Generating function approach (continued)

The boundary value method approach discussed on page 5 relies on

1) being able to chose an "appropriate" set of zeros of the kernel (Step 1).
2) For the zeros choosen in Step 1 on page 5, we need to show that the right hand side of (1) is analytic in the interior of E, is continuous in E and then formulate a solvable boundary value problem, e.g. Hilbert Riemann-Hilbert, Dirichlet, etc.
3) Typically, the solvable boundary value problems are defined on the unit circle. Thus we need to obtain an expression for the mapping $\psi: E \to D$. For smooth (bounded) contour one may employ Theodorsen's procedure, see [Section I.4 and Section IV.1.3, 2].

The combination of the above ingredients makes this approach technically challenging and its succesful application cannot be guaranteed.

We will illustrate in the sequel two approaches that can be used, under a certain structure, to solve the functional equation (1).

Uniformization method

The global idea of the uniformization method is as follows: Uniformize the kernel equation $K(x,y)=0$ by introducing a uniformizing variable, say p, by writing

$$x := x(p) \quad \text{and} \quad y := y(p).$$

Consider $P(x(p),0)$ and $P(0,y(p))$ and show they are analytic in certain p-regions. Furthermore, for p such that $|x(p)|, |y(p)| < 1$ and $K(x(p),y(p))=0$, solve

$$A(x(p),y(p))\, K(x(p),0) + B(x(p),y(p))\, K(0,y(p)) + C(x(p),y(p))\, K(0,0) = 0.$$

Uniformization method example : 2×2 switch

Consider the 2×2 switch model as introduced in the previous lecture. The objective here is to solve the functional equation (2) using a uniformizing variable, p.

Step 1: Let $x := p$ and $y := y(p)$ such that $p\,y(p) - r(p, y(p)) = 0$. More concretely, note that the kernel $K(x,y) = xy - r(x,y)$ is a quadratic polynomial in both x and y. Thus, for a fixed $x := p$, there are two zero roots in y, say $y_1(p)$ and $y_2(p)$, with

$$|y_2(p)| < |p| < |y_1(p)| \quad , \text{ for } |p| \geq 1 \text{ and } p \neq 1$$

Step 2: For $x := p$ and $y := y_1(p)$, Equation (2) reduces to

$$0 = (y_1(p) - 1)\, r(p,0)\, P(p,0) + (p-1)\, r(0, y_1(p))\, P(0, y_1(p)) +$$
$$+ (p-1)(y_1(p)-1)\, r(0,0)\, P(0,0) \Rightarrow$$
$$-(p-1)\, r(0, y_1(p))\, P(0, y_1(p)) = (y_1(p)-1)\, r(p,0)\, P(p,0) + (p-1)(y_1(p)-1)\, r(0,0)\, P(0,0). \quad (6)$$

Note that for $p = 1$ the right hand side of (6) is obviously finite and $y_1(1) = \left(\frac{2-p}{p}\right)^2 > 1 \left(y_2(1) = 1\right)$. So $y_1(1)$ is a simple pole of $P(0, y_1(p))$. Hence, $P(0,y)$ can be analytically continued out from $|y| \leq 1$ into $\{y : 1 \leq |y| < y_1(1)\}$.

Performing the same analysis as before for the uniformizing variable $y := q$ and $x = x(q)$ such that $x(q)\,q - r(x(q), q) = 0$, with

$$|x_2(q)| < |q| < |x_1(q)| \quad \text{for } |q| \geq 1 \text{ and } q \neq 1.$$

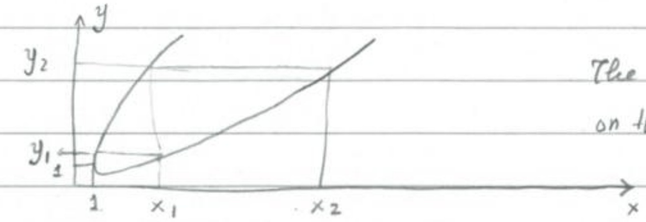
Furthermore, Equation (2) yields

$$0 = (q-1)\, r(x_1(q), 0)\, P(x_1(q), 0) + (x_1(q)-1)\, r(0,q)\, P(0,q) + (x_1(q)-1)(q-1)\, r(0,0)\, P(0,0) \Rightarrow$$
$$-(q-1)\, r(x_1(q), 0)\, P(x_1(q), 0) = (x_1(q)-1)\, r(0,q)\, P(0,q) + (x_1(q)-1)(q-1)\, r(0,0)\, P(0,0) \quad (7)$$

Note that for $q = y_1(1)$ the right hand side of (7) is bounded and thus $P(x,0)$ can be analytically continued out of $|x| \leq 1$ into $\{x : 1 \leq |x| < x_1(y_1(1))\}$.

