Part 2

The kernel method for (reflected) 2-D random walks

Overview

1. Nice solutions
   - Lattice path counting
   - A tandem queue with priority
   - A weird example
2. Singularity analysis
   - Demonstration by example
3. Ugly solutions (dark side of the kernel method)
   - Boundary value problems
   - Asymptotics for rare events

Example from combinatorics

See Knuth (1973) or Prodinger (2004).

- Start from the origin. Move from (n, i) to $(n+1, i \pm 1)$, except in the case $i = 0$, when you can only go to $(n+1, 1)$.
- How many paths leads from the origin to (n, i)?
- Let the generating function $f_i(u)$ describe all walks leading to (n, i). Then $[u^n]f_i(u)$ represents the number of walks from $(0, 0)$ to (n, i).

We can see that

$$\begin{aligned} f_i(u) &= uf_{i-1}(u) + uf_{i+1}(u), \quad i \geq 1, \\ f_0(u) &= 1 + uf_1(u) \end{aligned}$$

and with $F(u, z) = \sum_{i \geq 0} f_i(u) z^i$ this gives

$$F(u, z) - f_0(u) = uzF(u, z) + \frac{u}{z}[F(u, z) - f_0(u) - zf_1(u)]$$

$$F(u,z) = uzF(u,z) + \frac{u}{z}[F(u,z) - F(u,0)] + 1,$$

or better:

$$F(u,z) = \frac{uF(u,0) - z}{uz^2 - z + u}.$$

The denominator vanishes for

$$z(u) = \frac{1 \pm \sqrt{1-4u^2}}{2u}.$$

The root $z_0(u) = \frac{1-\sqrt{1-4u^2}}{2u}$ is the bad one, so for this root the numerator should vanish as well:

$$uF(u,0) = z_0(u)$$

leading to an explicit representation of $F(u,z)$.

A tandem queue with coupled processors

- Customers arrive at queue 1 according to a Poisson process with rate λ
- Each customer requires a two-stage service with exponential service times with mean ν_1^{-1} and ν_2^{-1}
- The total service rate is constant, 1 say
- Queue 1 gets p and queue 2 gets $1 - p$ of the service rate
- If one of the queues is empty, the other queue gets service rate 1

Preliminaries

Let $N_1(t)$ and $N_2(t)$ denote the queue lengths at time t. Let

$$\mathbb{P}(N_1 = n, N_2 = k) = \lim_{t \to \infty} \mathbb{P}(N_1(t) = n, N_2(t) = k)$$

We then aim at determining the bivariate generating function

$$P(x, y) = \mathbb{E}(x^{N_1} y^{N_2}) = \sum_{n \geq 0} \sum_{k \geq 0} \mathbb{P}(N_1 = n, N_2 = k) x^n y^k$$

Key functional equation

$$h_1 P(x,y) = h_2 P(x,0) + h_3 P(0,y) + h_4 P(0,0)$$

where

$$\begin{aligned}
h_1(x,y) &= (\lambda + p\,\nu_1 + (1-p)\,\nu_2)xy - \lambda x^2 y - p\,\nu_1 y^2 - (1-p)\,\nu_2 x \\
h_2(x,y) &= (1-p)\,[\nu_1\, y(y-x) + \nu_2\, x(y-1)] \\
h_3(x,y) &= p\,[\nu_2\, x(1-y) + \nu_1\, y(x-y)] \\
h_4(x,y) &= p\,\nu_2\, x(y-1) + (1-p)\,\nu_1\, y(x-y)
\end{aligned}$$

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h_1(x,y) &= (\lambda + p\,\nu_1 + (1-p)\,\nu_2)xy - \lambda x^2 y - p\,\nu_1 y^2 - (1-p)\,\nu_2 x \\
h_2(x,y) &= (1-p)\,[\nu_1\, y(y-x) + \nu_2\, x(y-1)] \\
h_3(x,y) &= -p \cdot h_2(x,y)/(1-p) \\
h_4(x,y) &= \nu_2\, x(y-1) - h_2(x,y)
\end{aligned}$$

With

$$\gamma(y) = \nu_1 y^2 / (\nu_1 y - \nu_2 y + \nu_2)$$

we have $h_2(\gamma(y), y) = 0$ and hence

$$P(\gamma(y), y) = \frac{h_4(\gamma(y), y)}{h_1(\gamma(y), y)} P(0, 0)$$

Letting $y \uparrow 1$ yields

$$P(0, 0) = 1 - \frac{\lambda}{\nu_1} - \frac{\lambda}{\nu_2}$$

The ergodicity condition is therefore

$$\rho = \frac{\lambda}{\nu_1} + \frac{\lambda}{\nu_2} < 1$$

Kernel method

This is just one of various ways that zero-pairs (x, y) will let vanish parts of the functional equation

$$h_1 P(x, y) = h_2 P(x, 0) + h_3 P(0, y) + h_4 P(0, 0)$$

The function h_1 is referred to as *kernel*, and choosing zeropairs (x, y) such that $h_1(x, y) = 0$ is known as the *kernel method*

