terms should not occur. We will illustrate this with the following classic example.

5.1.2 Example: the pendulum

Consider the motion of the pendulum, described[1] by the ordinary differential equation

$$\ddot{\theta} + K^2 \sin(\theta) = 0, \quad \text{with } \theta(0) = \varepsilon,\ \theta'(0) = 0,$$

where $0 < \varepsilon \ll 1$. We note that $\theta = O(\varepsilon)$ so we scale $\theta = \varepsilon\psi$ to get (after dividing by ε)

$$\ddot{\psi} + K^2(\psi - \tfrac{1}{6}\varepsilon^2\psi^3 + \dots) = 0, \quad \text{with } \psi(0) = 1,\ \psi'(0) = 0,$$

If we are interested in a solution only up to $O(\varepsilon^2)$ we can obviously ignore the higher order terms indicated by the dots.

Following the above procedure, we introduce the transformation $\tau = \omega t$ to obtain

$$\omega^2\psi'' + K^2\big(\psi - \tfrac{1}{6}\varepsilon^2\psi^3\big) = 0,$$

where the prime indicates now differentiation to τ. Since the essential small parameter is apparently ε^2, we expand

$$\omega = \omega_0 + \varepsilon^2\omega_1 + \dots, \qquad \psi = \psi_0 + \varepsilon^2\psi_1 + \dots,$$

[1] The equation may be simpified by rescaling time by $\tilde{t} = Kt$, such that factor K^2 cancels out.

and find, after substitution, the equations for the first two orders

$$\begin{aligned} &\omega_0^2\psi_0'' + K^2\psi_0 = 0, && \psi_0(0) = 1,\ \psi_0'(0) = 0, \\ &\omega_0^2\psi_1'' + K^2\psi_1 = -2\omega_0\omega_1\psi_0'' + \tfrac{1}{6}K^2\psi_0^3, && \psi_1(0) = 0,\ \psi_1'(0) = 0. \end{aligned}$$

Note that we are relatively free to choose ω_0, as long as it is $O(1)$. (It is only a coordinate transformation that would automatically be compensated in the equation.) Clearly, a good choice is $\omega_0 = K$ because this simplifies the formulas greatly. The solution ψ_0 is then

$$\psi_0 = \cos\tau, \quad \omega_0 = K,$$

leading to the following equation for ψ_1

$$\begin{aligned} \psi_1'' + \psi_1 &= 2K^{-1}\omega_1\cos\tau + \tfrac{1}{6}\cos^3\tau \\ &= 2K^{-1}\omega_1\cos\tau + \tfrac{1}{8}\cos\tau + \tfrac{1}{24}\cos 3\tau, \end{aligned}$$

since $\cos^3\tau = \frac{3}{4}\cos\tau + \frac{1}{4}\cos 3\tau$ (see section 8.4).

At this point it is essential to observe that the right-hand-side consists of two forcing terms: one with frequency 3 and one with 1, the resonance frequency of the left-hand-side. This resonance would lead to secular terms, as the solutions will behave like $\tau\sin(\tau)$ and $\tau\cos(\tau)$. This would spoil our approximation if we had no further degrees of freedom. However, this is where our rescaled time comes in! We know that by scaling with the *correct* frequency ω of the system there will be no secular terms. So we have to choose ω_1 such, that no secular terms arise.

Therefore, in order to suppress the occurrence of secular terms, the amplitude of the resonant forcing term should vanish, which yields the next order terms ω_1 and ψ_1. We thus have

$$\omega_1 = -\tfrac{1}{16}K$$

leading to

$$\psi_1 = A_1\cos\tau + B_1\sin\tau - \tfrac{1}{192}\cos 3\tau.$$

With the initial conditions this is

$$\psi_1 = \tfrac{1}{192}\big(\cos\tau - \cos 3\tau\big).$$

Altogether we have eventually

$$\theta(t) = \varepsilon\cos\tau + \tfrac{1}{192}\varepsilon^3\big(\cos\tau - \cos 3\tau\big) + O(\varepsilon^5), \quad \tau = K\big(1 - \tfrac{1}{16}\varepsilon^2 + O(\varepsilon^4)\big)t.$$

5.2 Exercises

5.2.1 A quadratically perturbed harmonic oscillator

Consider the following problem for $y(t, \varepsilon)$

$$y'' + y - y^2 = 0, \quad \text{with} \quad y(0) = \varepsilon, \quad y'(0) = 0$$

asymptotically for small positive parameter ε.

i) Determine a second-order (three term) straightforward expansion and discuss its uniformity for large t.
ii) Construct by means of the Lindstedt-Poincaré method ("method of strained coordinates") a second-order (three term) approximate solution.

5.2.2 A weakly nonlinear harmonic oscillator

Consider the following problem for $y(t, \varepsilon)$

$$y'' + (1 + y'^2)y = 0, \quad \text{with} \quad y(0) = \varepsilon, \quad y'(0) = 0$$

asymptotically for small positive parameter ε.

i) Determine a second-order (two term) straightforward expansion and discuss its uniformity for large t.
ii) Construct by means of the Lindstedt-Poincaré method ("method of strained coordinates") a second-order (two term) approximate solution.

5.2.3 A weakly nonlinear, quadratically perturbed harmonic oscillator

Consider the system governed by the equation of motion

$$y'' + y + \varepsilon\alpha y^2 = 0, \quad y(0) = 1, \quad y'(0) = 0,$$

asymptotically for $\varepsilon \to 0$, where $\alpha = O(1)$.

a) Determine a second-order (three term) straightforward expansion and discuss its uniformity for large t.

b) Determine a second-order (three term) expansion, valid for large t, by means of the Lindstedt-Poincaré method.

5.2.4 A coupled nonlinear oscillator

Determine a first-order uniformly valid expansion for the periodic solution of

$$u'' + u = \varepsilon(1 - z)u'$$
$$cz' + z = u^2$$

asymptotically for $\varepsilon \to 0$, where $c = O(1)$ is a positive constant and $u, z = O(1)$.

5.2.5 A weakly nonlinear 4th order oscillator

Determine a periodic solution to $O(\varepsilon)$ of the problem

$$u''' + u'' + u' + u = \varepsilon(1 - u^2 - (u')^2 - (u'')^2)(u'' + u')$$

asymptotically for $\varepsilon \to 0$, where $u = O(1)$.

5.2.6 A weakly nonlinear oscillator

Use Lindstedt-Poincaré's method to get a two-term asymptotic approximation $y = y(t)$ to the problem

$$y'' + y = \varepsilon y y'^2, \quad y(0) = 1, \;\; y'(0) = 0.$$

5.2.7 The Van der Pol oscillator

Consider the weakly nonlinear oscillator, described by the Van der Pol equation, for variable $y = y(t, \varepsilon)$ in t:

$$y'' + y - \varepsilon(1 - y^2)y' = 0$$

asymptotically for small positive parameter ε.

Construct by means of the Lindstedt-Poincaré method ("method of strained coordinates") a second-order (three term) approximation of a *periodic* solution.

Note that not all solutions are periodic (see for example the phase portrait in figure 8.2), so you have to make sure to start on the right trajectory. Otherwise, you are free to make the solution unique in any convenient way. Take for example initial conditions

$$y(0) = A(\varepsilon), \;\; y'(0) = 0$$

with $A(\varepsilon)$ to be determined.

Chapter 6

Matched Asymptotic Expansions

6.1 Theory

6.1.1 Singular perturbation problems

If the solution of the problem considered does not allow a regular expansion, the problem is singular and the solution has no uniform Poincaré expansion in the same variable. We will consider two classes of problems. In the first one the singular behaviour is of boundary layer type and the solution can be built up from locally regular expansions. The solution method is called "method of matched asymptotic expansions". In the other one more time or length scales occur together and a solution is constructed by considering these length scales as if they were independent. The solution method is called "method of multiple scales".

6.1.2 Matched asymptotic expansions

Very often it happens that a simplifying limit applied to a more comprehensive model gives a correct approximation for the main part of the domain, but not everywhere: the limit is *non-uniform*. This non-uniformity may be in space, in time, or in any other variable. For the moment we think of non-uniformity in space, let's say a small region near $x = 0$. If this region of non-uniformity is crucial for the problem, for example because it contains a boundary condition, or a source, the primary reduced problem (which does not include the region of non-uniformity) is not sufficient. This, however, does not mean that no use can be made of the inherent small parameter. The local nature of the non-uniformity itself gives often the possibility of another reduction. In such a case we call this a couple of limiting forms, "inner and outer problems", and are evidence of the fact that we have apparently physically two connected but different problems as far as the dominating mechanism is concerned. Depending on the problem, we now have two simpler problems, serving as boundary conditions to each other via continuity or *matching* conditions.

Non-uniform asymptotic approximations

If a function of x and ε is "essentially" (we will see later what that means) dependent of a combination like x/ε (or anything equivalent, like $(x - x_0)/\varepsilon^2$), then there exists no regular (that means: uniform) asymptotic expansion for all $x = O(1)$ considered. A different expansion arises when $x = O(\varepsilon)$, in other words after scaling $t = x/\varepsilon$ where $t = O(1)$. If the limit exists, we may see something like

$$\Phi(x, \varepsilon) = \varphi\left(\frac{x}{\varepsilon}, x, \varepsilon\right) \simeq \varphi(\infty, x, 0) + \ldots, \quad \Phi(\varepsilon t, \varepsilon) = \varphi(t, \varepsilon t, \varepsilon) \simeq \varphi(t, 0, 0) + \ldots.$$

where x is assumed fixed and non-zero.

Practical examples are

$$\begin{aligned}
\mathrm{e}^{-x/\varepsilon} + \sin(x + \varepsilon) &= 0 + \sin x + \varepsilon \cos x + \ldots \quad \text{on } x \in (0, \infty) \\
\mathrm{e}^{-t} + \sin(\varepsilon t + \varepsilon) &= \mathrm{e}^{-t} + \varepsilon(t + 1) + \ldots \quad \text{on } t \in [0, L] \\
\arctan\left(\frac{x}{\varepsilon}\right) + \tan(\varepsilon x) &= \frac{\pi}{2} + \varepsilon\left(x - \frac{1}{x}\right) + \ldots \quad \text{on } x \in (0, \infty) \\
\arctan(t) + \tan(\varepsilon^2 t) &= \arctan(t) + \varepsilon^2 t + \ldots \quad \text{on } t \in [0, L] \\
\frac{1}{x^2 + \varepsilon^2} &= \frac{1}{x^2} - \frac{\varepsilon^2}{x^4} + \ldots \quad \text{on } x \in (0, \infty) \\
\frac{1}{\varepsilon^2 t^2 + \varepsilon^2} &= \varepsilon^{-2} \frac{1}{1 + t^2} \quad \text{on } t \in [0, L]
\end{aligned}$$

where L is some constant. Of course, if x occurs only in a combination like x/ε, the asymptotic approximation becomes trivial after transformation, but that is only here for the example.

We call this expansion the *outer expansion*, principally valid in the "$x = O(1)$"-outer region. Now consider the *stretched coordinate*

$$t = \frac{x}{\varepsilon}.$$

If the transformed $\Psi(t, \varepsilon) = \Phi(x, \varepsilon)$ has a non-trivial regular asymptotic expansion, then we call this expansion the *inner expansion*, principally valid in the "$t = O(1)$"-inner region, or *boundary layer*. The adjective "non-trivial" is essential: the expansion must be *significant*, *i.e.* different from the outer-expansion rewritten in the inner variable t. This determines the choice (in the present examples) of the inner variable $t = x/\varepsilon$. The scaling $\delta(\varepsilon) = \varepsilon$ is the asymptotically *largest* gauge function with this property.

Note the following example, where we have *three* inherent length scales: $x = O(1)$, $x = O(\varepsilon)$, $x = O(\varepsilon^2)$ and therefore two (nested) boundary layers $x = \varepsilon t$ and $x = \varepsilon^2 \tau$,

$$\begin{aligned}
\log(x/\varepsilon + \varepsilon) &= -\log(\varepsilon) + \log(x) + \ldots\ldots \quad \text{on } x \in (0, \infty) \\
\log(t + \varepsilon) &= \log(t) + \frac{\varepsilon}{t} + \ldots \quad \text{on } t \in (0, L] \\
\log(\varepsilon\tau + \varepsilon) &= \log(\varepsilon) + \log(\tau + 1) \quad \text{on } \tau \in [0, L]
\end{aligned}$$

An important relation between an inner and an outer expansion is the hypothesis that they *match*: the respective regions of validity should, asymptotically, overlap (known as the *overlap hypothesis*). Algorithmically, one may express this as follows, known as Van Dyke's Rule. *The outer limit of the*

inner expansion should be equal to the inner limit of the outer expansion. In other words, the outer-expansion, rewritten in the inner-variable, has a regular series expansion, which is *equal* to the regular asymptotic expansion of the inner-expansion, rewritten in the outer-variable.

Suppose that we have an outer expansion $\mu_0\phi_0 + \mu_1\phi_1 + \dots$ in outer variable x and a corresponding inner expansion $\lambda_0\psi_0 + \lambda_1\psi_1 + \dots$ in inner variable t, where $x = \delta t$. Suppose we can re-expand the outer expansion in the inner variable and the inner expansion in the outer variable such that

$$\begin{aligned}\mu_0(\varepsilon)\varphi_0(\delta t) \quad &+ \mu_1(\varepsilon)\varphi_1(\delta t) \quad + \dots = \lambda_0(\varepsilon)\eta_0(t) + \lambda_1(\varepsilon)\eta_1(t) + \dots ,\\ \lambda_0(\varepsilon)\psi_0(x/\delta) &+ \lambda_1(\varepsilon)\psi_1(x/\delta) + \dots = \mu_0(\varepsilon)\theta_0(x) + \mu_1(\varepsilon)\theta_1(x) + \dots ,\end{aligned}$$

Then for matching the results should be equivalent to the order considered. In particular the expansion of η_k, written back in x,

$$\lambda_0(\varepsilon)\eta_0(x/\delta) + \lambda_1(\varepsilon)\eta_1(x/\delta) + \dots = \mu_0(\varepsilon)\zeta_0(x) + \mu_1(\varepsilon)\zeta_1(x) + \dots,$$

should be such that $\zeta_k = \theta_k$ for $k = 0, 1, \cdots$.

