In the previous lecture we showed that the 2×2 switch bi-variate generating function is meromorphic. This implies that the joint limiting distribution of the queue lengths can be written as

$$\pi_{n,m} = \sum_{i=0}^{\infty} c_i \alpha_i^{\,n} \beta_i^{\,m} \quad , \; n,m : n+m > N \qquad (11)$$

for some $N \in \mathbb{N}$ number, and $\pi_{n,m} = c_0 \alpha_0^{\,n} \beta_0^{\,m}$ when $n,m \to \infty$.

Note that $\frac{1}{\alpha_i}$ corresponds to x-poles and $\frac{1}{\beta_i}$ corresponds to y-poles of the bi-variate generating function

$$P(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \pi_{n,m} x^n y^m \quad , \; |x|, |y| \leq 1$$

However, it is not easy to retrieve (11) from the solution of the generating function. What we can instead do is use (11) "directy" on the balance equations so as to obtain an easy expression for the quantity of interest. This is achieved using the compensation approach.

Compensation approach

The compensation approach was developed by Ivo Adan [8] and aims at a direct solution for a sub-class of two-dimensional random walks in the quadrant:

i) Step size requirement: only transitions to neighboring states

ii) Forbidden steps requirement: no transitions in the interior to the North, North-East and East

iii) Homogeneity requirement: the same transitions occur in the interior, and similarly for the horizontal (respectively the vertical) boundary.

This approach exploits the fact that the balance equations in the interior are satisfied by linear (finite or infinite) combinations of product-forms, the parameters of which satisfy a kernel equation, and that need to be chosen such that the balance equations on the boundaries are satisfied as well. As it turns out, this can be done by alternatingly compensating for the errors on the two boundaries, which eventually may lead to an infinite series of product-forms.

Implementation steps of the compensation approach

Step 1: write the balance equations for the interior and substitute

$$\pi_{n,m} = \alpha^n \beta^m \qquad n,m > 0$$

in the balance equations. After simplifications the product-form satisfying the balance equations in the interior reveal a kernel equation

$$K(\alpha, \beta) = 0 \qquad (12)$$

which is quadratic in α and β. Equation (12) forms the basis; the points (α, β) on the curve characterize a continium of product-forms satisfying the inner equations.

Step 2 : construct a linear combination of elements in the basis, which is the formal solution to the balance equations. Here the word *formal* is used to indicate that (at this stage) we do not bother about the convergence of the solution. This aspect is treated in Step 3. The construction of a linear combination starts with a suitable initial term that satisfies the interior and the horizontal boundary. This choice reveals (typically) a unique choice for the α, say α_0. From (12) we choose the unique β, say β_0, that $|\alpha_0| > |\beta_0|$.

If this term also satisfies the balance equations for the vertical boundary then we have retrieved the limiting distribution. If this term does not satisfy the vertical boundary then we add another product-form term, say $\alpha_1^n \beta_0^m$, so as to compensate for the error, such that the sum of the two terms satisfies the balance equations on the vertical boundary. Adding the term $\alpha_1^n \beta_0^m$ may violate the balance equations on the horizontal boundary, in which case we would again add a product-form solution such that the sum of the three terms satisfies the horizontal boundary. Continue in this manner till the entire formal series is constructed.

Step 3 : show that the formal solution converges. This is split up into two parts

i) show that the sequences $\{\alpha_i\}_{i\in\mathbb{N}}$ and $\{\beta_i\}_{i\in\mathbb{N}}$ converge to zero exponentially fast

ii) show that the formal solution converges absolutely for all $n+m > N$.

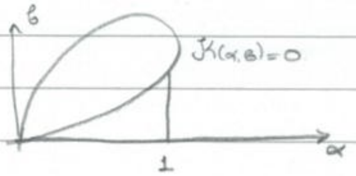
Step 4 : determine the normalization constant.