Priority for queue 1 ($p = 1$)

$$h_1 P(x,y) = h_2 P(x,0) + h_3 P(0,y) + h_4 P(0,0)$$

$$\begin{aligned}
h_1(x,y) &= (\lambda + p\,\nu_1 + (1-p)\,\nu_2)xy - \lambda x^2 y - p\,\nu_1 y^2 - (1-p)\,\nu_2 x \\
h_2(x,y) &= (1-p)\,[\nu_1\, y(y-x) + \nu_2\, x(y-1)] \\
h_3(x,y) &= p\,[\nu_2\, x(1-y) + \nu_1\, y(x-y)] \\
h_4(x,y) &= p\,\nu_2\, x(y-1) + (1-p)\,\nu_1\, y(x-y)
\end{aligned}$$

Priority for queue 1 ($p = 1$)

$$h_1 P(x, y) = h_3 P(0, y) + h_4 P(0, 0)$$

$$\begin{aligned} h_1(x, y) &= (\lambda + \nu_1)xy - \lambda x^2 y - \nu_1 y^2 \\ h_2(x, y) &= 0 \\ h_3(x, y) &= \nu_2 x(1 - y) + \nu_1 y(x - y) \\ h_4(x, y) &= \nu_2 x(y - 1) \end{aligned}$$

Priority for queue 1 ($p = 1$)

$$-y(\lambda x^2 - (\lambda + \nu_1)x + \nu_1 y) \cdot P(x, y) = h_3 P(0, y) + h_4 P(0, 0)$$

Now use

$$\xi(y) = \frac{\lambda + \nu_1 - \sqrt{(\lambda + \nu_1)^2 - 4\lambda\nu_1 y}}{2\lambda}$$

for which $h_1(\xi(y), y) = 0$. This yields

$$P(0, y) = -\frac{h_4(\xi(y), y)}{h_3(\xi(y), y)} P(0, 0)$$

and

$$P(x, y) = \frac{h_3(x, y)}{h_1(x, y)} P(0, y) + \frac{h_4(x, y)}{h_1(x, y)} P(0, 0)$$

The latter implies (with $\rho_1 = \lambda/\nu_1$)

$$P(x, 1) = \frac{1 - \rho_1}{1 - \rho_1 x} \quad \Rightarrow \quad \mathbb{P}(N_1 = n) = (1 - \rho_1)\rho_1^n$$

Priority for queue 2 ($p = 0$)

Again the functional equation greatly simplifies due to $h_3(x, y) = 0$. Then, for $\eta(x) = \nu_2/(\lambda + \nu_2 - \lambda x)$, we see that $h_1(x, \eta(x)) = 0$ and hence

$$P(x,0) = -\frac{h_4(x,\eta(x))P(0,0)}{h_2(x,\eta(x))}$$
$$= \frac{(\nu_1\nu_2 - \lambda\nu_1 x)(1-\rho)}{\lambda^2(x-x_*)(x-x^*)} = \frac{c_1}{x-x_*} + \frac{c_2}{x-x^*},$$

with

$$x^* = \frac{\lambda + \nu_1 + \nu_2 - \sqrt{(\lambda+\nu_1+\nu_2)^2 - 4\nu_1\nu_2}}{2\lambda}$$

and

$$c_1 = \frac{(\nu_1\nu_2 - \lambda\nu_1 x_*)(1-\rho)}{\lambda^2(x_* - x^*)}, \quad c_2 = \frac{(\nu_1\nu_2 - \lambda\nu_1 x^*)(1-\rho)}{\lambda^2(x_* - x^*)}.$$

Priority for queue 2 ($p = 0$)

This gives

$$P(x,1) = \frac{\nu_1}{\lambda x}\left[\frac{c_1}{x - x_*} + \frac{c_2}{x - x^*} - (1-\rho)\right]$$

and

$$\mathbb{P}(N_1 = n) \sim \frac{\nu_1^2 \lambda x^* - \nu_1^2 \nu_2}{\lambda^3 (x^* - x_*)(x^*)^2}(1-\rho)\left(\frac{1}{x^*}\right)^n.$$

A weird example

Consider the following random walk in the quarter plane

- In the interior of the state space, the walk steps $(1,0)$ w.p. $\frac{1-p}{3}$, $(0,-1)$ w.p. $\frac{1+p}{3}$ and $(-1,1)$ w.p. $\frac{1}{3}$.
- On the horizontal axis, the walk steps $(1,0)$ w.p. $\frac{1}{2}$ and $(-1,1)$ w.p. $\frac{1}{2}$.
- On the vertical axis, the walk steps $(1,0)$ w.p. $\frac{1-p}{2}$ and $(0,-1)$ w.p. $\frac{1+p}{2}$.