A simple but typical example is the following function on $x \in [0, \infty)$

$$f(x, \varepsilon) = \sin(x + \varepsilon) + \mathrm{e}^{-x/\varepsilon}\cos x$$

with outer expansion with $x = O(1)$

$$F(x, \varepsilon) = \sin x + \varepsilon\cos x - \tfrac{1}{2}\varepsilon^2\sin x - \tfrac{1}{6}\varepsilon^3\cos x + O(\varepsilon^4)$$

and inner expansion with boundary layer (*i.e.* inner) variable $t = x/\varepsilon = O(1)$

$$G(t, \varepsilon) = \mathrm{e}^{-t} + \varepsilon(t + 1) - \tfrac{1}{2}\varepsilon^2 t^2\,\mathrm{e}^{-t} - \tfrac{1}{6}\varepsilon^3(t + 1)^3 + O(\varepsilon^4).$$

The outer expansion in the inner variable

$$F(\varepsilon t, \varepsilon) = \sin(\varepsilon t) + \varepsilon\cos(\varepsilon t) - \tfrac{1}{2}\varepsilon^2\sin(\varepsilon t) - \tfrac{1}{6}\varepsilon^3\cos(\varepsilon t) + O(\varepsilon^4)$$

becomes re-expanded

$$F_{\text{in}}(t, \varepsilon) = \varepsilon(t + 1) - \tfrac{1}{6}\varepsilon^3(t + 1)^3 + O(\varepsilon^4)$$

which is, rewritten in x (and re-ordered in powers of ε), given by

$$F_{\text{in}}(x/\varepsilon, \varepsilon) = x - \tfrac{1}{6}x^3 + \varepsilon(1 - \tfrac{1}{2}x^2) - \tfrac{1}{2}\varepsilon^2 x - \tfrac{1}{6}\varepsilon^3 + O(\varepsilon^4).$$

The inner expansion in the outer variable

$$G(x/\varepsilon, \varepsilon) = x + \varepsilon + (1 - \tfrac{1}{2}x^2)\,\mathrm{e}^{-x/\varepsilon} - \tfrac{1}{6}(x + \varepsilon)^3 + O(\varepsilon^4)$$

becomes re-expanded

$$G_{\text{out}}(x, \varepsilon) = x - \tfrac{1}{6}x^3 + \varepsilon(1 - \tfrac{1}{2}x^2) - \tfrac{1}{2}\varepsilon^2 x - \tfrac{1}{6}\varepsilon^3 + O(\varepsilon^4).$$

Indeed is $G_{\text{out}}(x, \varepsilon)$ functionally equal to $F_{\text{in}}(x/\varepsilon, \varepsilon)$ to the order considered.

Another way to present matching is via an intermediate scaling. Conceptually, this remains closer to the idea of overlapping expansions than Van Dyke's matching rule, but in practice it is more laborious. Suppose we have an outer expansion $F(x, \varepsilon)$ in the outer variable x, and a corresponding inner expansion $G(t, \varepsilon)$ in the boundary layer variable t, where $x = \delta t$ and $\delta(\varepsilon) = o(1)$. Then for matching there should be an intermediate scaling $x = \eta\xi$, with[1] $\delta \ll \eta \ll 1$, such that under this scaling, the re-expanded outer expansion $[F(\eta\xi, \varepsilon)]_{\text{exp}}$ is equal (to the orders considered) to the re-expanded

[1] In other words, $\delta = o(\eta)$ and $\eta = o(1)$.

inner expansion $[G(\frac{\eta}{\delta}\xi, \varepsilon)]_{\text{exp}}$. Note that the result must not depend on the exact choice of η, and the expansions should be taken of high enough order.

With the above example we have with (for exampe) $\eta \sim \varepsilon^{\frac{1}{2}}$

$$\begin{aligned} F(\eta\xi, \varepsilon) &= \sin(\eta\xi) + \varepsilon\cos(\eta\xi) - \tfrac{1}{2}\varepsilon^2\sin(\eta\xi) - \tfrac{1}{6}\varepsilon^3\cos(\eta\xi) + O(\varepsilon^4) \\ &= \eta\xi + \varepsilon - \tfrac{1}{6}\eta^3\xi^3 - \tfrac{1}{2}\varepsilon\eta^2\xi^2 - \tfrac{1}{2}\varepsilon^2\eta\xi - \tfrac{1}{6}\varepsilon^3 + \tfrac{1}{120}\eta^5\xi^5 + \dots \end{aligned}$$

which is indeed to leading orders equal to

$$\begin{aligned} G(\tfrac{\eta}{\varepsilon}\xi, \varepsilon) &= \mathrm{e}^{-\eta\xi/\varepsilon} + \varepsilon(\tfrac{\eta}{\varepsilon}\xi + 1) - \tfrac{1}{2}\varepsilon^2(\tfrac{\eta}{\varepsilon}\xi)^2\,\mathrm{e}^{-\eta\xi/\varepsilon} - \tfrac{1}{6}\varepsilon^3(\tfrac{\eta}{\varepsilon}\xi + 1)^3 + O(\varepsilon^4). \\ &= \eta\xi + \varepsilon - \tfrac{1}{6}(\eta\xi + \varepsilon)^3 + \dots. \end{aligned}$$

The idea of matching is very important because it allows one to move smoothly from one regime into the other. The method of constructing local, but matching, expansions is therefore called "Matched Asymptotic Expansions" (MAE). An intermediate variable is typically used in evaluating integrals across a boundary layer (see below).

Constructing asymptotic solutions

The most important application of this concept of inner- and outer-expansions is that approximate solutions of certain differential equations can be constructed for which the limit under a small parameter is apparently non-uniform.

The main lines of argument for constructing a MAE solution to a differential equation satisfying some boundary conditions are as follows. Suppose we have the following (example) problem.

$$\varepsilon\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} + \frac{\mathrm{d}\varphi}{\mathrm{d}x} - 2x = 0, \qquad \varphi(0) = \varphi(1) = 2. \tag{6.1}$$

Assuming that the outer solution is $O(1)$ because of the boundary conditions, we have for the equation to leading order

$$\frac{\mathrm{d}\varphi_0}{\mathrm{d}x} - 2x = 0,$$

with solution

$$\varphi_0 = x^2 + A.$$

The integration constant A can be determined by the boundary condition $\varphi_0(0) = 2$ at $x = 0$ or $\varphi_0(1) = 2$ at $x = 1$, but not both, so we expect a boundary layer at either end. By trial and error we find that no solution can be constructed if we assume a boundary layer at $x = 1$, so, inferring a boundary layer at $x = 0$, we have to use the boundary condition at $x = 1$ and find

$$\varphi_0 = x^2 + 1.$$

The structure of the equation indeed suggests a correction of $O(\varepsilon)$, so we try the expansion

$$\varphi = \varphi_0 + \varepsilon\varphi_1 + \varepsilon^2\varphi_2 + \cdots.$$

For φ_1 this results into the equation

$$\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} + \frac{\mathrm{d}^2\varphi_0}{\mathrm{d}x^2} = 0,$$

with $\varphi_1(1) = 0$ (the $O(\varepsilon)$-term of the boundary condition), which has the solution

$$\varphi_1 = 2 - 2x.$$

Higher orders are straightforward:

$$\frac{\mathrm{d}\varphi_n}{\mathrm{d}x} = 0, \qquad \text{with } \varphi_n(1) = 0,$$

leading to solutions $\varphi_n \equiv 0$. We find for the outer expansion

$$\varphi = x^2 + 1 + 2\varepsilon(1 - x) + O(\varepsilon^N). \tag{6.2}$$

We continue with the inner expansion, and find near $x = 0$, an order of magnitude of the solution givne by $\varphi = \lambda\psi$, and a boundary layer thickness given by $x = \delta t$ (both λ and δ are to be determined)

$$\frac{\varepsilon\lambda}{\delta^2}\frac{\mathrm{d}^2\psi}{\mathrm{d}t^2} + \frac{\lambda}{\delta}\frac{\mathrm{d}\psi}{\mathrm{d}t} - 2\delta t = 0.$$

Both from the matching ($\varphi_{\text{outer}} \to 1$ for $x \downarrow 0$) and from the boundary condition ($\varphi(0) = 2$) we have to conclude that $\varphi_{\text{inner}} = O(1)$ and so $\lambda = 1$. Furthermore, the boundary layer has only a reason for existence if it comprises new effects, not described by the outer solution. From the heuristic *correspondence principle* we expect that (*i*) a meaningful rescaling corresponds with a *distinguished limit* or *significant degeneration*, while (*ii*) new effects are only included if we have a new equation; in this case if $(\mathrm{d}^2\psi/\mathrm{d}t^2)$ is included. So $\varepsilon\delta^{-2}$ must be at least as large as δ^{-1}, the largest of δ^{-1} and δ. From the principle that we look for the equation with the richest structure, it must be exactly as large, implying a boundary layer thickness $\delta = \varepsilon$. Thus we have the inner equation

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}t^2} + \frac{\mathrm{d}\psi}{\mathrm{d}t} - 2\varepsilon^2 t = 0.$$

From this equation it would *seem* that we have a series expansion without the $O(\varepsilon)$-term, since the equation for this order would be the same as for the leading order. However, from matching with the outer solution:

$$\varphi_{\text{outer}} \to 1 + 2\varepsilon + \varepsilon^2(t^2 - 2t) + \cdots \qquad (x = \varepsilon t,\ t = O(1)),$$

we see that an additional $O(\varepsilon)$-term is to be included. So we substitute the series expansion:

$$\varphi = \psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + \cdots \tag{6.3}$$

It is a simple matter to find

$$\begin{aligned}
&\frac{\mathrm{d}^2\psi_0}{\mathrm{d}t^2} + \frac{\mathrm{d}\psi_0}{\mathrm{d}t} = 0, \quad \psi_0(0) = 2 \ \longrightarrow\ \psi_0 = 2 + A_0(\mathrm{e}^{-t} - 1),\\
&\frac{\mathrm{d}^2\psi_1}{\mathrm{d}t^2} + \frac{\mathrm{d}\psi_1}{\mathrm{d}t} = 0, \quad \psi_1(0) = 0 \ \longrightarrow\ \psi_1 = A_1(\mathrm{e}^{-t} - 1),\\
&\frac{\mathrm{d}^2\psi_2}{\mathrm{d}t^2} + \frac{\mathrm{d}\psi_2}{\mathrm{d}t} = 2t, \quad \psi_2(0) = 0 \ \longrightarrow\ \psi_2 = t^2 - 2t + A_2(\mathrm{e}^{-t} - 1),
\end{aligned}$$

where the constants $A_0, A_1, A_2, \cdots$ are to be determined from the matching condition that inner and outer solution should be asymptotically equivalent in the region of overlap. We follow Van Dyke's

matching rule, and rewrite outer expansion (6.2) in inner variable t, inner expansion (6.3) in outer variable x, re-expand and rewrite the result in x. This results into

$$x^2 + 1 + 2\varepsilon(1-x) + O(\varepsilon^3) \simeq 1 + 2\varepsilon + x^2 - 2\varepsilon x + O(\varepsilon^3) \tag{6.4a}$$

$$\begin{aligned} &2 + A_0(\mathrm{e}^{-t} - 1) + \varepsilon A_1(\mathrm{e}^{-t} - 1) + \varepsilon^2\big(t^2 - 2t + A_2(\mathrm{e}^{-t} - 1)\big) + O(\varepsilon^3) \\ &\qquad\qquad \simeq 2 - A_0 - \varepsilon A_1 + x^2 - 2\varepsilon x - \varepsilon^2 A_2 + O(\varepsilon^3) \end{aligned} \tag{6.4b}$$

The resulting reduced expressions (6.4a) and (6.4b) must be functionally equivalent. A full matching is thus obtained if we choose $A_0 = 1$, $A_1 = -2$, $A_2 = 0$.

Composite expansion

If the boundary layer structure is simple enough, in particular if we have just a simple boundary layer with matching inner and outer expansions, it is possible to combine the separate expansions into a single uniform expansion, called a composite expansion.

Suppose we have an outer expansion $\phi = \mu_0\phi_0 + \mu_1\phi_1 + \dots$ in outer variable $x \in (0,1)$ and a corresponding inner expansion $\psi = \lambda_0\psi_0 + \lambda_1\psi_1 + \dots$ in inner variable $t \in [0,\infty)$, where $x = \delta t$ and $\delta(\varepsilon) = o(1)$. In view of matching, the overlapping parts

$$\hat{\phi}(x) = \big[\phi(\delta t)\big]_{t=x/\delta} = \big[\mu_0(\varepsilon)\varphi_0(\delta t) + \mu_1(\varepsilon)\varphi_1(\delta t) + \dots\big]_{t=x/\delta} = \big[\lambda_0(\varepsilon)\eta_0(t) + \lambda_1(\varepsilon)\eta_1(t) + \dots\big]_{t=x/\delta}$$

$$\hat{\psi}(x) = \psi(x/\delta) = \lambda_0(\varepsilon)\psi_0(x/\delta) + \lambda_1(\varepsilon)\psi_1(x/\delta) + \dots = \mu_0(\varepsilon)\theta_0(x) + \mu_1(\varepsilon)\theta_1(x) + \dots$$

are functionally equivalent to the order considered, *i.e.* $\hat{\phi} \simeq \hat{\psi}(x)$. This means that the combined expression

$$\phi(x) + \psi(x/\delta)$$

is for $x = O(1)$ asymptotically equal to $\phi(x) + \hat{\psi}(x)$, and for $x = O(\delta)$ asymptotically equal to $\psi(x) + \hat{\phi}(x)$. In both cases it is the overlapping part $\hat{\phi}(x)$ (or equivalently $\hat{\psi}(x)$) which is too much. The combined expansion

$$\Phi(x) = \phi(x) + \psi(x/\delta) - \hat{\phi}(x)$$

is thus valid both in the boundary layer and in the outer region.

As an example we may consider the previous problem (6.1), with solution (reformulated)

$$\begin{aligned} \phi(x) &= x^2 + 1 + 2\varepsilon(1-x) + O(\varepsilon^3) \\ \psi(t) &= 1 + \mathrm{e}^{-t} - 2\varepsilon(\mathrm{e}^{-t} - 1) + \varepsilon^2(t^2 - 2t) + O(\varepsilon^3) \\ \hat{\phi}(x) &= 1 + 2\varepsilon + x^2 - 2\varepsilon x + O(\varepsilon^3) \\ \Phi(x) &= x^2 + 1 + \mathrm{e}^{-x/\varepsilon} + 2\varepsilon - 2\varepsilon x - 2\varepsilon\,\mathrm{e}^{-x/\varepsilon} + O(\varepsilon^3) \end{aligned}$$

Approximate evaluation of integrals

Another application of MAE is integration. We split the integral halfway the region of overlap, and approximate the integrand by its inner and outer approximation. Take for example

$$f(x,\varepsilon) = \frac{\log(1+x)}{x^2 + \varepsilon^2}, \quad 0 \leqslant x < \infty, \quad 0 < \varepsilon \ll 1,$$

with outer expansion

$$f(x,\varepsilon) = \frac{\log(1+x)}{x^2+\varepsilon^2} = \frac{\log(1+x)}{x^2} - \varepsilon^2\frac{\log(1+x)}{x^4} + O(\varepsilon^4)$$

and inner expansion in boundary layer $x = \varepsilon t$

$$f(\varepsilon t,\varepsilon) = \frac{\log(1+\varepsilon t)}{\varepsilon^2(t^2+1)} = \frac{1}{\varepsilon^2}\Big(\frac{\varepsilon t - \frac{1}{2}\varepsilon^2 t^2 + O(\varepsilon^3)}{t^2+1}\Big) = \frac{1}{\varepsilon}\frac{t}{t^2+1} - \frac{\frac{1}{2}t^2}{t^2+1} + O(\varepsilon).$$

If we introduce a function $\eta = \eta(\varepsilon)$ with $\varepsilon \ll \eta \ll 1$ (note that eventually the detailed choice of η is and should be immaterial), and split up the integration interval $[0,\infty) = [0,\eta] \cup [\eta,\infty)$, we find

$$\begin{aligned}\int_0^\infty \frac{\log(1+x)}{x^2+\varepsilon^2}\,\mathrm{d}x &\simeq \int_0^{\eta/\varepsilon} \frac{t}{t^2+1} + O(\varepsilon)\,\mathrm{d}t + \int_\eta^\infty \frac{\log(1+x)}{x^2} + O(\varepsilon^2)\,\mathrm{d}x \\ &= \Big[\tfrac{1}{2}\log(1+t^2) + O(\varepsilon)\Big]_0^{\eta/\varepsilon} + \Big[\log x - \frac{1+x}{x}\log(1+x) + O(\varepsilon^2/x^2)\Big]_\eta^\infty \\ &= \Big(\log\eta + \tfrac{1}{2}\log(1+\varepsilon^2/\eta^2) - \log\varepsilon + O(\eta)\Big) + \Big(\log(1+\eta) - \log\eta + \frac{\log(1+\eta)}{\eta} + O(\varepsilon^2/\eta^2)\Big) \\ &\simeq -\log\varepsilon + 1.\end{aligned}$$

Implicit matching subtleties

An interesting detail in the matching process of boundary layer problems where the inner equation is a form of Newton's equation (for example exercises 6.2.8, 6.2.6, 6.2.13, and others) is the following. Consider a boundary layer equation in $Y(t) = Y_0(t) + \ldots$, $0 \leqslant t < \infty$, of the form

$$\tfrac{\partial^2}{\partial t^2}Y_0 + F'(Y_0) = 0,$$

which may be integrated to

$$\tfrac{1}{2}(\tfrac{\partial}{\partial t}Y_0)^2 + F(Y_0) = E.$$

If Y_0 should be matched for $t \to \infty$ to an outer solution $y(x)$ of $O(1)$ with $x = \varepsilon t$, then the integration constant E may be found by observing that $y_x \sim \varepsilon^{-1}Y_t = O(1)$, so the leading order Y_{0t} should vanish for large t. Hence $E = F(y(0))$. An important condition for consistency is that the final integral

$$\int_{Y_0(0)}^{Y_0} \frac{1}{\sqrt{E - F(\eta)}}\,\mathrm{d}\eta = \pm\sqrt{2}t$$

diverges (no square root singularity but at least a simple pole) at $\eta = y(0)$, in order to have $t \to \infty$.