Remarks

- The kernel equation (12) is equivalent to the kernel equation obtained from the functional equation, see Step 1 of the generating function approach. In particular,

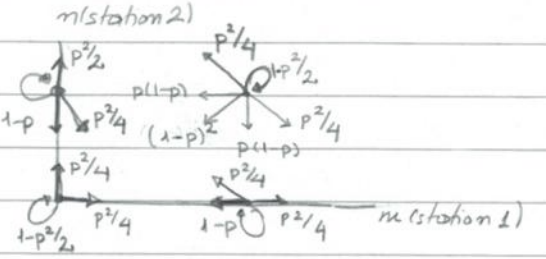
$$K(\alpha,\beta)=0 \iff K(x,y)=0 \quad \text{for } x=\frac{1}{\alpha} \text{ and } y=\frac{1}{\beta}$$

- In case of a meromorphic generating function, we show that $K(x,y)=0$ for $|x|,|y|>1$ is shaped like a parabola. Correspondingly, $K(\alpha,\beta)=0$ is a close contour that goes through $(0,0)$



Compensation approach example : 2x2 switch

Consider the 2x2 switch model as introduced in the previous lectures.



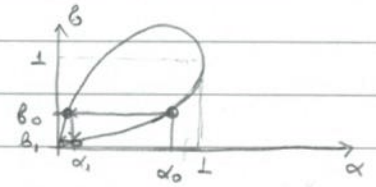
transition diagram with self-loops
($r_1 = r_2 = p$ and $t_{ij} = \frac{1}{2}$ $i,j = 1,2$)

Substituting $\pi_{m,n} = \alpha^m \beta^n$ in the balance equations in the interior $(m,n>0)$ after division by common powers yields

$$\left(1-\frac{p^2}{2}\right)\alpha\beta = \frac{p^2}{4}\beta^2 + \frac{p^2}{4}\alpha^2 + p(1-p)\alpha\beta^2 + p(1-p)\alpha^2\beta + (1-p)^2\alpha^2\beta^2 \qquad (13)$$

Check that this equivalent to $K(\frac{1}{\alpha}, \frac{1}{\beta}) = \frac{1}{\alpha\beta} - (1-p+\frac{p}{2}(\frac{1}{\alpha}+\frac{1}{\beta}))^2 = 0$.

Because in the last step we will re-nomalize, the terms α and β are required to satisfy $0<|\alpha|<1$ and $0<|\beta|<1$.



Substituting $\pi_{m,n} = \alpha^m \beta^n$ in the balance equations for the interior and the horizontal boundary, yields

$$\alpha_0 = \frac{p^2/4}{p^2/4 + p(1-p) + (1-p)^2} = \frac{p^2/4}{1 - 3p^2/4}$$

Then, from (13)

$$\beta_0 = \frac{p^2/4\ \alpha_0}{p^2/4 + p(1-p) + (1-p)^2\alpha_0}$$

Note that the term $\alpha_0^m \beta_0^n$ does not satisfy the vertical boundary. To compensate for this error we add $c_1 \alpha_1^m \beta_0^n$ (note that we only have the freedom to choose the α-term). We request $\alpha_0^m \beta_0^n + d_1 \alpha_1^m \beta_0^n$ satisfies the interior and the vertical boundary. This indicates that α_1 should be chosen as the companion root of α_0. Furthermore, the balance equation on the vertical boundary yields

$$d_1 = -\frac{1-\alpha_1}{1-\alpha_0}.$$

Note that the term $d_1 \alpha_1^m \beta_0^n$ does not satisfy the horizontal boundary, to this end we add one more product-form, say $c_1 \alpha_1^m \beta_1^n$. The coefficient can be obtained from the balance equations on the horizontal boundary

$$c_1 = -\frac{1-\beta_1}{1-\beta_0} d_1.$$

The new term violates the vertical boundary, so we keep on adding terms alternatingly. This results in the following formal solution

$$\alpha_0^m \beta_0^n + d_1 \alpha_1^m \beta_0^n + c_1 \alpha_1^m \beta_1^n + d_2 \alpha_2^m \beta_1^n + c_2 \alpha_2^m \beta_2^n + \cdots \qquad (14)$$

with

$$d_{i+1} = -\frac{(1-\alpha_{i+1})(1-\beta_i)}{(1-\alpha_0)(1-\beta_0)} \quad \text{and} \quad c_{i+1} = \frac{(1-\beta_{i+1})(1-\alpha_{i+1})}{(1-\alpha_0)(1-\beta_0)} \qquad i=0,1,\dots$$