This idea can be recursively applied back to (6)-(7) to further extend the analyticity of both $P(x,0)$ and $P(0,y)$ besides at the points in which these functions have simple poles. Note that in every finite domain in $\mathbb{C}$ the functions $P(x,0)$ and $P(0,y)$ have a finite number of poles $\Rightarrow$ $P(x,0)$ and $P(0,y)$ is meromorphic.

Say $y_1 = y_1(1)$, $x_1 = x_1(y_1)$, $y_2 = y_1(x_1)$, $x_2 = x_1(y_2)$, ... the sequence of simple poles we have obtained.



The sequences of simple poles $\{x_n\}_{n\geq 1}$, $\{(y_n)\}_{n\geq 1}$ on the parabola defined by $xy - r(x,y) = 0$.

Now for $p \in \mathbb{C} \setminus \{(x_n)\}_{n\geq 1}$, Equation (6) can be written as

$$P(p,0) = -\frac{p-1}{y_1(p)-1}\,\frac{r(0,y_1(p))}{r(p,0)}\,P(0,y_1(p)) - (p-1)\,\frac{r(0,0)}{r(p,0)}\,P(0,0) \tag{8}$$

Note that there are p such that $r(p,0)=0$, thus $P(0,y_1(p))=0$. We can easily see that $r(p,0)=0 \Rightarrow p = 2(1-\frac{1}{p})^{2}$ a double root. Let $y_1^* = y_1(2(1-\frac{1}{p}))$. Similarly to before we define the sequences $\{(x_n^*)\}_{n\geq 1}$, $\{(y_n)\}_{n\geq 1}$ for which $P(x_n^*,0)=0$ and $P(0,y_n^*)=0$, and x_n^*, y_n^* are double roots.

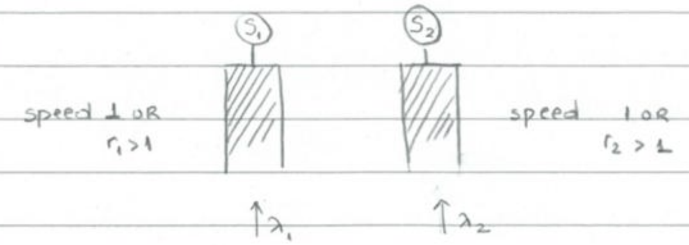
Based on all above information

$$P(x,0) = P(1,0)\,\frac{\prod_{n=1}^{\infty}(1-\frac{1}{x_n})}{\prod_{n=1}^{\infty}(1-\frac{x}{x_n})}\,\frac{\prod_{n=1}^{\infty}(1-\frac{x}{x_n^*})^2}{\prod_{n=1}^{\infty}(1-\frac{1}{x_n^*})^2}, \quad x \in \mathbb{C}.$$

The expression for $P(1,0)$ was obtained on page 6.

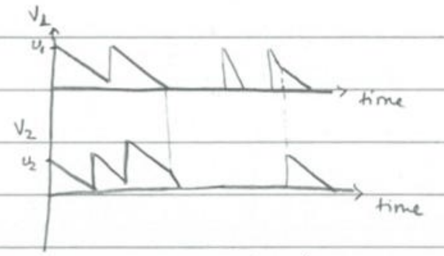
Similarly, we obtain $P(0,y)$, from that we can reconstruct $P(x,y)$.

Uniformization method example : Coupled processors.



Consider two M/G/1 queues with arrival rates λ_1 and λ_2, general service distribution B_1, and B_2, coupled via speeds. The speed of server i is 1 if the other queue is not empty and r_i if the other queue is empty, $r_i > 1$, $i = 1,2$.

Let V_i denote the workload at processor i, $i = 1,2$.



Let $\Psi(s_1, s_2) = \mathbb{E}\left[e^{-s_1 V_1 - s_2 V_2}\right]$, then we can produce the functional equation of the bi-variate Laplace transform, which reads as follows

$$\underbrace{\left(\lambda_1(1-\tilde{B}_1(s_1)) - s_1 + \lambda_2(1-\tilde{B}_2(s_2)) - s_2\right)}_{K(s_1,s_2)}\Psi(s_1,s_2) = -\left((r_1-1)s_1 + (r_2-1)s_2\right)\Psi(0,0) + \left((r_1-1)s_1 - s_2\right)\Psi(s_1,0) + \left((r_2-1)s_2 - s_1\right)\Psi(0,s_2) \qquad (9)$$

with $\tilde{B}_i(s_i) = \mathbb{E}\left(e^{-s_i B_i}\right)$, $i = 1,2$, and $s_i \in \mathbb{C}_+$ (i.e. $\mathrm{Re}\, s_i \geq 0$).