We illustrate this by the following example. The singular boundary value problem

$$\varepsilon^2 y'' + y^2 = K(x), \quad y(0) = 0, \quad y(1) = 0$$

where $K(x) > c > 0$ is $O(1)$ and sufficiently smooth, has boundary layers of $O(\varepsilon)$ near $x = 0$ and $x = 1$. We consider $x = 0$. (The other is analogous.)

An outer approximation $y = y_0 + \dots$ is readily found to be

$$y_0(x) = \pm\sqrt{K(x)},$$

with sign to be decided. Write for notational convenience $K(0) = k^2$. The leading order inner equation for $y(x) = Y(t) = Y_0 + \dots$, where $x = \varepsilon t$, is

$$Y_0'' + Y_0^2 = k^2, \quad Y_0(0) = 0.$$

As argued above, for matching it is required that $Y_0(t) \to \pm k$ and $Y_0' \to 0$. We integrate

$$\tfrac{1}{2}(Y_0')^2 + \tfrac{1}{3}Y_0^3 - k^2 Y_0 = E = \pm(\tfrac{1}{3}k^3 - k^3) = \tfrac{2}{3}k^3.$$

Since Y_0 is small for $t \to 0$ and $(Y_0')^2 > 0$, the sign of E can only be positive, and *thus* outer solution $y_0(x)$ must be negative. Furthermore

$$\tfrac{2}{3}k^3 - \tfrac{1}{3}Y_0^3 + k^2 Y_0 = \tfrac{1}{3}(Y_0 + k)^2(2k - Y_0).$$

Noting that Y_0' has to be negative, we can finish as usual to find explicitly

$$\int_{Y_0}^{0} \frac{1}{(\eta + k)\sqrt{2k - \eta}}\,\mathrm{d}\eta = \tfrac{2}{3}\sqrt{3}\left[\operatorname{artanh}\left(\sqrt{\frac{2k - Y_0}{3k}}\right) - \operatorname{artanh}\left(\sqrt{\tfrac{2}{3}}\right)\right] = \sqrt{\tfrac{2}{3}k}\,t$$

such that

$$Y_0(t) = 2k - 3k\tanh^2\left(\sqrt{\tfrac{1}{2}k}\,t + \operatorname{artanh}\left(\sqrt{\tfrac{2}{3}}\right)\right)$$

Logarithmic switchback

It is not always evident from just the structure of the equation what the necessary expansion will look like. Sometimes it is well concealed and we are only made aware of an invalid initial choice by a matching failure. In fact, it is also the matching process itself that reveals us the required sequence of scaling functions. An example of such a back reaction is known as *logarithmic switchback*.

Consider the following problem for $y = y(x, \varepsilon)$ on the unit interval.

$$\varepsilon y'' + x(y' - y) = 0,\ 0 < x < 1, \qquad y(0, \varepsilon) = 0, \quad y(1, \varepsilon) = \mathrm{e}.$$

The outer solution appears to have the expansion

$$y(x, \varepsilon) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + O(\varepsilon^3).$$

By trial and error, the boundary layer appears to be located near $x = 0$, so the governing equations and boundary conditions are then

$$\begin{aligned} y_0' - y_0 &= 0, & y_0(1) &= \mathrm{e}, \\ y_n' - y_n &= -x^{-1} y_{n-1}'', & y_n(1) &= 0, \end{aligned}$$

with general solution

$$y_n(x) = A_n\,\mathrm{e}^x + \int_x^1 z^{-1}\,\mathrm{e}^{x-z}\,y_{n-1}''(z)\,\mathrm{d}z,$$

such that

$$\begin{aligned}
y_0(x) &= \mathrm{e}^x,\\
y_1(x) &= -\,\mathrm{e}^x \ln(x),\\
y_2(x) &= \mathrm{e}^x\big(\tfrac{1}{2}\ln(x)^2 + \tfrac{3}{2} - 2x^{-1} + \tfrac{1}{2}x^{-2}\big),
\end{aligned}$$

etc. The boundary layer thickness is found from the assumed scaling $x = \varepsilon^m t$ and noting that $y = O(1)$ because of the matching with the outer solution. This leads to the significant degeneration of $m = \frac{1}{2}$, or $x = \varepsilon^{\frac{1}{2}} t$. The boundary layer equation for $y(x,\varepsilon) = Y(t,\varepsilon)$ is thus

$$Y'' + tY' - \varepsilon^{\frac{1}{2}} tY = 0, \qquad Y(0,\varepsilon) = 0.$$

The obvious choice of expansion of Y in powers of $\varepsilon^{\frac{1}{2}}$ is not correct, as the found solution does not match with the outer solution. Therefore, we consider the outer solution in more detail for small x. When $x = \varepsilon^{\frac{1}{2}} t$, we have for the outer solution

$$y(\varepsilon^{\frac{1}{2}} t,\varepsilon) = 1 + \varepsilon^{\frac{1}{2}} t + \varepsilon\big(-\tfrac{1}{2}\ln\varepsilon + \tfrac{1}{2}t^2 - \ln t + \tfrac{1}{2}t^{-2} + \dots\big) + O(\varepsilon^{\frac{3}{2}}\ln\varepsilon) \tag{6.5}$$

(The dots indicate powers of t^{-2} that appear with higher order y_n.) So we apparently need at least

$$Y(t,\varepsilon) = Y_0(t) + \varepsilon^{\frac{1}{2}} Y_1(t) + \varepsilon\ln(\varepsilon) Y_2(t) + \varepsilon Y_3(t) + o(\varepsilon),$$

with equations and boundary conditions

$$\begin{aligned}
Y_0'' + tY_0' &= 0, & Y_0(0) &= 0,\\
Y_1'' + tY_1' &= tY_0, & Y_1(0) &= 0,\\
Y_2'' + tY_2' &= 0, & Y_2(0) &= 0,\\
Y_3'' + tY_3' &= tY_1, & Y_3(0) &= 0,
\end{aligned}$$

etc. Hence, the inner expansion is given by

$$\begin{aligned}
Y_0(t) &= A_0\,\mathrm{erf}\big(\tfrac{t}{\sqrt{2}}\big),\\
Y_1(t) &= A_1\,\mathrm{erf}\big(\tfrac{t}{\sqrt{2}}\big) + A_0\big[t\,\mathrm{erf}\big(\tfrac{t}{\sqrt{2}}\big) + 2\big(\tfrac{2}{\pi}\big)^{\frac{1}{2}}(\mathrm{e}^{-\frac{1}{2}t^2} - 1)\big],\\
Y_2(t) &= A_2\,\mathrm{erf}(\tfrac{t}{\sqrt{2}}),\\
Y_3(t) &= A_3\,\mathrm{erf}(\tfrac{t}{\sqrt{2}}) + \int_0^t \mathrm{e}^{-\frac{1}{2}z^2}\int_0^z \mathrm{e}^{\frac{1}{2}\xi^2}\,\xi Y_1(\xi)\,\mathrm{d}\xi\,\mathrm{d}z.
\end{aligned}$$

Unfortunately, Y_3 cannot be expressed in closed form. However, for demonstration it is sufficient to derive the behaviour of Y_3 for large t. As $\mathrm{erf}(z) \to 1$ exponentially fast for $z \to \infty$, we obtain

$$Y_1(t) = A_0 t + A_1 - 2\big(\tfrac{2}{\pi}\big)^{\frac{1}{2}} A_0 + \text{ exponentially small terms.}$$

If Y_3 behaves for large t algebraically, then $tY_3' \gg Y_3''$, so $Y_3' = Y_1 - t^{-1}Y_3'' \simeq A_0 t$. By successive substitution it follows that

$$Y_3(t) = \tfrac{1}{2}A_0 t^2 + (A_1 - 2\big(\tfrac{2}{\pi}\big)^{\frac{1}{2}} A_0)t - A_0\ln(t) + \dots$$

For matching of the inner solution, we introduce the intermediate variable $\eta = \varepsilon^{-\alpha}x = \varepsilon^{\frac{1}{2}-\alpha}t$ where $0 < \alpha < \frac{1}{2}$, and compare with expression (6.5). We have

$$\begin{aligned} A_0 + \varepsilon^{\frac{1}{2}}\big(A_1 - 2\big(\tfrac{2}{\pi}\big)^{\frac{1}{2}}A_0\big) + \varepsilon^{\alpha}A_0\eta + \varepsilon\ln(\varepsilon)A_2 + \tfrac{1}{2}\varepsilon^{2\alpha}A_0\eta^2 \\ + \varepsilon^{\frac{1}{2}+\alpha}\big(A_1 - 2\big(\tfrac{2}{\pi}\big)^{\frac{1}{2}}A_0)\eta - \varepsilon A_0\ln\eta + \varepsilon(\tfrac{1}{2}-\alpha)A_0\ln\varepsilon \\ \equiv 1 + \varepsilon^{\alpha}\eta + \tfrac{1}{2}\varepsilon^{2\alpha}\eta^2 - \varepsilon\ln\eta - \alpha\varepsilon\ln(\varepsilon) + \tfrac{1}{2}\varepsilon^{2-2\alpha}\eta^{-2}. \end{aligned}$$

Noting that $2 - 2\alpha > 1$, we find a full matching with

$$A_0 = 1, \quad A_1 = 2\big(\tfrac{2}{\pi}\big)^{\frac{1}{2}}, \quad A_2 = -\tfrac{1}{2}.$$

This problem is an example where intermediate matching is preferable.

Prandtl's boundary layer analysis.

The start of modern boundary layer theory is Prandtl's analysis in 1904 of the canonical problem of uniform incompressible low-viscous flow of main flow speed U_∞, viscosity μ and density ρ_0, along a flat plate of length L. Consider the stationary 2D Navier-Stokes equations for incompressible flow for velocity (u, v) and pressure p

$$u_x + v_y = 0, \quad \rho_0(uu_x + vu_y) = -p_x + \mu(u_{xx} + u_{yy}), \quad \rho_0(uv_x + vv_y) = -p_y + \mu(v_{xx} + v_{yy}),$$

subject to boundary conditions $u = v = 0$ at $y = 0$, $0 < x < L$. Make dimensionless $u := U_\infty u$, $v := U_\infty v$, $p := \rho_0 U_\infty^2 p$, $x := Lx$, $y := Ly$. (The scaling of the pressure may not be evident, but is due to the fact that the low-viscous problem is inertia dominated, so the pressure gradient, which is really a reaction force, should balance the inertia terms.) We are left with the dimensionless *Reynolds number* $Re = \rho_0 U_\infty L/\mu$. Since Re is supposed to be large, we write $\varepsilon = Re^{-1}$ small. We obtain

$$u_x + v_y = 0, \quad uu_x + vu_y = -p_x + \varepsilon(u_{xx} + u_{yy}), \quad uv_x + vv_y = -p_y + \varepsilon(v_{xx} + v_{yy}),$$

subject to boundary conditions $u = v = 0$ at $y = 0$, $0 < x < 1$. The leading order outer solution for $y = O(1)$ is given by $(u, v, p) = (1, 0, 0)$, but this solution does not satisfy the boundary condition $u = 0$ at $y = 0$ along the plate. So we anticipate a boundary layer in y, such that the viscous friction εu_{yy} contributes. When we scale $x = X$, $y = \varepsilon^n Y$, $u = U$, $v = \varepsilon^m V$, and $p = P$, we find

$$\begin{aligned} U_X + \varepsilon^{m-n}V_Y = 0, \quad UU_X + \varepsilon^{m-n}VU_Y = -P_X + \varepsilon U_{XX} + \varepsilon^{1-2n}U_{YY}, \\ \varepsilon^m UV_X + \varepsilon^{2m-n}VV_Y = -\varepsilon^{-n}P_Y + \varepsilon^{1+m}V_{XX} + \varepsilon^{1+m-2n}V_{YY}. \end{aligned}$$

This yields the distinguished limit $m = n = \frac{1}{2}$, with the significant degeneration

$$U_X + V_Y = 0, \quad UU_X + VU_Y = U_{YY}, \quad P_Y = 0,$$

known as *Prandtl's boundary layer equations*. Since $P = P(X)$ has to match to the outer solution $p =$ constant (for this particular flat plate problem), pressure gradient $P_X = 0$ and disappears to leading order. Very quickly after Prandtl's introduction of his boundary layer equations, Blasius (1906) was

able to reduce the equation to an ordinary differential equation by means of a similarity solution for the stream function ψ, with $U = \psi_Y$ and $V = -\psi_X$, of the form

$$\psi(X, Y) = \sqrt{2X}\, f(\eta), \qquad \eta = \frac{Y}{\sqrt{2X}},$$

leading to Blasius' equation

$$f''' + f f'' = 0.$$

Prandtl's boundary layer equations, but with other boundary conditions, are also valid in the viscous wake behind the plate $x > 1$, $y = O(\varepsilon^{1/2})$ (Goldstein, 1930). The trailing edge region around $x = 1$, $y = 0$, however, is far more complicated (Stewartson, 1969). Here the boundary layer structure consists of three layers $y = O(\varepsilon^{5/8})$, $O(\varepsilon^{4/8})$, $O(\varepsilon^{3/8})$ within $x - 1 = O(\varepsilon^{3/8})$. This is known as Stewartson's triple deck.

The rôle of matching

It is important to note that a matching is possible at all! Only a part of the terms can be matched by selection of the undetermined constants. Other terms are already equal, without free constants, and there is no way to repair a possibly incomplete matching here. This is an important consistency check on the found solution, at least as long as no real proof is available. If no matching appears to be possible, almost certainly one of the assumptions made with the construction of the solution has to be reconsidered. Particularly notorious are logarithmic singularities of the outer solution, as we saw above. See for other examples [13].