Aziz, Starobinski and Thiran (2008):

Theorem

This model is unstable for $p=0$ *and stable for* $p \in (0,1]$*.*

Denote the joint stationary probabilities by $\mathbb{P}(N_1 = n, N_2 = k)$ and let

$$P(x, y) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \mathbb{P}(N_1 = n, N_2 = k) x^n y^k$$

for which we have

$$h_1(x, y)P(x, y) = h_2(x, y)P(x, 0) + h_3(x, y)P(0, y) + h_4(x, y)P(0, 0), \tag{1}$$

with

$$\begin{aligned}
h_1(x, y) &= 6xy - 2(1-p)x^2y - 2y^2 - 2(1+p)x, \\
h_2(x, y) &= (1+2p)x^2y + y^2 - 2(1+p)x, \\
h_3(x, y) &= (1-p)x^2y - 2y^2 + (1+p)x, \\
h_4(x, y) &= (2+p)x^2y - y^2 - (1+p)x.
\end{aligned}$$

See FIM, Section 1.3, for a general description of how to derive such functional equations.

Denote the joint stationary probabilities by

$$\pi(n,k) = \mathbb{P}(N_1 = n, N_2 = k) = \lim_{t\to\infty} \mathbb{P}(N_1(t) = n, N_2(t) = k)$$

Theorem

For the case $p = 1$ the stationary distribution of the random walk has a closed-form solution with $\pi(0,0) = \pi(0,1) = (2-\sqrt{2})/6$, $\pi(1,0) = (\sqrt{2}-1)/3$ and

$$\pi(n,k) = \left(\frac{1}{\sqrt{2}}\right)^n \left(1 - \frac{1}{\sqrt{2}}\right)^{k+1}, \quad n,k \geq 1, \tag{2}$$

$$\pi(n,0) = \frac{2}{3}\left(1 - \frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)^n, \quad n \geq 2, \tag{3}$$

$$\pi(0,k) = \frac{1}{3}\left(1 - \frac{1}{\sqrt{2}}\right)^k, \quad k \geq 2. \tag{4}$$

Proof

For the case $p = 1$ the balance equations for $n \geq 2$ read

$$\pi(n,k) = \frac{1}{3}\pi(n+1,k-1) + \frac{2}{3}\pi(n,k+1), \quad k \geq 2, \tag{5}$$

$$\pi(n,1) = \frac{1}{2}\pi(n+1,0) + \frac{2}{3}\pi(n,2), \tag{6}$$

$$\pi(n,0) = \frac{2}{3}\pi(n,1) + \frac{1}{2}\pi(n-1,0). \tag{7}$$

First substitute a trial solution $\pi(n,k) = C \cdot \alpha^n \beta^k$ and $\pi(n,0) = C \cdot d \cdot \alpha^n \beta^k$ into (5) and (6), and divide by $\alpha^n \beta^{k-1}$ to obtain

$$\beta = \frac{1}{3}\alpha + \frac{2}{3}\beta^2, \tag{8}$$

$$\beta = \frac{1}{2}d\alpha + \frac{2}{3}\beta^2, \tag{9}$$

and hence $d = \frac{2}{3}$. Substituting the same trial solution into (7) yields (upon some rewriting)

$$\alpha = \alpha\beta + \frac{1}{2}. \tag{10}$$

Note that it follows from (10) and $\alpha < 1$ that $\beta < \frac{1}{2}$.

Combining these equations gives an equation for β

$$2\beta^3 - 5\beta^2 + 3\beta - \frac{1}{2} = 0 \tag{11}$$

with solutions $\frac{1}{2}(2-\sqrt{2}), \frac{1}{2}, \frac{1}{2}(2+\sqrt{2})$. Therefore, the values of α and β that lead to a convergent solution of the stationary distribution are given by

$$\alpha = \frac{1}{\sqrt{2}}, \quad \beta = 1 - \frac{1}{\sqrt{2}}. \tag{12}$$

We now need to match this trial solution with the remaining balance equations:

$$\pi(0,k) = \frac{1}{3}\pi(1,k-1) + \pi(0,k+1), \quad k \geq 2, \tag{13}$$

$$\pi(0,1) = \frac{1}{2}\pi(1,0) + \pi(0,2), \tag{14}$$

$$\pi(1,0) = \frac{2}{3}\pi(1,1) + \pi(0,0). \tag{15}$$

Substituting the trial solution $\pi(n, k) = C \cdot \gamma \cdot \beta^k$ into (13) yields $\gamma\beta = \gamma\beta^2 + \frac{1}{3}\alpha$ and hence (with α, β as in (12))

$$\gamma = \frac{\alpha}{3\beta(1-\beta)} = \frac{2}{6 - 3\sqrt{2}}.$$

Combining (14), (15) and $\pi(0,0) = \pi(0,1)$ yields $\pi(0,0) = \frac{1}{3}C$. Summing over all probabilities identifies the normalization constant as $C = 1 - \frac{1}{\sqrt{2}}$, which completes the proof.