All in all, (14) assumes the form

$$x_{m,n} = \frac{1}{(1-\alpha_0)(1-\beta_0)} \sum_{i=0}^{\infty} (1-\beta_i)\beta_i^n \left[(1-\alpha_i)\alpha_i^m - (1-\alpha_{i+1})\alpha_{i+1}^m \right] \qquad (15)$$

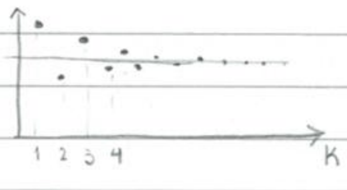
Next we check if we could have started from the vertical boundary, this will lead to a second formal expression starting from $(\tilde{\alpha}_0, \tilde{\beta}_0)$ symmetric to (15), say $\tilde{x}_{m,n}$.
Assume $\pi_{m,n} = c\, x_{m,n} + \tilde{c}\, \tilde{x}_{m,n}$ for $m+n>0$. Then, using the balance equations for states (0,1) and (1,0), together with the normalizing equation, yields

$$\pi_{m,n} = c_0 x_{m,n} + \tilde{c}_0 \tilde{x}_{m,n}, \quad m+n>0. \qquad (16)$$

with $c_0 = (1-\alpha_0)(1-\beta_0)$ and $\tilde{c}_0 = (1-\tilde{\alpha}_0)(1-\tilde{\beta}_0)$.

Remarks

- We can numerically use Equation (16) for the states far from the origin and then solve the small system of balance equations close to the origin so as to obtain an accurate solution.
- For states far from the origin we can approximate $\pi_{m,n}$ by including only few terms of the formal series expression.
- The formal series constitutes an asymptotic expansion. Assume we incorporate K terms of Equation (14), then the inclusion of the K+1-term improves the accuracy of the approximation.



Connection of previous approaches with matrix geometric

Assume the random walk framework of the compensation approach as presented on page 15. We show in this section how one can connect the generating function approach, the compensation approach and matrix geometric techniques.

Intermezzo

Consider a QBD with finite phases, say $m<\infty$. The stability condition (sufficient & necessary) of such a QBD can be obtained by the drift condition

$$\underline{x}' A_1 \mathbb{1} < \underline{x}' A_{-1} \mathbb{1},$$

with A_1, A_0, A_{-1} the transition matrices capturing the rates to a higher level, to the same level and to a lower level, respectively, and $\underline{x}'$ a row vector obtained as the unique solution $\underline{x}'(A_1 + A_0 + A_{-1}) = 0$, such that $\underline{x}'\mathbb{1} = 1$, and $\mathbb{1}$ a column vector of ones.

We know that the limiting distribution can be obtained as

$$\underline{\pi}_{n+1} = \underline{\pi}_n R, \quad n \geq 0,$$

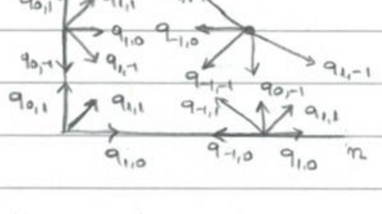
with $\underline{\pi}_n = (\pi_{n,0}, \pi_{n,1}, \ldots, \pi_{n,m})$. If the rate is diagonizable, then the limiting distribution is written as

$$\underline{\pi}_n = \sum_{k=1}^{m} \underline{h}_k \alpha_k^n, \quad n \geq 0$$

with $\underline{h}_k$ the eigenvectors of matrix R and α_k the corresponding right eigenvalue.
The above result is based on the analysis presented in [Section 1.6, 9].

We show here a natural extension of the above result in case of infinite dimension "diagonizable" matrix R.

Let



assumptions: $q_{1,0} + q_{1,1} + q_{0,1} > 0$ (out of state 0)

$q_{-1,-1} + q_{0,-1} + q_{1,-1} > 0$ (down/South)

$q_{-1,-1} + q_{-1,0} + q_{-1,1} > 0$. (left/West)

G^H be the generator in case n is the level and G^V be the generator in case m is the level.