Summarizing, matching of inner- and outer expansion plays an important rôle in the following ways:

i) it provides information about the sequence of order (gauge) functions $\{\mu_k\}$ and $\{\lambda_k\}$ of the expansions;

ii) it allows us to determine unknown constants of integration;

iii) it provides a check on the consistency of the solution, giving us confidence in the correctness.

6.2 Exercises

6.2.1 Singularly perturbed ordinary differential equations

Determine the asymptotic approximation of solution $y(x,\varepsilon)$ (1st or 1st+2nd leading order terms for positive small parameter $\varepsilon \to 0$) of the following singularly perturbed problems.

α and β are non-zero constants, independent of ε.

Provide arguments for the determined boundary layer thickness and location, and show how free constants are determined by the matching procedure.

a)
$$\varepsilon y'' - y' = 2x, \qquad y(0,\varepsilon) = \alpha, \quad y(1,\varepsilon) = \beta.$$

b)
$$\varepsilon y' + y^2 = \cos(x), \qquad y(0,\varepsilon) = 0, \quad 0 \leqslant x \leqslant \tfrac{1}{2}\pi.$$

c)
$$\varepsilon y'' + (2x+1)y' + y^2 = 0, \qquad y(0,\varepsilon) = \alpha, \quad y(1,\varepsilon) = \beta.$$

6.2.2 A hidden boundary layer structure

The problem for $\phi = \phi(x,\varepsilon)$ and $\varepsilon > 0$, $x \geqslant 0$, $F(x)$ smooth, given by

$$\varepsilon^2\phi'' - F(x)\phi = 0, \qquad \phi(0) = a, \quad \phi'(0) = b$$

is difficult to analyse asymptotically for small ε (why?). We therefore transform the problem.

a) Rewrite the problem into one for $y(x)$, where $y(x)$ given by
$$\phi(x) = a \exp\left(\frac{1}{\varepsilon}\int_0^x y(z)\,\mathrm{d}z\right).$$
What is the initial condition for y?

b) Assume that $F(0) \neq 0$ and $F(x) \geqslant 0$ along the interval of interest $[0, L]$. Formulate a formal asymptotic solution of $y = y(x,\varepsilon)$ for small ε up to and including $O(\varepsilon)$.
Hint: the equation $y' + y^2 = 1$ has the solution $\tanh(x - C)$.

c) Apply this to the asymptotic solution for $F(x) = \mathrm{e}^x$.

6.2.3 MAE and integration

Consider the function

$$f(x,\varepsilon) = \mathrm{e}^{-x/\varepsilon}(1+x) + \pi\cos(\pi x + \varepsilon) \quad \text{for} \quad 0 \leqslant x \leqslant 1.$$

a) Construct an outer and inner expansion of f with error $O(\varepsilon^3)$.

b) Integrate f from $x = 0$ to 1 exactly and expand the result up to $O(\varepsilon^3)$.

c) Compare this with the integral that is obtained by integration of the inner and outer expansions following the method described in Example 15.30.

6.2.4 Friedrichs' model problem

A variation on Friedrichs' (1942) model problem for a boundary layer in a viscous fluid is

$$\varepsilon y'' = \big(a(x) - y\big)' \quad \text{for} \quad 0 \leqslant x \leqslant 1,$$

where $y(0) = 0$, $y(1) = 1$, and $a(x)$ is a given strictly positive smooth function independent of ε, and therefore of order 1. Find a two-term inner and outer expansion of the solution of this problem.

6.2.5 The Michaelis-Menten model

A classic enzyme-reaction model, for the first time proposed by Michaelis en Menten (1913), considers a substrate (concentration S) reacting with an enzyme (concentration E) to an enzyme-substrate complex (concentration C), that on its turn dissociates into the final product (concentration P) and the enzyme. The reaction of the substrate to the complex is described in time t by the system

$$\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} &= -k_1 E S + k_{-1} C + k_2 C,\\
\frac{\mathrm{d}S}{\mathrm{d}t} &= -k_1 E S + k_{-1} C,\\
\frac{\mathrm{d}C}{\mathrm{d}t} &= k_1 E S - k_{-1} C - k_2 C,\\
\frac{\mathrm{d}P}{\mathrm{d}t} &= k_2 C,
\end{aligned}$$

with initial values $S(0) = S_0$, $C(0) = 0$, $E(0) = E_0$ and $P(0) = 0$. The parameters k_1, k_{-1} and k_2 are reaction rates: k_1 of the forward reaction, k_{-1} of the backward reaction, and k_2 of the dissociation.

a. If $[S] = [C] = [E] = [P]$ mol/m^3, and $[T] =$ s, what are the dimensional units of k_1, k_{-1} and k_2?

b. Expressed in de problem variables S, C, E, P en t, and the problem parameters E_0, S_0, k_1, k_{-1} and k_2, how many dimensionless quantities has this problem?
Note: "mol" is already dimensionless and does not count as separate unit.

c. Show that $E = E_0 - C$. Ignore the equation for P. Make S, C and t dimensionless such that we obtain a system of the form

$$\begin{aligned}
\frac{\mathrm{d}s}{\mathrm{d}\tau} &= -s + sc + \lambda c,\\
\varepsilon\frac{\mathrm{d}c}{\mathrm{d}\tau} &= s - sc - \mu c,
\end{aligned}$$

with $s(0) = 1$, $c(0) = 0$.

d. Consider the resulting problem asymptotically for $\varepsilon \to 0$. We see that there are two time scales (which?). The short one corresponds with the transient switch-on effects, which behave mathematically like a boundary layer in time. Solve the problem asymptotically to leading and first order in ε.

6.2.6 Stirring a cup of tea

When we stir a cup of tea, the surface of the fluid deforms until equilibrium is attained between gravity, centrifugal force and surface tension. This last force is only important near the wall.

Consider for this problem the following model problem.

A cylinder (radius a, axis vertically) with fluid (density ρ, surface tension σ) rotates around its axis $\vec{e}_z$ (angular velocity Ω) in a gravity field $-g\vec{e}_z$. By the gravity and the centrifugal force the surface deforms to something that looks like a paraboloid. Within a small neighbourhood of the cylinder wall the contact angle α is felt by means of the surface tension.

Because of symmetry we can describe the surface by a radial tangent angle ψ with the horizon, parametrized by arc length s, such that $s = 0$ corresponds wit the axis, and $s = L$ with the wall of the cylinder. L is unknown.

Select the origin on the axis at the surface, such that he vertical and radial coordinate are given by

$$Z(s) = \int_0^s \sin\psi(s')\,\mathrm{d}s'$$
$$R(s) = \int_0^s \cos\psi(s')\,\mathrm{d}s'.$$

The necessary balance between hydrostatic pressure and surface tension yields the equation

$$p_0 - \rho g Z + \tfrac{1}{2}\rho\Omega^2 R^2 = -\sigma\Big(\frac{\mathrm{d}\psi}{\mathrm{d}s} + \frac{\sin\psi}{R}\Big)$$

with unknown p_0. Other boundary conditions are

$$\psi(0) = 0, \quad \psi(L) = \alpha, \quad R(L) = a.$$

a. Make dimensionless with a: $s = at$, $R = ar$, $Z = az$, $L = a\lambda$, and introduce

$$\varepsilon^2 = \frac{\sigma}{\rho g a^2}, \quad \beta = \frac{p_0}{\rho g a}. \quad \mu = \frac{\Omega^2 a}{g}.$$

   Identify the dimensionless constants in terms of standard dimensionless numbers.

b. Solve the resulting problem asymptotically for $\varepsilon \to 0$. Assume that $\mu = O(1)$. Note that β and λ are unknown and therefore part of the solution.

6.2.7 A singularly perturbed nonlinear problem

Find a composite expansion of the solution of the following boundary value problem

$$\varepsilon y'' + 2y' + y^3 = 0$$

along $0 < x < 1$, where $y(0) = 0$ and $y(1) = \frac{1}{2}$.

6.2.8 Groundwater flow

Through a long strip of ground of width L between two canals (water level h_0 and h_1) the ground water seeps slowly from one side to the other.

Select a coordinate system such that the Z-axis is parallel to the long axis of the strip and the canals, the Y-axis is vertical, and the X-axis perpendicular to both. $X = 0$ corresponds with canal 0, and $X = L$ with canal 1. Assume that the groundwater level is constant in Z-direction.

Assume that the layer of ground lies on top of a semipermeable layer at level$Y = 0$, while the ground water level is given by $Y = h(X)$.

The water leaks through the semi-permeable layer at a rate proportional to the local hydrostatic pressure. As this pressure is on its turn proportional to water level h, this yields a flux density αh, where α is a constant.

Water comes in by precipitation (rain). Fluctuations in precipitation are assumed to be averaged away by the slow groundwater flow, such that the fluxdensity N from this precipitation is constant in time. Assume that variations in overgrowth and buildings may result into a position dependent $N = N(X)$.

Between two neighbouring positions X and $X + \mathrm{d}X$ there exists a small difference in height and therefore in pressure. According to Darcy's law this creates a flow with a velocity proportional to the pressure difference, and dependent of the porosity of the ground. As the pressure difference is the same along the full height, the flow velocity is uniform, and we have

$$p(X) - p(X + \mathrm{d}X) \;\sim\; h(X) - h(X + \mathrm{d}X) \;\sim\; v(X)\,\mathrm{d}X,$$

and the horizontal fluxdensity is proportional to

$$v = -D\frac{\mathrm{d}h}{\mathrm{d}X}$$

where D is in general a function of position.

The flux balance along a slice $\mathrm{d}X$ is then given by $\left[Dh\frac{\mathrm{d}h}{\mathrm{d}X}\right]_X^{X+\mathrm{d}X} = (\alpha h - N)\,\mathrm{d}X$, or

$$\frac{\mathrm{d}}{\mathrm{d}X}\left(Dh\frac{\mathrm{d}h}{\mathrm{d}X}\right) = \alpha h - N$$

a. We consider the situation with $h_0 = 0$, and$D$ is constant. Make dimensionless with L, h_1 and α: $X = Lx$, $h(X) = h_1\phi(x)$, $N(X) = \alpha h_1 K(x)$, and introduce the positive dimensionless parameter
$$\varepsilon = \frac{Dh_1}{2\alpha L^2}.$$

b. Assume heavy rain, such that $K(x) = O(1)$. solve the resulting problem asymptotically for $\varepsilon \to 0$.

c. Assume little rain, such that $K(x) = \varepsilon\kappa(x)$, with $\kappa = O(1)$. Solve the resulting problem asymptotically for $\varepsilon \to 0$. Take good care at $x = 1$. The boundary layer is rather complicated with a layered structure.

d. What changes when we take the slightly more general case of $D = D(X)$?

6.2.9 Heat conduction

Consider steady-state heat conduction in the rectangular region $0 \leqslant x' \leqslant L$, $-D \leqslant y' \leqslant D$. Assume that the temperature is prescribed along the edges $x' = 0$ and $x' = L$ and that the edges $y' = \pm D$ are insulated. We are interested in the problem for a slender geometry, *i.e.* $\varepsilon = D/L \ll 1$. If we normalize x with respect to L and y with respect to D, we need to solve on the rectangle $0 \leqslant x \leqslant 1$, $-1 \leqslant y \leqslant 1$ the equation

$$\varepsilon^2 T_{xx} + T_{yy} = 0, \quad T(0, y, \varepsilon) = f(y), \quad T(1, y, \varepsilon) = g(y), \quad T_y(x, \pm 1, \varepsilon) = 0$$

1. Construct an outer expansion in the form

$$T(x, y, \varepsilon) = T_0(x, y) + \varepsilon^2 T_1(x, y) + O(\varepsilon^4)$$

2. Construct an appropriate inner expansion along the edges $x = 0$ and $x = 1$.
3. Verify that matching is possible and determine the unknown constants.
4. Solve the problem exactly and compare this with the results found.

6.2.10 Sign problems

A small parameter multiplying the highest derivative does not guarantee that boundary or interior layers are present.

After solving the following problem, explain why the method of matched asymptotic expansions cannot be used (in a straightforward manner) to find an asymptotic approximation to the solution.

(a)

$$\varepsilon^2 y'' + \omega^2 y = 0$$

along $0 < x < 1$ and $\omega \neq 0$.

(b)

$$\varepsilon^2 y'' = y'$$

along $0 < x < 1$, where $y'(0) = -1$ and $y(1) = 0$.

6.2.11 A chemical reaction-diffusion problem (singular limit)

Reconsider the chemical reaction-diffusion problem of problem 3.3.11 (page 46)

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}c}{\mathrm{d}r}\right) = \phi^2 c^n, \quad 0 < r < 1,$$
$$c(1) = 1, \quad c'(0) = 0,$$

but now for the asymptotic behaviour of c when $\varepsilon = \phi^{-1} \to 0$. Solve first the exact solution for $n = 1$ to guess the general structure. Find the leading order inner and outer solution.
Hint. Introduce $y = rc$.

6.2.12 A boundary layer problem with variable coefficients

Suppose that $y(x)$ satisfies the boundary value problem

$$\varepsilon y'' + a(x)y' + b(x)y = 0,$$

with $0 < \varepsilon \ll 1$ and

$$y(0) = A, \quad y(1) = B.$$

Suppose also that $a(x)$ and $b(x)$ are analytic in $[0, 1]$ (*i.e.* have convergent Taylor series in any point $\in [0, 1]$.

a) If $a > 0$, find an approximate solution and show that it has a boundary layer at $x = 0$.

b) If $a < 0$, find an approximate solution and show that it has a boundary layer at $x = 1$.

c) Finally, if $a(x_0) = 0$ for $x_0 \in (0, 1)$, where $a < 0$ for $x < x_0$ and $a > 0$ for $x > x_0$, show that no boundary layer at the end points can exist, and therefore that an interior layer must exist at x_0.

   Suppose that

$$\beta = \frac{b(x_0)}{a'(x_0)}.$$

   Show that as $x \downarrow x_0$, resp. $x \uparrow x_0$, the outer solution in $x < x_0$, resp. $x > x_0$ satisfy

$$y \simeq c_\pm |x - x_0|^{-\beta},$$

   where the constants $c_\pm$ are known, but in general are not the same.

   Hence show by rescaling x and y as

$$y(x) = \left(\frac{\varepsilon}{a'(x_0)}\right)^{-\frac{1}{2}\beta} Y(X), \quad x = x_0 + \left(\frac{\varepsilon}{a'(x_0)}\right)^{\frac{1}{2}} X,$$

   the equation can be approximately written in the transition region as

$$Y'' + XY' + \beta Y = 0$$

   with matching conditions

$$Y \sim c_\pm |X|^{-\beta} \quad \text{as} \quad X \to \pm\infty.$$

   Solve this problem for $\beta = -1$. (Use Maple or Mathematica.)

6.2.13 The stiffened catenary revisited

A cable, suspended between the points $X = 0$, $Y = 0$ and $X = D$, $Y = 0$, is described as a linear elastic, geometrically non-linear inextensible bar of weight Q per unit length.