Corollary

For the case $p = 1$ *we have the marginal distributions*

$$\mathbb{P}(N_1 = n) = \frac{7\sqrt{2} - 8}{6} \left(\frac{1}{\sqrt{2}} \right)^n, \quad n \geq 1, \tag{16}$$

$$\mathbb{P}(N_2 = k) = \left(\frac{1}{3} + \frac{1}{\sqrt{2}} \right) \left(1 - \frac{1}{\sqrt{2}} \right)^k, \quad k \geq 1, \tag{17}$$

$\mathbb{P}(N_1 = 0) = \frac{1}{3\sqrt{2}} \approx 0.2357$ *and* $\mathbb{P}(N_2 = 0) = \frac{2+\sqrt{2}}{6} \approx 0.5690$*.*

In terms of generating functions we thus find that

$$P(x,y) = \frac{2}{3}\frac{2-\sqrt{2}+(\sqrt{2}-1)(x+y)-(3-2\sqrt{2})xy}{(2-\sqrt{2}x)(2-(2-\sqrt{2})y)} \qquad (18)$$

and

$$P(x,0) = \frac{2-\sqrt{2}+(\sqrt{2}-1)x}{6-3\sqrt{2}x}, \quad P(0,y) = \frac{2-\sqrt{2}+(\sqrt{2}-1)y}{6-3(2-\sqrt{2})y},$$

and it is straightforward to check that these functions satisfy the functional equation (1). Starting from the functional equation (1) and deriving, in a direct way, (18) as its solution is an open problem and would be of interest from a methodological perspective. Who can tell me how to do this?

Intermezzo

The tandem queue with $p = 0, 1$ are typical examples of the *kernel method* as it is known in the field of combinatorics:

- Prodinger (2004), Pemantle & Wilson (2008), Flajolet & Sedgewick (2008), Bousquet-Melou (2000-2008) and many many more works

The kernel method has also a long history in two-queue models:

- *join-the-shortest-queue* Kingman (1961), *serve-the-longest-queue* Flatto (1989), *coupled processors* Fayolle & Iasnogorodski (1979)

These queueing models are among the most difficult random walks in the quarter plane and typically lead to a solution in terms of (Riemann-Hilbert) *boundary value problems*:

- Malyshev (1972, pioneering work), Cohen (1988, survey) and textbooks by Cohen & Boxma (1983), Fayolle, Iasnogorodski & Malyshev (1999), JvL (2005), JvL & Resing (2006), JvL & Guillemin (2009)

The tandem queue with $p \in (0,1)$ yields a random walk that requires the boundary value technique

The solution of $P(x,y)$ will be difficult and does not allow for explicit inversion

We therefore aim at deriving expressions of the type

$$\mathbb{P}(N_1 = n) \sim f(n) \cdot \zeta^{-n}$$

This requires:

1. A full solution of $P(x,y)$, and $P(x,1) = \sum_{n=0}^{\infty} \mathbb{P}(N_1 = n) x^n$
2. Determining the dominant singularity ζ of $P(x,1)$
3. Obtaining asymptotics using singularity analysis

Asymptotics for priority case

Change the notation (my sincere apologies!) according to $\nu_1 = \nu_2 = \mu_1 + \mu_2$ and $p = \mu_1/(\mu_1 + \mu_2)$ and assume $\lambda + \mu_1 + \mu_2 = 1$ The functional equation becomes

$$h_1(x,y)P(x,y) = h_2(x,y)P(x,0) + h_3(x,y)P(0,y) + h_4(x,y)P(0,0)$$

where

$$\begin{aligned} h_1(x,y) &= xy - \lambda x^2 y - \mu_1 y^2 - \mu_2 x, \\ h_2(x,y) &= \mu_2(y^2 - x), \\ h_3(x,y) &= \mu_1(x - y^2), \\ h_4(x,y) &= \mu_1 x(y-1) + \mu_2 y(x-y). \end{aligned}$$

and in case we give priority to station 1, $\mu_2 = 0$ and things simplify.