Then the stochastic process is ergodic iff all four conditions hold

i) the MC $A^H = A_{-1}^H + A_0^H + A_1^H$ is ergodic, and let $\underline{x}^H$ be the unique solution of

$$\underline{x}^H A^H = 0 \quad \text{and} \quad \underline{x}^H \mathbb{1} = 1$$

ii)
$$\underline{x}^H A_1^H \mathbb{1} < \underline{x}^H A_{-1}^H \mathbb{1}$$

iii) the MC $A^V = A_{-1}^V + A_0^V + A_1^V$ is ergodic, and let $\underline{x}^V$ be the unique solution of

$$\underline{x}^V A^V = 0 \quad \text{and} \quad \underline{x}^V \mathbb{1} = 1.$$

iv) $\quad \underline{x}^{\nu} A_1^{\nu} \underline{\mathbb{1}} < \underline{x}^{\nu} A_{-1} \underline{\mathbb{1}}$

Let $\underline{\pi}_n = (\pi_{n,0}, \pi_{n,1}, \cdots)$, $n \geq 0$. It is known that

$$\underline{\pi}_{n+1} = \underline{\pi}_n R$$

and R is obtained as the minimal non-negative solution to the matrix quadratic equation

$$A_1{}^{H} + A_0{}^{H} R + A_{-1}{}^{H} R^2 = 0.$$

Instead of solving the above matrix quadratic equation we show how the eigenvalues and eigenvectors are connected with the product-form terms calculated using the compensation approach.

Procedure

Step 1: $\pi(x,y) = \sum x^n \underline{\pi}_n (1\ y\ y^2 \cdots)' = \underline{\pi}_0 (1\ y\ y^2 \cdots)' + \underline{\pi}_1 (x^{-1} I - R)^{-1} (1\ y\ y^2 \cdots)'$

Step 2: Substitute the above expression into the functional equation

$$K(x,y)\pi(x,y) + A(x,y)\pi(x,0) + B(x,y)\pi(0,y) + C(x,y)\pi(0,0) = 0$$

yields

$$\underline{\pi}_1 (x^{-1} I - R)^{-1} \left(K(x,y)(1\ y\ y^2 \cdots)' + A(x,y)(1\ 0\ 0 \cdots)' \right) \tag{17}$$
$$= -\underline{\pi}_0 \left((K(x,y) + B(x,y))(1\ y\ y^2 \cdots)' + (A(x,y) + C(x,y))(1\ 0\ 0 \cdots)' \right)$$

Note that $(x^{-1}I - R)^{-1}$ is a formal expression and should be interpreted as an operator and not as the inverse of a matrix.

Due to the fact that $\pi_{n,m} = c_0 \alpha_0^n \beta_0^m + \sum_{i=1}^{\infty} c_i \alpha_i^n (\beta_{i-1}^m + \beta_i^m \frac{d_i}{c_i})$ we can extend Equation (17) meromorphically on $\mathbb{C} \times \mathbb{C}$, except for the poles $\{\alpha_i\}$ and $\{\beta_i\}$.

Note that equivalently

$$\underline{\pi}_n = \sum_{i=0}^{\infty} \alpha_i^n \underline{h}_i$$

which immediately implies (using $\underline{\pi}_{n+1} = \underline{\pi}_n R$) that $\{\alpha_i\}$ are the eigenvalues of R and $\underline{h}_i$ the corresponding eigenvectors.

Step 3: Setting $x = \frac{1}{\alpha_0}$ in (17) we note that the right hand side of (19) is bounded which implies that $K(x,y)(1\ y\ y^2 \cdots) + A(x,y)(1\ 0\ 0 \cdots) = 0$. This reveals the starting solution α_0 with $|\alpha_0| < 1$ (note that $|x^{-1}| > 1$). Then, we can recursively build from $K(x,y) = 0$ the entire sequence of α's and β's.

Step 4: Calculation of the coefficients $\{c_i\}$ and $\{d_i\}$. This can be obtained by considering the functional equation or by considering the balance equations.

[8] IJBF Adan (1991). A Compensation Approach for Queuing problems. PhD dissertation, Eindhoven University of Technology.
[9] MF Neuts (1981). Matrix-Geometric Solutions in Stochastic Models: An Algorithmic Approach. The John Hopkins University Press.