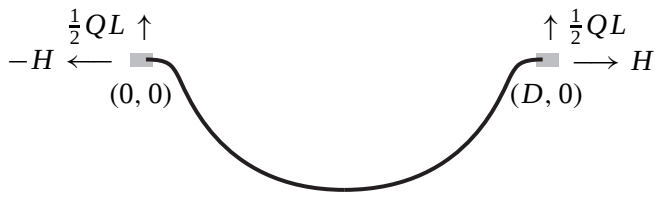


Figure 6.1: A suspended cable

At the suspension points the cable is horizontally clamped such that the cable hangs in the vertical plane through the suspension points.

The total length L of the cable is much larger than D, while the bending stiffness EI is relatively small, such that the cable is slack.

In order to keep the cable in position, the suspension points apply a reaction force, with horizontal component H resp. $-H$, and a vertical component V, resp. $QL - V$. From symmetry we already have $V = \frac{1}{2}QL$, but H is unknown.

With s the arc length along the cable, $\psi(s)$ the tangent angle with the horizon, and $X(s), Y(s)$ the cartesian co-ordinates of a point on the cable, the shape of the cable is given by

$$
\begin{aligned}
&EI\frac{\mathrm{d}^2\psi}{\mathrm{d}s^2} = H\sin\psi - (Qs - V)\cos\psi \\
&\psi(0) = \psi(L) = 0 \\
&X(L) = \int_0^L \cos\psi(s)\,\mathrm{d}s = D \\
&Y(L) = \int_0^L \sin\psi(s)\,\mathrm{d}s = 0
\end{aligned}
$$

a. Make dimensionless with L: $s = Lt$, $X = Lx$, $Y = Ly$, $D = Ld$, and introduce $\varepsilon^2 = EI/QL^3$, $h = H/QL$.

b. Solve the resulting problem asymptotically for $\varepsilon \to 0$. Assume $d = O(1)$, $h = O(1)$.

   As posed, d is known and h is unknown, and so $h = h(\varepsilon, d)$. It may be more convenient to deal with the inverse problem first, where h is known, and d results. Of course, then is $d = d(\varepsilon, h)$. Finally, after having found the relation between d and h (asymptotically), we can solve this for h and given d.

Chapter 7

Multiple Scales and the WKB method

7.1 Theory

7.1.1 General procedure

Suppose a function $\varphi(x, \varepsilon)$ depends on more than one length scale acting together, for example x, εx, and $\varepsilon^2 x$. Then the function does not have a regular expansion on the full domain of interest, $x \leqslant O(\varepsilon^{-2})$ say. It is not possible to bring these different length scales together by a simple coordinate transformation, like in the method of slow variation or the Lindstedt-Poincaré method, or to split up our domain in subdomains like in the method of matched asymptotic expansions. Therefore we have to find another way to construct asymptotic expansions, valid in the full domain of interest. The approach that is followed in the *method of multiple scales* is at first sight rather radical: the various length scales are temporarily considered as independent variables: $x_1 = x, x_2 = \varepsilon x, x_3 = \varepsilon^2 x$, and the original function φ is identified with a more general function $\psi(x_1, x_2, x_3, \varepsilon)$ depending on a higher dimensional independent variable.

$$\varphi(x, \varepsilon) = A(\varepsilon)\,\mathrm{e}^{-\varepsilon x}\cos(x + \theta(\varepsilon)) \;\text{ becomes }\; \psi(x_1, x_2, \varepsilon) = A(\varepsilon)\,\mathrm{e}^{-x_2}\cos(x_1 + \theta(\varepsilon)). \qquad \square$$

Since this identification is not unique, we may add constraints such that this auxiliary function ψ does have a Poincaré expansion on the full domain of interest. After having constructed this expansion, it may be associated to the original function along the line $x_1 = x, x_2 = \varepsilon x, x_3 = \varepsilon^2 x$.

The technique, utilizing this difference between small scale and large scale behaviour is the method of multiple scales. As with most approximation methods, this method has grown out of practice, and works well for certain types of problems. Typically, the multiple scale method is applicable to problems with on the one hand a certain global quantity (energy, power), which is conserved or almost conserved, controlling the amplitude, and on the other hand two rapidly interacting quantities (kinetic and potential energy), controlling the phase. Usually, this describes slowly varying waves, affected by small effects during a long time. Intuitively, it is clear that over a short distance (a few wave lengths) the wave only sees constant conditions and will propagate approximately as in the constant case, but over larger distances it will somehow have to change its shape in accordance with its new environment.

7.1.2 A practical example: a damped oscillator

We will illustrate the method by considering a damped harmonic oscillator

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 2\varepsilon \frac{\mathrm{d}y}{\mathrm{d}t} + y = 0 \quad (t \geqslant 0), \qquad y(0) = 0, \ \frac{\mathrm{d}y(0)}{\mathrm{d}t} = 1 \tag{7.1}$$

with $0 < \varepsilon \ll 1$. The exact solution is readily found to be

$$y(t) = \mathrm{e}^{-\varepsilon t} \, \frac{\sin\bigl(\sqrt{1-\varepsilon^2}\, t\bigr)}{\sqrt{1-\varepsilon^2}}. \tag{7.2}$$

A naive approximation of this $y(t)$, for small ε and fixed t, would give

$$y(t) = \sin t - \varepsilon t \sin t + O(\varepsilon^2), \tag{7.3}$$

which appears to be useful for $t = O(1)$ only. For large t the approximation becomes incorrect:

1) if $t \geq O(\varepsilon^{-1})$ the second term is of equal importance, or larger, as the first term and nothing is left over of the slow exponential decay;
2) if $t \geq O(\varepsilon^{-2})$ the phase has an error of $O(1)$, or larger, giving an approximation of which even the sign may be in error.

We would obtain a far better approximation if we adopted two different time variables, *viz.* $T = \varepsilon t$ and $\tau = \sqrt{1-\varepsilon^2}\, t$, and changed to $y(t, \varepsilon) = Y(\tau, T, \varepsilon)$ where

$$Y(\tau, T, \varepsilon) = \mathrm{e}^{-T} \, \frac{\sin(\tau)}{\sqrt{1-\varepsilon^2}}.$$

It is easily verified that a Taylor series of Y in ε yields a regular expansion for all t.

If we construct a straightforward approximate solution directly from equation (7.1), we would get the same approximation as in (7.3), which is too limited for most applications. However, knowing the character of the error, we may try to avoid them and look for the auxiliary function Y, instead of y. As we, in general, do not know the occurring time scales, their determination becomes part of the problem.

Suppose we can expand

$$y(t, \varepsilon) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \cdots . \tag{7.4}$$

Substituting in (7.1) and collecting equal powers of ε gives

$$\begin{aligned} O(\varepsilon^0): \ & \frac{\mathrm{d}^2 y_0}{\mathrm{d}t^2} + y_0 = 0 && \text{with } \ y_0(0) = 0, \ \frac{\mathrm{d}y_0(0)}{\mathrm{d}t} = 1, \\ O(\varepsilon^1): \ & \frac{\mathrm{d}^2 y_1}{\mathrm{d}t^2} + y_1 = -2\frac{\mathrm{d}y_0}{\mathrm{d}t} && \text{with } \ y_1(0) = 0, \ \frac{\mathrm{d}y_1(0)}{\mathrm{d}t} = 0. \end{aligned}$$

We then find

$$y_0(t) = \sin t, \qquad y_1(t) = -t \sin t, \qquad \textit{etc.}$$

which reproduces indeed expansion (7.3). The straightforward, Poincaré type, expansion (7.4) breaks down for large t, when $\varepsilon t \geqslant O(1)$. It is important to note that this caused by the fact that any y_n is excited in its eigenfrequency (by the "source"-terms $-2\mathrm{d}y_{n-1}/\mathrm{d}t$), resulting in resonance. We recognise

the generated algebraically growing terms of the type $t^n \sin t$ and $t^n \cos t$, called *secular terms* (definition 5.1.1). Apart from being of limited validity, the expansion reveals nothing of the real structure of the solution, and we change our strategy to looking for an auxiliary function dependent on different time scales. We start with the hypothesis that, next to a fast time scale t, we have the slow time scale

$$T := \varepsilon t. \tag{7.5}$$

Then we identify the solution y with a suitably chosen other function Y that depends on both variables t and T

$$Y(t, T; \varepsilon) := y(t; \varepsilon). \tag{7.6}$$

There exist infinitely many functions $Y(t, T, \varepsilon)$ that are equal to $y(t, \varepsilon)$ along the line $T = \varepsilon t$ in (t, T)-space. So we have now some freedom to prescribe additional conditions. With the unwelcome appearance of secular terms in mind it is natural to think of conditions, to be chosen such that no secular terms occur when we construct an approximation.

Since the time derivatives of y turn into partial derivatives of Y, *i.e.*

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial Y}{\partial t} + \varepsilon \frac{\partial Y}{\partial T}, \tag{7.7}$$

equation (7.1) becomes for Y

$$\frac{\partial^2 Y}{\partial t^2} + Y + 2\varepsilon\Big(\frac{\partial Y}{\partial t} + \frac{\partial^2 Y}{\partial t \partial T}\Big) + \varepsilon^2\Big(\frac{\partial^2 Y}{\partial T^2} + 2\frac{\partial Y}{\partial T}\Big) = 0. \tag{7.8}$$

Assume the expansion

$$Y(t, T, \varepsilon) = Y_0(t, T) + \varepsilon Y_1(t, T) + \varepsilon^2 Y_2(t, T) + \cdots \tag{7.9}$$

and substitute this into (7.8) to obtain to leading orders

$$\begin{aligned} &\frac{\partial^2 Y_0}{\partial t^2} + Y_0 = 0, \\ &\frac{\partial^2 Y_1}{\partial t^2} + Y_1 = -2\frac{\partial Y_0}{\partial t} - 2\frac{\partial^2 Y_0}{\partial t \partial T}, \end{aligned}$$

with initial conditions

$$\begin{aligned} &Y_0(0, 0) = 0, \qquad \frac{\partial}{\partial t} Y_0(0, 0) = 1, \\ &Y_1(0, 0) = 0, \qquad \frac{\partial}{\partial t} Y_1(0, 0) = -\frac{\partial}{\partial T} Y_0(0, 0). \end{aligned}$$

The solution for Y_0 is easily found to be

$$Y_0(t, T) = A_0(T) \sin(t + \theta_0(T)) \qquad \text{with} \quad A_0(0) = 1, \theta_0(0) = 0. \tag{7.10}$$

This gives a right-hand side for the Y_1-equation of

$$-2\Big(A_0 + \frac{\partial A_0}{\partial T}\Big) \cos(t + \theta_0) + 2A_0 \frac{\partial \theta_0}{\partial T} \sin(t + \theta_0).$$

No secular terms occur (no resonance between Y_1 and Y_0) if these terms vanish:

$$A_0 + \frac{\partial A_0}{\partial T} = 0 \quad \text{yielding} \quad A_0 = \mathrm{e}^{-T}, \quad \frac{\partial \theta_0}{\partial T} = 0 \quad \text{yielding} \quad \theta_0 = 0. \tag{7.11}$$

Together we have indeed constructed an approximation of (7.2), valid for $t \leq O(\varepsilon^{-1})$.

$$y(t, \varepsilon) = \mathrm{e}^{-\varepsilon t} \sin t + O(\varepsilon). \tag{7.12}$$

Note (this is typical of this approach), that we determined Y_0 only on the level of Y_1, but without having to solve Y_1 itself.

The present approach is by and large the multiple scale technique in its simplest form. Variations on this theme are sometimes necessary. For example, we have not completely got rid of secular terms. On a longer time scale ($t = O(\varepsilon^{-2})$) we have again resonance in Y_2 because of the "source" $\mathrm{e}^{-T} \sin t$, yielding terms $O(\varepsilon^2 t)$. We see that a second time scale $T_2 = \varepsilon^2 t$ is necessary. From the exact solution we may infer that these longer time scales are not really independent and it may be worthwhile to try a fast time of strained coordinates type: $\tau = \omega(\varepsilon)t = (1+\varepsilon^2\omega_1+\varepsilon^4\omega_4+\ldots)t$. In the present example we would recover $\omega(\varepsilon) = \sqrt{1-\varepsilon^2}$.

7.1.3 The WKB Method: Slowly varying fast time scale

The method of multiple scales fails when the slow variation is caused by external effects, like a slowly varying problem parameter. In this case the nature of the slow variation is not the same for all time, but may vary. This is demonstrated by the following example. Consider the problem

$$\ddot{x} + \kappa(\varepsilon t)^2 x = 0, \quad x(0, \varepsilon) = 1, \quad \dot{x}\,(0, \varepsilon) = 0,$$

where $\kappa = O(1)$. It seems plausible to assume 2 time scales: a fast one $O(\kappa^{-1}) = O(1)$ and a slow one $O(\varepsilon^{-1})$. So we introduce next to t the slow scale $T = \varepsilon t$, and rewrite $x(t, \varepsilon) = X(t, T, \varepsilon)$. We expand $X = X_0 + \varepsilon X_1 + \ldots$, and obtain $X_0 = A_0(T)\cos(\kappa(T)t - \theta_0(T))$. Suppressing secular terms in the equation for X_1 requires $A_0' = \kappa' t - \theta_0' = 0$, which is impossible.

Here, the fast time scale is slowly varying itself and the fast variable is to be strained locally by a suitable strain function. This sounds complicated, but is in fact quite simple: we introduce a fast time scale via a slowly varying function. Often, it is convenient to write this function in the form of an integral, because it always appears in the equations after differentiation.

The introduction of a slow time scale together with the slowly varying fast time scale, is generally associated with the WKB method. Usually is the WKB Assumption (Ansatz) restricted to the assumption of the solution of a particular form related to waves.

$$\tau = \int^t \omega(\varepsilon t', \varepsilon)\,\mathrm{d}t' = \frac{1}{\varepsilon}\int^T \omega(z, \varepsilon)\,\mathrm{d}z, \quad \text{where } T = \varepsilon t, \tag{7.13}$$

while for $x(t, \varepsilon) = X(\tau, T, \varepsilon)$ we have

$$\dot{x} = \omega X_\tau + \varepsilon X_T \qquad \text{and} \qquad \ddot{x} = \omega^2 X_{\tau\tau} + \varepsilon\omega_T X_\tau + 2\varepsilon\omega X_{\tau T} + \varepsilon^2 X_{TT}. \tag{7.14}$$

After expanding $X = X_0 + \varepsilon X_1 + \dots$ and $\omega = \omega_0 + \varepsilon\omega_1 + \dots$ we obtain

$$\begin{aligned} &\omega_0^2 X_{0\tau\tau} + \kappa^2 X_0 = 0, \\ &\omega_0^2 X_{1\tau\tau} + \kappa^2 X_1 = -2\omega_0\omega_1 X_{0\tau\tau} - \omega_{0T} X_{0\tau} - 2\omega_0 X_{0\tau T}. \end{aligned} \tag{$*$}$$

The leading order solution is $X_0 = A_0(T)\cos(\lambda(T)\tau - \theta_0(T))$, where $\lambda = \kappa/\omega_0$. The right-hand side of $(*)$ is then

$$2\omega_0 A_0\lambda(\omega_1\lambda + \lambda_T\tau - \theta_{0T})\cos(\lambda\tau - \theta_0) + (A_0\lambda)^{-1}(\omega_0 A_0^2\lambda^2)_T \sin(\lambda\tau - \theta_0).$$

Suppression of secular terms requires $\lambda_T = 0$. Without loss of generality we can take $\lambda = 1$, or $\omega_0 = \kappa$. Then we need $\omega_1 = \theta_{0T}$, which just yields that $\lambda\tau - \theta_0 = \tau - \theta_0 = \varepsilon^{-1}\int^T \omega(z)\,\mathrm{d}z - \int^T \omega_1(z)\,\mathrm{d}z = \varepsilon^{-1}\int^T \omega_0(z)\,\mathrm{d}z + O(\varepsilon)$. In other words, we may just as well take $\omega_1 = 0$ and $\theta_0 =$ a constant. Finally we have $\omega_0 A_0^2\lambda^2 = \kappa A_0^2 =$ a constant.