As earlier, we find that

$$P(x,y) = \frac{\rho_1 x(1-\xi(y)) + x - y}{(\rho_1+1)x - \rho_1 x^2 - y} P(0,y), \tag{19}$$

where $\rho_i = \lambda/\mu_i$ and

$$P(0,y) = \frac{(1-y)P(0,0)}{1-y-\rho_2 y(1-\xi(y))}, \tag{20}$$

($P(0,0) = 1-\rho_1-\rho_2$) and

$$\xi(y) = \frac{1+\rho_1}{2\rho_1}\left(1-\sqrt{1-4\rho_1 y/(1+\rho_1)^2}\right). \tag{21}$$

From this it follows that

$$P(1,y) = \frac{1-y+\rho_1(1-\xi(y))}{1-y} P(0,y). \tag{22}$$

The function $\xi(y)$ represents the pgf of the number of customers served in a busy period of an $M/M/1$ queue with arrival rate λ and service rate μ_1. Denote this random variable by ξ. Then:

$$\mathbb{P}(\xi = n) = \frac{1}{n}\binom{2n-2}{n-1}\frac{\rho_1^{n-1}}{(1+\rho_1)^{2n-1}}, \quad n = 1, 2, \dots. \tag{23}$$

Stirling's approximation $n! \sim n^n e^{-n}\sqrt{2\pi n}$ yields

$$\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{\pi n}}, \tag{24}$$

and thus

$$\begin{aligned}
\mathbb{P}(\xi = n) &\sim \frac{1}{n}\frac{2^{2n-2}}{\sqrt{\pi(n-1)}}\frac{1+\rho_1}{\rho_1}\frac{\rho_1^n}{(1+\rho_1)^{2n}} \\
&= \frac{1+\rho_1}{2\rho_1}\frac{1}{2\sqrt{\pi n^3}}\frac{\sqrt{n}}{\sqrt{n-1}}\left(\frac{4\rho_1}{(1+\rho_1)^2}\right)^n \\
&\sim \frac{1+\rho_1}{2\rho_1}\frac{1}{2\sqrt{\pi n^3}}\left(\frac{4\rho_1}{(1+\rho_1)^2}\right)^n.
\end{aligned} \tag{25}$$

We are primarily interested in $\mathbb{P}(N_2 = n)$ for n large, but we don't have an explicit inversion of the pgf. Therefore, we resort to singularity analysis.
For general α we have that

$$[z^n](1-z)^{-\alpha} = (-1)^n \binom{\alpha}{n} = \binom{n+\alpha-1}{n} = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}, \tag{26}$$

where $\Gamma(z)$ is the Gamma function defined for $\mathrm{Re}(z) > 0$ as

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \mathrm{d}t. \tag{27}$$

Applying Stirling's approximation $\Gamma(n+1) \sim n^n e^{-n} \sqrt{2\pi n}$ then gives (see e.g. Flajolet & Sedgewick 2009)

$$[z^n](1-z)^{-\alpha} = \frac{n^{\alpha-1}}{\Gamma(\alpha)} \Big(1 + \mathcal{O}(1/n)\Big). \tag{28}$$

We now apply the above result to $\xi(y)$. Denote by $\mathbb{C}_y$ the complex y-plane and observe that the function $\xi(y)$ is analytic in $\mathbb{C}_y\backslash[(1+\rho_1)^2/(4\rho_1),\infty)$, i.e. it has a branch point at

$$y_B = \frac{(1+\rho_1)^2}{4\rho_1}. \tag{29}$$

This particular case is covered by (28) with $\alpha = -1/2$, which gives

$$\begin{aligned}
\mathbb{P}(\xi = n) = [y^n]\xi(y) &= -\frac{1+\rho_1}{2\rho_1}[y^n]\sqrt{1-4\rho_1 y/(1+\rho_1)^2} \\
&= -\frac{1+\rho_1}{2\rho_1}\left(\frac{4\rho_1}{(1+\rho_1)^2}\right)^n [y^n]\sqrt{1-y} \\
&= -\frac{1+\rho_1}{2\rho_1}\left(\frac{4\rho_1}{(1+\rho_1)^2}\right)^n \frac{n^{-3/2}}{\Gamma(-1/2)}\Big(1+\mathcal{O}(1/n)\Big) \\
&= \frac{1+\rho_1}{2\rho_1}\left(\frac{4\rho_1}{(1+\rho_1)^2}\right)^n \frac{1}{2\sqrt{\pi n^3}}\Big(1+\mathcal{O}(1/n)\Big)
\end{aligned} \tag{30}$$

Note that (30) yields (25).

We now turn to the function $P(1, y)$ and study the asymptotic behavior of

$$[y^n]P(1, y) = \mathbb{P}(N_2 = n)$$

by means of singularity analysis. The singularities of $P(1, y)$ consist of the branch point y_B and zeros of the denominator of the right-hand side of (20):

$$1 - y - \rho_2 y(1 - \xi(y)) = 0. \tag{31}$$

The question then is which singularity has the smallest modulus, since the singularity of $P(1, y)$ with the smallest modulus is dominant and determines the asymptotic behavior of the coefficients of $P(1, y)$, i.e. $\mathbb{P}(N_2 = n)$, for large values of n. Because we already know that $P(1, y)$ has a branch point in y_B, it remains to be investigated whether $P(1, y)$ has a pole in $1 < |y| < y_B$, so whether (31) has a solution in $1 < |y| < y_B$.