For more details on the model and the derivations for Equation (9) the interested reader is referred to [2], part III, Chapter III.3

Based on the approach we discussed in the previous lecture we can undertake two steps :

Step 1) Consider the zeros of the kernel $K(s_1, s_2) = 0$

Step 2) Solve the "boundary value problem".

For Step 1, notice that the kernel is seperable, i.e. $K(s_1, s_2) = f_1(s_1) + f(s_2)$, with $f_i(s_i) = \lambda_i(1 - \tilde{B}_i(s_i)) - s_i$. So

$$K(s_1, s_2) = f_1(s_1) + \omega + f_2(s_2) - \omega.$$

Uniformization method

Look for zeros $f_1(s_1) + \omega = f_2(s_2) - \omega = 0$. According to [7], under certain conditions $f_1(s_1) + \omega = 0$ has exactly one zero $s_1 = \delta_1(\omega)$ for $\mathrm{Re}\,\omega \geq 0$, $\omega \neq 0$, $\mathrm{Re}\,s_1 \geq 0$, with multiplicity one. Furthermore $\delta_1(\omega)$ is analytic in $\mathrm{Re}\,\omega > 0$ and continuous in $\mathrm{Re}\,\omega \geq 0$.

Similarly $f_1(s_1) - \omega = 0$ has exactly one zero $s_2 = \delta_2(\omega)$ for $\mathrm{Re}\,\omega \leq 0$, $\omega \neq 0$, $\mathrm{Re}\,s_1 \geq 0$ with multiplicity one. Furthermore $\delta_2(\omega)$ is analytic in $\mathrm{Re}\,\omega < 0$ and continuous in $\mathrm{Re}\,\omega \leq 0$.

Setting $s_i = \delta_i(\omega)$ in Equation (9) yields

$$0 = -\big((r_1-1)\delta_1(\omega) + (r_2-1)\delta_2(\omega)\big)\Psi(0,0) + \big((r_1-1)\delta_1(\omega) - \delta_2(\omega)\big)\Psi(\delta_1(\omega),0) + \\ + \big((r_2-1)\delta_2(\omega) - \delta(\omega)\big)\Psi(0,\delta_2(\omega)).$$

$$\Rightarrow \left(1-\frac{1}{r_1}\right)\delta_1(\omega) - \frac{1}{r_1}\delta_2(\omega)\left[\frac{1}{r_2}\big(\Psi(\delta_1(\omega),0) - \Psi(0,0)\big) - \frac{\Psi(0,0)}{r_1 r_2}\frac{1}{1-\frac{1}{r_1}-\frac{1}{r_2}}\right] =$$

$$= \left(1-\frac{1}{r_2}\right)\delta_2(\omega) - \frac{1}{r_2}\delta_1(\omega)\left[\frac{1}{r_1}\big(\Psi(0,\delta_2(\omega)) - \Psi(0,0) - \frac{\Psi(0,0)}{r_1 r_2}\frac{1}{1-\frac{1}{r_1}-\frac{1}{r_2}}\right] \qquad (10)$$

with $\frac{1}{r_1} + \frac{1}{r_2} \neq 1$.

A careful analysis, shows that the LHS of (10) is analytic in $\mathrm{Re}\,\omega > 0$ and continuous in $\mathrm{Re}\,\omega \geq 0$, while the RHS of (10) is analytic in $\mathrm{Re}\,\omega < 0$ and continuous in $\mathrm{Re}\,\omega \leq 0$.

Using now Liouville's Theorem, it is evident that both sides are constants in ω. Thus, this determines $\Psi(\delta_1(\omega),0)$ for $\mathrm{Re}\,\omega \geq 0$ and $\Psi(0,\delta_2(\omega))$ for $\mathrm{Re}\,\omega \leq 0$. Subsequently, we can determine $\Psi(s_1,0)$ for $\mathrm{Re}\,s_1 \geq 0$ and $\Psi(0,s_2)$ for $\mathrm{Re}\,s_2 \leq 0$. Finally, $\Psi(s_1,s_2)$ follows from (9).

References

[7] JW Cohen (1982) The Single Server Queue. North-Holland Publications.