For linear wave-type problems we may anticipate the structure of the solution and assume the so-called WKB hypothesis (after Wentzel, Kramers and Brillouin) or *ray approximation*

$$y(t,\varepsilon) = A(T,\varepsilon)\,\mathrm{e}^{\mathrm{i}\varepsilon^{-1}\int_0^T \omega(\tau,\varepsilon)\,\mathrm{d}\tau}\,. \tag{7.15}$$

The method is again illustrated by the example of the damped oscillator (7.1), but now in complex form, so we consider the real part of (7.15). After substitution and suppressing the exponential factor, we get

$$\begin{aligned} &(1-\omega^2)A + \mathrm{i}\varepsilon\Big(2\omega\frac{\partial A}{\partial T} + \frac{\partial\omega}{\partial T}A + 2\omega A\Big) + \varepsilon^2\Big(\frac{\partial^2 A}{\partial T^2} + 2\frac{\partial A}{\partial T}\Big) = 0, \\ &\mathrm{Re}(A) = 0, \quad \mathrm{Re}(\mathrm{i}\omega A + \varepsilon A') = 0 \ \text{ at } \ T = 0. \end{aligned} \tag{7.16}$$

Unlike in the multiple scales method the secular terms will not be explicitly suppressed, at least not to leading order. The underlying additional condition here is that the solution of the present type *exists* and that each higher order correction is no more secular than its predecessor. The solution is expanded as

$$\begin{aligned} A(T;\varepsilon) &= A_0(T) + \varepsilon A_1(T) + \varepsilon^2 A_2(T) + \cdots \\ \omega(T;\varepsilon) &= \omega_0(T) + \varepsilon^2\omega_2(T) + \cdots\,. \end{aligned} \tag{7.17}$$

Note that ω_1 may be set to zero since the factor $\exp(\mathrm{i}\int_0^T \omega_1(\tau)\,\mathrm{d}\tau)$ may be incorporated in A. By a similar argument, *viz.* by re-expanding the exponential for small ε, all other terms $\omega_2, \omega_3, \dots$ could be absorbed by A (this is often done). This is perfectly acceptable for the time scale $T = O(1)$, but for larger times we will not be able to suppress higher order secular terms. So we will find it more convenient to include these terms and use them whenever convenient.

We substitute the expansions and collect equal powers of ε to obtain to $O(\varepsilon^0)$

$$(1-\omega_0^2)A_0 = 0 \tag{7.18a}$$

with solution $\omega_0 = 1$ (or -1, but that is equivalent for the result). To $O(\varepsilon^1)$ we have then

$$A_0' + A_0 = 0 \ \text{ with } \ \mathrm{Re}(A_0) = 0,\ \mathrm{Im}(\omega_0 A_0) = -1 \text{ at } T = 0, \tag{7.18b}$$

with solution $A_0 = -\mathrm{i}\,\mathrm{e}^{-T}$. To order $O(\varepsilon^2)$ the equation reduces to

$$A_1' + A_1 = -\mathrm{i}(\tfrac{1}{2} + \omega_2)\,\mathrm{e}^{-T}, \quad \text{with } \mathrm{Re}(A_1) = 0,\ \mathrm{Im}(\omega_0 A_1) = \mathrm{Re}(A_0') \text{ at } T = 0, \tag{7.18c}$$

with solution

$$\omega_2 = -\tfrac{1}{2}, \quad A_1 = 0. \tag{7.18d}$$

Note that if we had chosen $\omega_2 = 0$, the solution would be $A_1 = -\frac{1}{2}T\,\mathrm{e}^{-T}$. Although by itself correct for $T = O(1)$, it renders the asymptotic hierarchy invalid for $T \geqslant O(1/\varepsilon)$ and is therefore better avoided. The solution that emerges is indeed consistent with the exact solution.

The air-damped resonator.

In dimensionless form this is given by

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \varepsilon \frac{\mathrm{d}y}{\mathrm{d}t}\left|\frac{\mathrm{d}y}{\mathrm{d}t}\right| + y = 0, \qquad \text{with} \quad y(0) = 1,\ \frac{\mathrm{d}y(0)}{\mathrm{d}t} = 0. \tag{$*$}$$

By rewriting the equation into the form

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\left[\tfrac{1}{2}(y')^2 + \tfrac{1}{2}y^2\right] = -\varepsilon (y')^2 |y'|$$

and assuming that y and $y' = O(1)$, it may be inferred that the damping acts on a time scale of $O(\varepsilon^{-1})$. So we conjecture the presence of the slow time variable $T = \varepsilon t$ and introduce a new dependent variable Y that depends on both t and T. We have

$$T = \varepsilon t, \qquad y(t, \varepsilon) = Y(t, T, \varepsilon), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial Y}{\partial t} + \varepsilon \frac{\partial Y}{\partial T},$$

and obtain for equation ($*$)

$$\frac{\partial^2 Y}{\partial t^2} + Y + \varepsilon\left(2\frac{\partial^2 Y}{\partial t \partial T} + \frac{\partial Y}{\partial t}\left|\frac{\partial Y}{\partial t}\right|\right) + O(\varepsilon^2) = 0$$
$$Y(0, 0, \varepsilon) = 1, \quad \left(\frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial T}\right) Y(0, 0, \varepsilon) = 0.$$

The error of $O(\varepsilon^2)$ results from the approximation $\frac{\partial}{\partial t}Y + \varepsilon \frac{\partial}{\partial T}Y = \frac{\partial}{\partial t}Y + O(\varepsilon)$, and is of course only valid outside a small neighbourhood of the points where $\frac{\partial}{\partial t}Y = 0$. We expand

$$Y(t, T, \varepsilon) = Y_0(t, T) + \varepsilon Y_1(t, T) + O(\varepsilon^2),$$

to find for the leading order

$$\frac{\partial^2 Y_0}{\partial t^2} + Y_0 = 0, \quad \text{with} \quad Y_0(0, 0) = 1,\ \frac{\partial}{\partial t} Y_0(0, 0) = 0.$$

The solution is given by

$$Y_0 = A_0(T)\cos(t - \Theta_0(T)), \quad \text{where} \quad A_0(0) = 1,\ \Theta_0(0) = 0.$$

For the first order we have the equation

$$\begin{aligned}\frac{\partial^2 Y_1}{\partial t^2} + Y_1 &= -2\frac{\partial^2 Y_0}{\partial t \partial T} - \frac{\partial Y_0}{\partial t}\left|\frac{\partial Y_0}{\partial t}\right| \\ &= 2\frac{\mathrm{d}A_0}{\mathrm{d}T}\sin(t-\Theta_0) - 2A_0\frac{\mathrm{d}\Theta_0}{\mathrm{d}T}\cos(t-\Theta_0) + A_0^2\sin(t-\Theta_0)|\sin(t-\Theta_0)|,\end{aligned}$$

with corresponding initial conditions. The secular terms are suppressed if the first harmonics of the right-hand side cancel. For this we use the Fourier series expansion

$$\sin(t)\,|\sin(t)| = -\frac{8}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)t}{(2n-1)(2n+1)(2n+3)}.$$

We obtain the equations

$$2\frac{\mathrm{d}A_0}{\mathrm{d}T} + \frac{8}{3\pi}A_0^2 = 0 \quad \text{and} \quad \frac{\mathrm{d}\Theta_0}{\mathrm{d}T} = 0,$$

with solution $\Theta_0(T) = 0$ and

$$A_0(T) = \frac{1}{1+\frac{4}{3\pi}T}.$$

Altogether we have the approximate solution

$$y(t,\varepsilon) = \frac{\cos(t)}{1+\frac{4}{3\pi}\varepsilon t} + O(\varepsilon).$$

This approximation appears to be remarkably accurate. See Figure 7.1 where plots, made for a parameter value of $\varepsilon = 0.1$, of the approximate and a numerically "exact" solution are hardly distinguishable. A maximum difference is found of 0.03.

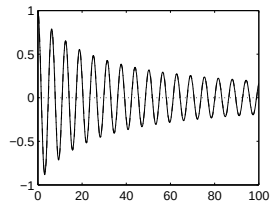


Figure 7.1: Plots of the approximate and a numerically "exact" solution $y(t, \varepsilon)$ of the air-damped resonator problem for $\varepsilon = 0.1$.

7.2 Exercises

7.2.1 Non-stationary Van der Pol oscillator

Consider the weakly nonlinear oscillator, described by the Van der Pol equation, for variable $y = y(t, \varepsilon)$ in t:

$$y'' + y - \varepsilon(1 - y^2)y' = 0$$

asymptotically for small positive parameter ε.

Construct by means of the method of multiple scales a first-order approximate solution. You are free to choose convenient (non-trivial) initial values.

7.2.2 The air-damped, unforced pendulum

For sufficiently high Reynolds numbers, the air-damped pendulum may be described by

$$m\,\ddot{\phi} + C\,\dot{\phi}\,|\,\dot{\phi}\,| + K\sin\phi = 0, \quad \phi(0) = \varepsilon, \;\; \dot{\phi}\,(0) = 0,$$

where $\varepsilon > 0$ is small and problem parameters m, K and C are positive. Assume that $C/m = O(\varepsilon)$. Use the method of multiple scales to get an asymptotic approximation of $\phi = \phi(t, \varepsilon)$ for $\varepsilon \to 0$.

7.2.3 The air-damped pendulum, harmonically forced near resonance

When an oscillator of resonance frequency ω_0 is excited harmonically, with a frequency ω near ω_0, the resulting steady state amplitude may be much larger than the forcing amplitude. Nonlinear effects may be called into action and limit the amplitude, which otherwise (in the linear model) would have been unbounded at resonance. In the following we will study an air-damped oscillator with harmonic forcing near resonance. The chosen parameter values are such that the resulting amplitude is just large enough to be bounded by the nonlinear damping.

a) Consider the damped harmonic oscillator with harmonic forcing

$$m\,\ddot{\phi} + K\phi = F\cos(\omega t).$$

Parameters m, K and F are positive. Find the steady state solution, *i.e.* the solution harmonically varying with frequency ω.

b) Consider the air-damped version

$$m\,\ddot{\phi} + C\,\dot{\phi}\,|\,\dot{\phi}\,| + K\phi = F\cos(\omega t),$$

where problem parameters m, K, C and F are positive. C and F are small in a way that $F = \varepsilon K$ and $C = \varepsilon m\beta$ where ε is small. The resonance frequency of the undamped linearised problem is $\omega_0 = \sqrt{K/m}$, while $\omega/\omega_0 = \Omega = 1 + \varepsilon\sigma$ with detuning parameter $\sigma = O(1)$. We are interested in the (bounded) steady state, and initial conditions are unimportant. Use similar techniques as used with the methods of multiple scales and Lindstedt-Poincaré to get an asymptotic approximation of $\phi = \phi(t)$ for $\varepsilon \to 0$.
Hint: make t dimensionless by $\tau = \omega t$. Write the leading order solution in the form $\phi_0 = A_0\cos(\tau - \tau_0)$ and find A_0 as a function of σ from ϕ_1. Note that $A_0(\sigma) \to 0$ if $\sigma \to \pm\infty$, which is in agreement with (a).

c) The same problem as above but with a nonlinear restoring force $K \sin\phi$, *i.e.*

$$m\,\ddot{\phi} + C\,\dot{\phi}\,|\,\dot{\phi}\,| + K\sin\phi = F\cos(\omega t),$$

while we now choose $F = \varepsilon^3 K$, $C = \varepsilon m\beta$ and $\omega/\omega_0 = \Omega = 1 + \varepsilon^2\sigma$. Note that we have to rescale ϕ.
The main difference with (b) is that A_0 cannot be expressed explicitly in σ, but if we plot σ as a function of A_0, we can recognise the physical solutions that satisfy $A_0(\sigma) \to 0$ if $\sigma \to \pm\infty$.

7.2.4 Homogenisation as a Multiple Scales problem

Consider a slow flow (like groundwater) or diffusion of matter in a medium with a fine local structure, of which the properties (porosity etc.) vary slowly on a larger scale. Usually we are eventually interested in the large scale behaviour. In this case it makes sense to separate the small and large scales, and see if the effect of the small scale behaviour can be represented by a large scale medium property, by way of a local averaging process of the small scale medium properties. This approach is called *homogenisation*, and can be considered as an application of the method of multiple scales.

Take the following model problem of diffusion of a concentration u in a medium with property a, driven by an external source f.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{a(x,\varepsilon)}\frac{\mathrm{d}}{\mathrm{d}x}u(x,\varepsilon)\right) = f(x).$$

a varies quickly with a slowly varying averaged value in a way to be described below. Introduce the fast variable t, such that $x = \varepsilon t$. Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{a(t;\varepsilon)}\frac{\mathrm{d}}{\mathrm{d}t}u(t,\varepsilon)\right) = \varepsilon^2 f(\varepsilon t)$$

Suppose

$$a(t,\varepsilon) = \alpha(x) + \beta(t;x)$$

such that

$$\int^t \beta(\tau,x)\,\mathrm{d}\tau = \text{integrable for } t\to\infty.$$

For the moment we start with assuming α is constant and $\beta = \beta(t)$. Assume the existence of the regular (= uniform Poincaré) asymptotic expansion in the <u>independent</u> variables t and x

$$u(t,\varepsilon) = U(t,x,\varepsilon) = U_0(t,x) + \varepsilon U_1(t,x) + \varepsilon^2 U_2(t,x) + \ldots$$

Regularity implies a uniform asymptotic sequence of the terms, so $U_0, U_1, U_2, \cdots = O(1)$ for $x \leqslant O(1)$ and $t \leqslant O(1/\varepsilon)$.

Note: usually this is not uniform on an interval with boundary conditions. At the ends we will have boundary layers $x = O(\varepsilon)$. These will be ignored here.

Derive the following homogenised equation in the slow variable only

$$U_0''(x) = \alpha f(x).$$

Indicate how to proceed for higher orders.

7.2.5 The non-linear pendulum with slowly varying length

Consider a pendulum, moving in the (x, y)-plane, of a mass m that is connected to a hinge at $(0, 0)$ by an idealised massless rod of length L, which is varied slowly in time (slow compared to the typical frequency of the fixed-length system). Denote by θ the angle between the rod and the vertical.