First observation

Lemma

If

$$\rho_2 < \frac{2\rho_1(1-\rho_1)}{(1+\rho_1)^2} =: \rho_c, \tag{32}$$

then the only solution to (31) *in the region* $|y| < y_B$ *is given by* $y = 1$.

Proof Rouché's theorem. □

Candidate solutions

We seek for solutions to (31), or

$$1 - \frac{4\rho_1}{(1+\rho_1)^2} y = \left(\frac{2\rho_1}{1+\rho_1} \left(\frac{1}{\rho_2 y} - \frac{1}{\rho_2} - 1 \right) + 1 \right)^2 . \qquad (33)$$

The relevant solution is given by

$$y_P = \frac{\rho_2 - \rho_1 - \rho_1\rho_2 + \sqrt{4\rho_1\rho_2^2 + (\rho_2 - \rho_1 - \rho_1\rho_2)^2}}{2\rho_2^2} . \qquad (34)$$

Lemma

If $\rho_2 < \rho_c$, the dominant singularity of the function $P(1, y)$ is the branch point y_B. If $\rho_2 > \rho_c$, the dominant singularity of the function $P(1, y)$ is the pole y_P. If $\rho_2 = \rho_c$, the dominant singularity of the function $P(1, y)$ is $y_B = y_P$.

Lemma

$$P(1,y) \approx \begin{cases} P(1,y_B) + \gamma_1\sqrt{1-y/y_B}, & \rho_2 < \rho_c, \\ \gamma_2/\sqrt{1-y/y_B}, & \rho_2 = \rho_c, \\ \gamma_3/(1-y/y_P), & \rho_2 > \rho_c, \end{cases} \qquad (35)$$

where $P(1,y) \approx f(y)$ *indicates that* $P(1,y)/f(y) \to 1$ *when* y *tends to its dominant singularity* y_B *or* y_P*, and*

$$\begin{aligned} \gamma_1 &= -\frac{2P(0,0)\rho_1(1+\rho_1)(\rho_2+2\rho_1\rho_2+\rho_1^2(4+\rho_2))}{(\rho_2+\rho_1^2(2+\rho_2)-2(1-\rho_2)\rho_1)^2}, \\ \gamma_2 &= \frac{2P(0,0)\rho_1(1-\rho_1)}{\rho_2(1+\rho_1)^2}, \\ \gamma_3 &= \frac{P(0,0)}{y_P}\cdot\frac{1-y_P+\rho_1(1-\xi(y_P))}{-1-\rho_2(1-\xi(y_P))+\rho_2 y_P\xi'(y_P)}. \end{aligned}$$

Applying (28) for $\alpha = -1/2, 1/2$ and 1 then yields

Theorem

(a) If $\rho_2 < \rho_c$,

$$\mathbb{P}(N_2 = n) \sim \gamma_1 \frac{-1}{2\sqrt{\pi n^3}} \left(\frac{1}{y_B}\right)^n .$$

(b) If $\rho_2 = \rho_c$,

$$\mathbb{P}(N_2 = n) \sim \gamma_2 \frac{1}{2\sqrt{\pi n}} \left(\frac{1}{y_B}\right)^n .$$

(c) If $\rho_2 > \rho_c$,

$$\mathbb{P}(N_2 = n) \sim \gamma_3 \left(\frac{1}{y_P}\right)^n ,$$

Back to the tandem queue, and rare events

Methods for tail asymptotics

- Generating function methods: Malyshev 1972, 1973; Flatto and McKean 1977; Fayolle and Iasnogorodski 1979; Fayolle, King and Mitrani 1982; Cohen and Boxma 1983; Flatto and Hahn 1984; Flatto 1985; Fayolle, Iasnogorodski and Malyshev 1991; Wright 1992; Kurkova and Suhov 2003; JvL 2005; Morrison 2007; JvL-Guillemin 2009;
- Probabilistic methods: McDonald 1999; Borovkov and Mogul'skii 2001; Foley and McDonald 2001, 2005-2009, Miyazawa 2008-2009
- Matrix analytic methods: Takahashi, Fujimoto and Makimoto 2001; Haque 2003; Miyazawa 2004; Miyazawa and Zhao 2004; Kroese, Scheinhardt and Taylor 2004; Haque, Liu and Zhao 2005; Motyer and Taylor 2006; Li, Miyazawa and Zhao 2007; He, Li and Zhao 2008
- Combinatorics: Bousquet-Melou 2005-2009; Mishna 2006-2009; Hou and Mansour 2008, and many more...