At time t the position, velocity and acceleration of the mass are given by

$$x(t) = L\sin\theta, \quad x'(t) = L\theta'\cos\theta + L'\sin\theta,\ x''(t) = L\theta''\cos\theta + 2L'\theta'\cos\theta - L\theta'^2\sin\theta + L''\sin\theta$$
$$y(t) = -L\cos\theta,\ y'(t) = L\theta'\sin\theta - L'\cos\theta,\ y''(t) = L\theta''\sin\theta + 2L'\theta'\sin\theta + L\theta'^2\cos\theta - L''\cos\theta$$

The balancing forces are then inertia, equal to m times the acceleration, gravity gm in downward y-direction, and a reaction force mR in the direction of the rod. If we regroup the forces in tangential and longitudinal direction and divide by m, we obtain the equations

$$L\theta'' + 2L'\theta' + g\sin\theta = 0$$
$$L'' - L\theta'^2 - g\cos\theta = R$$

For now, we are only interested in θ as function of time when L is given.

a) Assume that L is of the order of L_0, θ is of the order of $\sqrt{\varepsilon}$, where ε is small, and equal to the ratio between the inherent time scale of the pendulum $\sqrt{L_0/g}$ and the typical time scale of the variations of L, say λ. In other words:

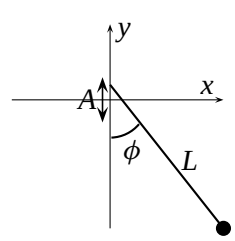
$$L = L\left(\frac{t}{\lambda}\right), \qquad \varepsilon = \frac{\sqrt{L_0/g}}{\lambda}$$

Make the problem dimensionless, scale the variables in an appropriate way, and expand the equations up to and including terms of $O(\theta^3)$.

b) Solve for $\theta = \theta(t)$ asymptotically for small ε by the WKB method.

7.2.6 Kapitza's Pendulum

Denote the vertical axis as y and the horizontal axis as x so that the motion of the pendulum happens in the (x, y)-plane. The following notation will be used: ω and A are the driving frequency and amplitude of the vertical oscillations of the suspension, g is the acceleration of gravity, L is the length of the rigid and light pendulum, m is the mass of the bob and $\omega_0 = \sqrt{g/L}$ is the frequency of the free pendulum.



Denoting the angle between pendulum and downward direction as ϕ, the position $x = \xi$, $y = \eta$ of the pendulum at time t is

$$\xi(t) = L\sin\phi$$
$$\eta(t) = -L\cos\phi - A\cos\omega t$$

The potential energy of the pendulum due to gravity is defined by its vertical position as

$$E_{\text{pot}} = -mg(L\cos\phi + A\cos\omega t)$$

The kinetic energy in addition to the standard term $\frac{1}{2}mL^2\,\dot{\phi}^2$ describing the velocity of a mathematical pendulum is the contribution due to the vibrations of the suspension

$$E_{\text{kin}} = \tfrac{1}{2}mL^2\,\dot{\phi}^2 + mAL\omega\sin(\omega t)\sin(\phi)\,\dot{\phi} + \tfrac{1}{2}mA^2\omega^2\sin^2(\omega t)$$

The total energy is then $E = E_{\text{kin}} + E_{\text{pot}}$ and the Lagrangian is $\mathcal{L}(t,\phi,\dot{\phi}) = E_{\text{pot}} - E_{\text{kin}}$. The motion of the pendulum satisfies the Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \frac{\partial\mathcal{L}}{\partial\phi}$$

which is

$$\ddot{\phi} = -L^{-1}(g + A\omega^2\cos\omega t)\sin\phi.$$

Assume that the driving amplitude A is small compared to L and frequency ω is large compared to the free frequency ω_0, in such a way that $A\omega/L\omega_0 = O(1)$. We make this explicit by writing $\varepsilon = \omega_0/\omega$ and $A/L = \varepsilon\mu$. If we rescale $\tau = \omega t$, we obtain

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} = -(\varepsilon^2 + \varepsilon\mu\cos\tau)\sin\phi.$$

From the structure of the equation we may infer that $\phi = \phi(\tau, T, \varepsilon)$ has a fast time scale τ and a slow time scale $T = \varepsilon\tau$. Finish the analysis by assuming that ϕ can be written as the sum of a slowly varying large part and a fast varying small part

$$\phi(\tau, T, \varepsilon) = \phi_0(T) + \varepsilon\phi_1(\tau, T) + \varepsilon^2\phi_2(\tau, T) + \ldots$$

Apply a consistency condition for ϕ_2 being bounded for $\tau \to \infty$. Find an equation for ϕ_0 and an expression for ϕ_1. Under what condition on μ are there two stationary solutions ϕ_0? Try to analyse the stability in $\phi_0 = \pi$, the inverted pendulum.

7.2.7 Rays in a slowly varying atmosphere

Waves (sound or light; the mathematics is the same) in a varying medium with wave velocity $c(\boldsymbol{x})$ is described by the wave equation

$$\nabla\cdot(c^2(\boldsymbol{x})\nabla\phi) - \frac{\partial^2\phi}{\partial t^2} = 0$$

for a suitable quantity (like pressure or potential) ϕ. We consider only time-harmonic, plane waves in the scalar variable x. This makes it a 1D problem. We write

$$\phi(x,t) = \mathrm{Re}(\hat{\phi}(x)\,\mathrm{e}^{\mathrm{i}\omega t})$$

while we assume that the medium varies over a typical length scale L, such that

$$c(x) = c_0\gamma\left(\frac{x}{L}\right)$$

where we assume that γ is a smooth function.

a. The waves have a typical wave length $\lambda = 2\pi/k$, where wave number $k = \omega/c_0$. Make the problem dimensionless on the wave length scale, *i.e.* on c_0/ω and introduce the dimensionless parameter $\varepsilon = c_0/\omega L$. Since we assumed that the medium varies slowly, *i.e.* on the scale of the wave (there is no other scale in the problem), this parameter ε is small. We see that there are essentially two length scales in the problem. For notational convenience we retain x and ϕ and write $\phi = \phi(x, \varepsilon)$.
b. Approximate γ for small ε. Can you solve the resulting equation?
c. On what x interval (in order of magnitude) will this approximation be valid?
d. In order to incorporate the slow and fast length scales, we assume the so-called WKB Ansatz

$$\phi(x, \varepsilon) = A(\varepsilon x)\, \mathrm{e}^{-\mathrm{i}\varepsilon^{-1}\theta(\varepsilon x)} .$$

Substitute ϕ in the equation, and expand to $O(1)$ and $O(\varepsilon)$. Can you solve θ and A?

Chapter 8

Some Mathematical Auxiliaries

8.1 Phase plane

Phase portrait and phase plots

A differentiable function $\phi(t)$, defined on some (not necessarily finite) interval $t \in [a, b]$, can be portrayed by the parametric curve (x, y) in $\mathbb{R}^2$, where $x = \phi(t)$ and $y = \phi'(t)$. This curve is called a phase portrait or phase plot of ϕ, and the (ϕ, ϕ')-plane is called a phase plane.

Phase plots are particularly useful if ϕ is defined by a differential equation from which relations between ϕ and ϕ' can be obtained, but exact solutions are not or not easily found.

Important examples are

$$\phi(t) = A\cos(\omega t), \quad \phi'(t) = -\omega A\sin(\omega t), \quad \text{with} \quad \omega^2\phi^2 + \phi'^2 = \omega^2 A^2,$$

leading to an ellipse as phase plot. A variant is

$$\phi(t) = A\,\mathrm{e}^{-ct}\cos(\omega t), \quad \phi'(t) = \sqrt{(\omega^2 + c^2)}\,A\,\mathrm{e}^{-ct}\sin(\omega t - \arctan(\omega/c) - \tfrac{1}{2}\pi)$$

leading to an elliptic spiral, converging to the origin if $c > 0$ and diverging to infinity if $c < 0$.

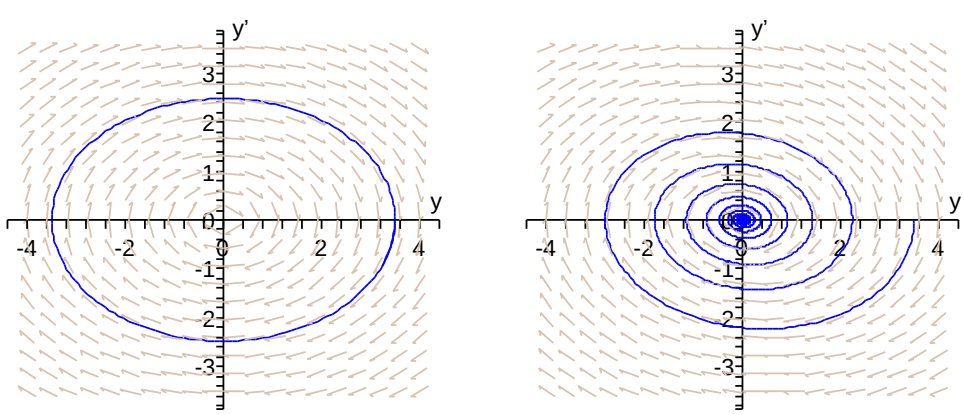


Figure 8.1: Elliptic (periodic) and spiral (damped) phase plots.

Phase plot to illustrate the solutions of differential equation

A differential equation like the harmonic equation

$$y'' + \omega^2 y = 0$$

is simple enough to be solved exactly by $y(t) = A\cos(\omega t) + B\sin(\omega t)$, leading to periodic (circular or elliptic) phase plots (see above). More difficult, in particular nonlinear, differential equations cannot be solved exactly, and solutions have to be found (in general) numerically. The plot of a single solution, however, does not tell us much about the whole family of all possible solutions. In such a case it is instructive to create a phase plot. Take for example the Van der Pol equation

$$y'' + y - \varepsilon(1 - y^2)y' = 0.$$

For small enough $\|(y, y')\|$, the nonlinear term is on average negative and acts as a source leading to an increase. For large enough $\|(y, y')\|$, the nonlinear term is on average positive and acts as a sink leading to a decay. From outside inwards and from inside outwards, these solutions converge to a periodic solution with (for small ε) an amplitude of about 2.

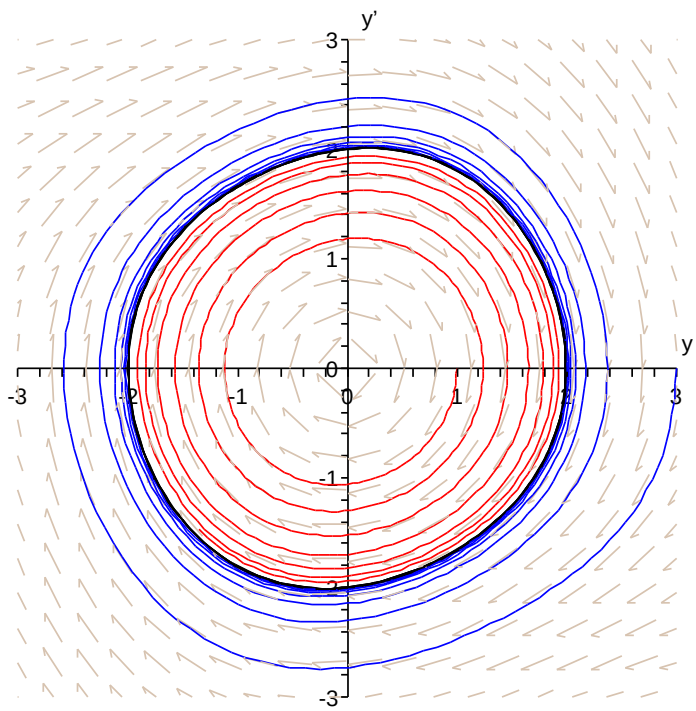


Figure 8.2: A phase plot of the Van der Pol equation, with $\varepsilon = 0.1$ and solutions starting form $y' = 0$ with $y = 1$ (red) and $y = 3$ (blue), respectively.

Stability of stationary solutions

One of the most important applications of the phase plot is the stability analysis of stationary solutions of 2nd order autonomous ordinary differential equations. Consider the equation

$$y'' = F(y, y'),$$

then we can rewrite this as a system by identifying $x_1 = y$ and $x_2 = y'$ with

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ F(x_1, x_2) \end{bmatrix}.$$

If the system has stationary solutions, they satisfy $x_2 = 0$ and $F(x_1, 0) = 0$. Assume a stationary solution $(x_1, x_2) = (X_0, 0)$. Consider perturbation around it of the form $x_1 = X_0 + \xi$, $x_2 = \eta$, where $\|(\xi, \eta)\|$ is small. Then after linearisation

$$F(x_1, x_2) = a\xi + b\eta + \ldots, \quad a = \frac{\partial}{\partial x}F(X_0, 0), \quad b = \frac{\partial}{\partial y}F(X_0, 0),$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} \eta \\ a\xi + b\eta \end{bmatrix} + \cdots = \begin{bmatrix} 0 & 1 \\ a & b \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix} + \ldots$$

The matrix has (possibly complex) eigenvalues

$$\lambda_{1,2} = \tfrac{1}{2}b \pm \sqrt{a + \tfrac{1}{4}b^2}.$$

The solutions of the linearised system are typically a linear combination of $\mathrm{e}^{\lambda_1 t}$ and $\mathrm{e}^{\lambda_2 t}$. Depending on the signs of $\lambda_{1,2}$, this results in local behaviour in the phase plane of ellipses (neutrally stable), converging spirals (stable) or diverging spirals (unstable).

Van der Pol's transformation

An interesting class of problems is the nonlinear oscillator

$$y'' + k^2 y + \varepsilon y' g(y, y') = 0.$$

With $g(y, y') = y^2 - 1$ is the Van der Pol equation a famous example. After transformation $t := kt$ and $x_1 = y$, $x_2 = y'$ we have

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - \varepsilon x_2\, g(x_1, x_2). \end{aligned}$$

Considerable progress can be made if we write the solution in polar coordinates of the phase plane:

$$x_1 = r\sin\varphi, \quad x_2 = r\cos\varphi.$$

This leads to

$$\begin{aligned} \dot{r}\,\sin\varphi + r\,\dot{\varphi}\,\cos\varphi &= r\cos\varphi \\ \dot{r}\,\cos\varphi - r\,\dot{\varphi}\,\sin\varphi &= -r\sin\varphi - \varepsilon r\cos\varphi\,\tilde{g}(r, \varphi). \end{aligned}$$

After eliminating r and φ we have

$$\begin{aligned} \dot{r} &= -\varepsilon r\cos^2\varphi\,\tilde{g}(r, \varphi) \\ \dot{\varphi} &= 1 + \varepsilon\sin\varphi\cos\varphi\,\tilde{g}(r, \varphi). \end{aligned}$$

Since $\varepsilon r\cos^2\varphi \geqslant 0$, the growth ($\dot{r} > 0$) or decay ($\dot{r} < 0$) of the solution depends entirely on the sign of $\tilde{g}$. A consequence is that if $\tilde{g}$ is positive for large r and negative for small r (like the Van der Pol equation), the expanding and contracting phase plots, not being able to cross each other, have to result in (at least) one closed contour (a so-called limit cycle), *i.e.* a periodic solution.

8.2 Newton's equation

An interesting equation that we encounter rather often is Newton's equation

$$y'' + V'(y) = 0, \quad y(0) = y_0, \quad y'(0) = y_1,$$

where V (in mechanical context a potential) is a sufficiently smooth given function of y. The interesting aspect is that the equation does not depend on y' and therefore can be integrated to

$$\tfrac{1}{2}(y')^2 + V(y) = E = \tfrac{1}{2}y_1^2 + V(y_0),$$

with integration constant E. In mechanical context this relation amounts to conservation of total energy E, being the sum of kinetic energy $\frac{1}{2}(y')^2$ and potential energy $V(y)$.

Note that this relation between y and y' is sufficient to construct phase plots for various values of E. For those values of E, where these phase plots correspond to closed curves, we know in advance that the corresponding solutions are periodic, which is important information.

We can eliminate y' and obtain

$$y' = \pm\sqrt{2}\,\sqrt{E - V(y)}.$$

Furthermore, we can even determine y implicitly formally

$$\int_{y_0}^{y} \frac{1}{\sqrt{E - V(s)}}\,\mathrm{d}s = \pm\sqrt{2}\,t$$

and with some luck we can integrate this integral explicitly. Note that a full integration may depend on the value of E.

A simple but important example is

$$y'' + k^2 y = 0$$

with ellipses in the phase plane described by

$$\tfrac{1}{2}(y')^2 + \tfrac{1}{2}k^2y^2 = E$$

leading to

$$\int_{y_0}^{y} \frac{1}{\sqrt{E - \frac{1}{2}k^2s^2}}\,\mathrm{d}s = \left[\frac{\sqrt{2}}{k}\arctan\left(\frac{\frac{1}{2}\sqrt{2}\,ks}{\sqrt{E - \frac{1}{2}k^2s^2}}\right)\right]_{y_0}^{y} = \pm\sqrt{2}\,t.$$

The integral describes one period ($+$ for one half and $-$ for another half), that can be extended. Hence we obtain the expected $y = y_0\cos(kt) + y_1k^{-1}\sin(kt)$.

Another, less trivial example is

$$y'' + y - y^3 = 0$$

with

$$\tfrac{1}{2}(y')^2 + \tfrac{1}{2}y^2 - \tfrac{1}{4}y^4 = E.$$

Elementary analysis reveals that this relation yields in the phase plane closed curves around the origin if $0 < E \leqslant \frac{1}{4}$. Hence, there are periodic solutions for those values of E.

8.3 Normal vectors of level surfaces

A convenient way to describe a smooth surface $\mathcal{S}$ is by means of a suitable smooth function $S(\boldsymbol{x})$, where $\boldsymbol{x} = (x, y, z)$, chosen such that the level surface $S(\boldsymbol{x}) = 0$ coincides with $\mathcal{S}$. So $S(\boldsymbol{x}) = 0$ if and only if $\boldsymbol{x} \in \mathcal{S}$. (Example: $x^2 + y^2 + z^2 - R^2 = 0$ for a sphere; $z - h(x, y) = 0$ for a landscape.)

Then for closely located points $\boldsymbol{x}, \boldsymbol{x} + \boldsymbol{h} \in \mathcal{S}$ we have

$$S(\boldsymbol{x} + \boldsymbol{h}) = S(\boldsymbol{x}) + \boldsymbol{h} \cdot \nabla S(\boldsymbol{x}) + O(\boldsymbol{h}^2) \simeq \boldsymbol{h} \cdot \nabla S(\boldsymbol{x}) = 0.$$

Since $\boldsymbol{h}$ is (for $\boldsymbol{h} \to 0$) a tangent vector of $\mathcal{S}$, it follows that ∇S at $S = 0$ is a normal of $\mathcal{S}$ (provided $\nabla S \neq 0$). We write $\boldsymbol{n} \sim \nabla S\big|_{S=0}$.

8.4 Trigonometric relations

The real or imaginary parts of the binomial series $(\mathrm{e}^{\mathrm{i}x} \pm \mathrm{e}^{-\mathrm{i}x})^n = \sum_{k=0}^{n} (\pm)^k \binom{n}{k} \mathrm{e}^{\mathrm{i}(n-2k)x}$ easily yield trigonometric relations, useful for recognising resonance terms:

$$\begin{aligned}
\sin^2 x &= \tfrac{1}{2}\big(1 - \cos 2x\big),\\
\sin\ x\ \cos\ x &= \tfrac{1}{2}\sin 2x,\\
\cos^2 x &= \tfrac{1}{2}\big(1 + \cos 2x\big),\\[2ex]
\sin^3 x &= \tfrac{1}{4}\big(3\sin x - \sin 3x\big),\\
\sin^2 x\ \cos\ x &= \tfrac{1}{4}\big(\cos x - \cos 3x\big),\\
\sin\ x\ \cos^2 x &= \tfrac{1}{4}\big(\sin x + \sin 3x\big),\\
\cos^3 x &= \tfrac{1}{4}\big(3\cos x + \cos 3x\big),\\[2ex]
\sin^4 x &= \tfrac{1}{8}\big(3 - 4\cos 2x + \cos 4x\big),\\
\sin^3 x\ \cos\ x &= \tfrac{1}{8}\big(2\sin 2x - \sin 4x\big),\\
\sin^2 x\ \cos^2 x &= \tfrac{1}{8}\big(1 - \cos 4x\big),\\
\sin\ x\ \cos^3 x &= \tfrac{1}{8}\big(2\sin 2x + \sin 4x\big).\\
\cos^4 x &= \tfrac{1}{8}\big(3 + 4\cos 2x + \cos 4x\big),\\[2ex]
\sin^5 x &= \tfrac{1}{16}\big(10\sin x - 5\sin 3x + \sin 5x\big),\\
\sin^4 x\ \cos\ x &= \tfrac{1}{16}\big(2\cos x - 3\cos 3x + \cos 5x\big),\\
\sin^3 x\ \cos^2 x &= \tfrac{1}{16}\big(2\sin x + \sin 3x - \sin 5x\big),\\
\sin^2 x\ \cos^3 x &= \tfrac{1}{16}\big(2\cos x - \cos 3x - \cos 5x\big),\\
\sin\ x\ \cos^4 x &= \tfrac{1}{16}\big(2\sin x + 3\sin 3x + \sin 5x\big),\\
\cos^5 x &= \tfrac{1}{16}\big(10\cos x + 5\cos 3x + \cos 5x\big).
\end{aligned}$$

8.5 Units, dimensions and dimensionless numbers

Basic units			
Name	Symbol	Physical quantity	Unit
meter	m	length	m
kilogram	kg	mass	kg
second	s	time	s
ampere	A	electric current	A
kelvin	K	temperature	K
candela	cd	luminous intensity	cd
mole	mol	amount of substance	1
hertz	Hz	frequency	$1/\mathrm{s}$
newton	N	force, weight	$\mathrm{kg\,m/s^2}$
pascal	Pa	pressure, stress	$\mathrm{N/m^2}$
joule	J	energy, work, heat	$\mathrm{N\,m}$
watt	W	power	$\mathrm{J/s}$
radian	rad	planar angle	1
steradian	sr	solid angle	1
coulomb	C	electric charge	$\mathrm{A\,s}$
volt	V	electric potential	$\mathrm{kg\,m^2/s^3A}$
ohm	Ω	electric resistance	$\mathrm{kg\,m^2/s^3A^2}$
siemens	S	electric conductance	$1/\Omega$
lumen	lm	luminous flux	$\mathrm{cd\,sr}$
lux	lx	illuminance	$\mathrm{lm/m^2}$

Basic variables			
Quantity	Relation	Unit	Dimensions
stress	force/area	$\mathrm{N/m^2 = Pa}$	$\mathrm{kg\,m^{-1}\,s^{-2}}$
pressure	force/area	$\mathrm{N/m^2 = Pa}$	$\mathrm{kg\,m^{-1}\,s^{-2}}$
Young's modulus	stress/strain	$\mathrm{N/m^2 = Pa}$	$\mathrm{kg\,m^{-1}\,s^{-2}}$
Lamé parameters λ and μ	stress/strain	$\mathrm{N/m^2 = Pa}$	$\mathrm{kg\,m^{-1}\,s^{-2}}$
strain	displacement/length	1	1
Poisson's ratio	transverse strain/axial strain	1	1
density	mass/volume	$\mathrm{kg/m^3}$	$\mathrm{kg\,m^{-3}}$
velocity	length/time	$\mathrm{m/s}$	$\mathrm{m\,s^{-1}}$
acceleration	velocity/time	$\mathrm{m/s^2}$	$\mathrm{m\,s^{-2}}$
(linear) momentum	mass $\times$ velocity	$\mathrm{kg\,m/s}$	$\mathrm{kg\,m\,s^{-1}}$
force	momentum/time	N	$\mathrm{kg\,m\,s^{-2}}$
impulse	force $\times$ time	$\mathrm{N\,s}$	$\mathrm{kg\,m\,s^{-1}}$
angular momentum	distance $\times$ mass $\times$ velocity	$\mathrm{kg\,m^2/s}$	$\mathrm{kg\,m^2\,s^{-1}}$
moment (of a force)	distance $\times$ force	$\mathrm{N\,m}$	$\mathrm{kg\,m^2\,s^{-2}}$
work	force $\times$ distance	$\mathrm{N\,m = J}$	$\mathrm{kg\,m^2\,s^{-2}}$
heat	work	J	$\mathrm{kg\,m^2\,s^{-2}}$
energy	work	$\mathrm{N\,m = J}$	$\mathrm{kg\,m^2\,s^{-2}}$
power	work/time, energy/time	$\mathrm{J/s = W}$	$\mathrm{kg\,m^2\,s^{-3}}$
heat flux	heat rate/area	$\mathrm{W/m^2}$	$\mathrm{kg\,s^{-3}}$
heat capacity	heat change/temperature change	$\mathrm{J/K}$	$\mathrm{kg\,m^2\,s^{-2}\,K^{-1}}$
specific heat capacity	heat capacity/unit mass	$\mathrm{J/K\,kg}$	$\mathrm{m^2\,s^{-2}\,K^{-1}}$
thermal conductivity	heat flux/temperature gradient	$\mathrm{W/m\,K}$	$\mathrm{kg\,m\,s^{-3}\,K^{-1}}$
dynamic viscosity	shear stress/velocity gradient	$\mathrm{kg/m\,s}$	$\mathrm{kg\,m^{-1}\,s^{-1}}$
kinematic viscosity	dynamic viscosity/density	$\mathrm{m^2/s}$	$\mathrm{m^2\,s^{-1}}$
surface tension	force/length	$\mathrm{N/m}$	$\mathrm{kg\,s^{-2}}$

Dimensionless numbers			
Name	Symbol	Definition	Description
Archimedes	Ar	$g\Delta\rho L^3/\rho\nu^2$	particles, drops or bubbles
Arrhenius	Arr	E/RT	chemical reactions
Biot	Bi	hL/κ	heat transfer at surface of body
Biot	Bi	$h_D L/D$	mass transfer
Bodenstein	Bo	VL/D_{ax}	mass transfer with axial dispersion
Bond	Bo	$\rho g L^2/\sigma$	gravity against surface tension
Capillary	Ca	$\mu V/\sigma$	viscous forces against surface tension
Dean	De	$(VL/\nu)(L/2r)^{1/2}$	flow in curved channels
Eckert	Ec	$V^2/C_P\Delta T$	kinetic energy against enthalpy difference
Euler	Eu	$\Delta p/\rho V^2$	pressure resistance
Fourier	Fo	$\alpha t/L^2$	heat conduction
Fourier	Fo	Dt/L^2	diffusion
Froude	Fr	$V/(gL)^{1/2}$	gravity waves
Galileo	Ga	$gL^3\rho^2/\mu^2$	gravity against viscous forces
Grashof	Gr	$\beta\Delta T g L^3/\nu^3$	natural convection
Helmholtz	He	$\omega L/c = kL$	acoustic wave number
Kapitza	Ka	$g\mu^4/\rho\sigma^3$	film flow
Knudsen	Kn	λ/L	low density flow
Lewis	Le	α/D	combined heat and mass transfer
Mach	M	V/c	compressible flow
Nusselt	Nu	hL/κ	convective heat transfer
Ohnesorge	Oh	$\mu/(\rho L\sigma)^{1/2}$	viscous forces, inertia and surface tension
Péclet	Pe	VL/α	forced convection heat transfer
Péclet	Pe	VL/D	forced convection mass transfer
Prandtl	Pr	$\nu/\alpha = C_P\mu/\kappa$	convective heat transfer
Rayleigh	Ra	$\beta\Delta T g L^3/\alpha\nu$	natural convection heat transfer
Reynolds	Re	$\rho VL/\mu$	viscous forces against inertia
Schmidt	Sc	ν/D	convective mass transfer
Sherwood	Sh	$h_D L/D$	convective mass transfer
Stanton	St	$h/\rho C_P V$	forced convection heat transfer
Stanton	St	h_D/V	forced convection mass transfer
Stokes	S	ν/fL^2	viscous damping in unsteady flow
Strouhal	Sr	fL/V	hydrodynamic wave number
Weber	We	$\rho V^2 L/\sigma$	film flow, bubble formation, droplet breakup

Nomenclature		
Symbol	Description	Units
c	sound speed	m/s
C_P	specific heat	J/kg K
D	diffusion coefficient	m^2/s
Dax	axial dispersion coefficient	m^2/s
E	activation energy	J/mol
f	frequency	1/s
g	gravitational acceleration	m/s^2
h	heat transfer coefficient	W/m^2 K
h_D	mass transfer coefficient	m/s
k	wave number = ω/c	1/m
L	length	m
p, Δp	pressure	Pa
R	universal gas constant	J/mol K
r	radius of curvature	m
T, ΔT	temperature	K
t	time	s
V	velocity	m/s
$\alpha = \kappa/\rho C_P$	thermal diffusivity	m^2/s
β	coef. of thermal expansion	K^{-1}
κ	thermal conductivity	W/m K
λ	molecular mean free path	m
μ	dynamic viscosity	Pa s
$\nu = \mu/\rho$	kinematic viscosity	m^2/s
ρ, $\Delta\rho$	density	kg/m^3
σ	surface tension	N/m
ω	circular frequency = $2\pi f$	1/s

8.6 Quotes

1. *The little things are infinitely the most important.* (Sherlock Holmes.)
2. *Entia non sunt multiplicanda praeter necessitatem = Entities should not be multiplied beyond necessity ≈ Other things being equal, simpler explanations are generally better than more complex ones.* (W. Ockham.)
3. *Formulas are wiser than man.* (J. de Graaf.)
4. *Nothing is as practical as a good theory.* (J.R. Oppenheimer.)
5. *An approximate answer to the right question is worth a great deal more than a precise answer to the wrong question.* (J. Tukey.)
6. *An exact solution of an approximate model is not better than an approximate solution of an exact model.* (section 2.)
7. *Never make a calculation until you know the answer: make an estimate before every calculation, try a simple physical argument (symmetry! invariance! conservation!) before every derivation, guess the answer to every puzzle.* (J.A. Wheeler.)
8. *The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas, like the colours or the words must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics.* (G.H. Hardy.)
9. *Divide each difficulty into as many parts as is feasible and necessary to resolve it.* (R. Descartes.)
10. *You make experiments and I make theories. Do you know the difference? A theory is something nobody believes, except the person who made it. An experiment is something everybody believes, except the person who made it.* (A. Einstein.)
11. *It is the theory which decides what we can observe.* (A. Einstein.)
12. *As far as the laws of mathematics refer to reality, they are not certain, as far as they are certain, they do not refer to reality.* (A. Einstein.)
13. *Science is nothing without generalisations. Detached and ill-assorted facts are only raw material, and in the absence of a theoretical solvent, have but little nutritive value.* (Lord Rayleigh)
14. *We need vigour, not rigour!* (anonym.)
15. *It is the nature of all greatness not to be exact.* (E. Burke.)
16. *The capacity to learn is a gift; The ability to learn is a skill; The willingness to learn is a choice.* (F. Herbert.)

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