Key functional equation

$$h_1 P(x,y) = h_2 P(x,0) + h_3 P(0,y) + h_4 P(0,0)$$

where

$$\begin{aligned}
h_1(x,y) &= (\lambda + p\,\nu_1 + (1-p)\,\nu_2)xy - \lambda x^2 y - p\,\nu_1 y^2 - (1-p)\,\nu_2 x \\
h_2(x,y) &= (1-p)\,[\nu_1\, y(y-x) + \nu_2\, x(y-1)] \\
h_3(x,y) &= -p \cdot h_2(x,y)/(1-p) \\
h_4(x,y) &= \nu_2\, x(y-1) - h_2(x,y)
\end{aligned}$$

A closer look at the kernel

We have that $h_1(X_{\pm}(y), y) = 0$ with

$$X_{\pm}(y) = \frac{1}{2y}\left((\hat{r}y - 1/r_2) \pm \sqrt{d_2(y)}\right)$$

where $\hat{r} = 1 + 1/r_1 + 1/r_2$, $r_1 = \lambda/(p\nu_1)$, $r_2 = \lambda/((1-p)\nu_2)$ and $d_2(y) = (\hat{r}y - 1/r_2)^2 - 4y^3/r_1$

- $d_2(y)$ has three roots in $\mathbb{R}$: $0 < y_1 < y_2 \leq 1 < y_3$
- $d_2(y) > 0$ for $y \in (-\infty, y_1) \cup (y_2, y_3)$
- $d_2(y) < 0$ for $y \in (y_1, y_2) \cup (y_3, \infty)$

Similarly, $h_1(x, Y_\pm(x)) = 0$ for

$$Y_\pm(x) = \frac{r_1}{2}\left((\hat{r} - x)x \pm \sqrt{d_1(x)}\right)$$

where $d_1(x) = ((\hat{r} - x)x)^2 - 4x/(r_1 r_2)$

- $d_1(x)$ has four real roots: $x_1 = 0 < x_2 \leq 1 < x_3 < x_4$
- $d_1(x) > 0$ for $x \in (-\infty, x_1) \cup (x_2, x_3) \cup (x_4, \infty)$
- $d_1(x) < 0$ for $x \in (x_1, x_2) \cup (x_3, x_4)$.

Analytic continuation

Lemma

The function $X^*(y)$ *defined in* $\mathbb{C} \setminus ([y_1, y_2] \cup [y_3, \infty))$ *by*

$$X^*(y) = \begin{cases} X_+(y) & \text{when } y \in \{z : \Re(z) \leq y_2, \Im(d_2(z^+)) < 0\} \cup (-\infty, y_1) \\ X_-(y) & \text{otherwise} \end{cases}$$

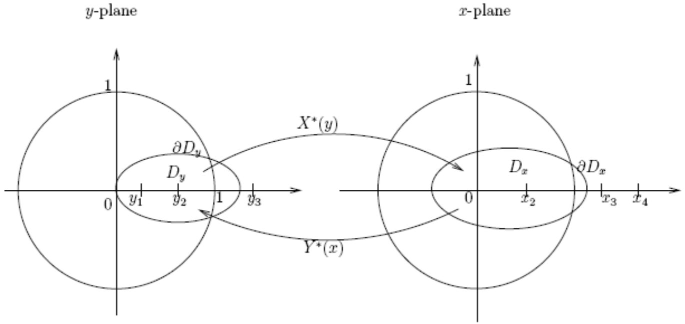
where $z^+ = \Re(z) + i|\Im(z)|$, *is analytic*

Lemma

The function $Y^*(x)$ *defined in* $\mathbb{C} \setminus ([x_1, x_2] \cup [x_3, x_4])$ *by*

$$Y^*(x) = \begin{cases} Y_+(x) & \text{when } x \in \{z : \Re(z) \leq x_2, \Im(d_1(z^+)) < 0\} \cup (-\infty, x_1) \\ Y_+(x) & \text{when } x \in \{z : \Re(z) \geq x_3, \Im(d_2(z^+)) > 0\} \cup (x_4, \infty) \\ Y_-(x) & \text{otherwise} \end{cases}$$

is analytic



Theorem

The function $X^*(y)$ *is a conformal mapping from* D_y *onto* D_x*.*
The reciprocal function is $Y^*(x)$

Lemma

We have $X^*(\partial D_y) \subset [x_1, x_2]$ *and* $Y^*(\partial D_x) \subset [y_1, y_2]$

Boundary value problem

When $h_1(x, y) = 0$ we have that

$$P(x, 0) = \frac{p}{1-p} P(0, y) - (1-\rho) \frac{h_4(x, y)}{h_2(x, y)}$$

and hence for $x \in \partial D_x$ and $y = Y^*(x)$

$$\Im\left(P(x, 0)\right) = \Im\left(- \frac{(1-\rho) h_4(x, y)}{h_2(x, y)} \right)$$

This is a classical Riemann-Hilbert problem. Simple calculations yield

$$\Im\left(\frac{h_4(x, y)}{h_2(x, y)} \right) = \frac{\nu_2 \lambda y (r_1 x^2 - y)}{2 i r_1 x (1-p) \mathcal{Q}_x(y)}$$

where $\mathcal{Q}_x(y) = \lambda \nu_1 y^2 + \nu_2(\nu_2 - \nu_1 + \lambda) y - \nu_2^2$

Theorem

The function $P(x,0)$ *is given by*

$$P(x,0) = \begin{cases} \frac{1}{2\pi i} \int_{\partial D_x} \frac{g_x(z)}{z-x} dz & \text{for} \quad x \in D_x, \\ g_x(x) + \frac{1}{2\pi i} \int_{\partial C_x} \frac{g_x(z)}{z-x} dz & \text{for} \quad x \in \mathbb{C} \setminus D_x, \end{cases} \tag{36}$$

where C_x *is a contour in* D_x *surrounding the slit* $[x_1, x_2]$ *and such that the function* g_x *given by*

$$g_x(x) = (1-\rho) \frac{\nu_2 Y^*(x)(p\nu_1 Y^*(x) - \lambda x^2)}{(1-p)x\mathcal{Q}_x(Y^*(x))} \tag{37}$$

The function $P(x,0)$ *is a meromorphic function in* $\mathbb{C} \setminus [x_3, x_4]$ *with singularities at the solutions to the equation* $\mathcal{Q}_x(Y^*(x)) = 0$ *if they exist*

Resultants

When $h_1(x, y) = 0$ we have that

$$P(x, 0) = \frac{p}{1-p} P(0, y) - (1-\rho) \frac{h_4(x, y)}{h_2(x, y)}$$

The common solutions of the equations $h_1(x, y) = 0$ and $h_2(x, y) = 0$ are then potential singularities for the function $P(x, 0)$

The resultant in x of the polynomials $h_1(x, y)$ and $h_2(x, y)$ is a polynomial of degree 5

$$Q_y(x) = -\nu_2 \nu_1 (1-p)^2 x^2 (x-1) \mathcal{Q}_y(x)$$

where $\mathcal{Q}_y(x) = \lambda^2 x^2 - (\lambda + \nu_1 + \nu_2)\lambda x + \nu_1 \nu_2$. This quadratic polynomial has two roots, one of which

$$x^* = \frac{\lambda + \nu_1 + \nu_2 - \sqrt{(\lambda + \nu_1 + \nu_2)^2 - 4\nu_1\nu_2}}{2\lambda} \in (1, x_3]$$

The resultant in y of the polynomials $h_1(x, y)$ and $h_2(x, y)$ is a polynomial of degree 5

$$Q_x(y) = -\nu_1(1-p)^2 y^2 (y-1)\mathcal{Q}_x(y)$$

where $\mathcal{Q}_x(y) = \lambda\nu_1 y^2 + \nu_2(\nu_2 - \nu_1 + \lambda)y - \nu_2^2$. This quadratic polynomial has two roots, one of which

$$y^* = \frac{\nu_2}{2\lambda\nu_1}\left(-(\nu_2 - \nu_1 + \lambda) + \sqrt{(\nu_2 - \nu_1 + \lambda)^2 + 4\lambda\nu_1}\right) \in (1, y_3]$$

Lemma

The equation $Q_x(Y^*(x)) = 0$ *has a solution in* $(-\infty, x_3]$, *which is necessarily equal to* $x^* \in (1, x_3]$, *if and only if* $y^* = Y^*(x^*)$

What were we doing again?

Let us return to

$$P(x,1) = \sum_{n=0}^{\infty} \mathbb{P}(N_1 = n) x^n$$

for which the key functional equation gives

$$P(x,1) = \nu_1 \frac{(1-p)P(x,0) - pP(0,1) - (1-p)(1-\rho)}{\lambda x - p\nu_1}$$

The dominant singularity of $P(x,1)$ will thus be one of the following three candidates:

1. $x = x_3$
2. $x = x^*$
3. $x = \frac{p\nu_1}{\lambda} = \frac{1}{r_1}$

Lemma

If $r_2 \leq 1$*, then*

$$(1-p)P\left(r_1^{-1},0\right) - pP(0,1) - (1-p)(1-\rho) = 0$$

and $1/r_1$ *is removable. If* $r_2 > 1$ *(and then* $r_1 \leq 1$ *by stability) we have*

$$(1-p)P\left(r_1^{-1},0\right) - pP(0,1) - (1-p)(1-\rho) < 0$$

and the point $1/r_1$ *is a singularity of* $P(x,1)$

Theorem

I. If $y^* = Y^*(x^*)$ *and* $x^* < x_3$*, which can occur only if* $r_1 \leq 1$*, then*

$$\mathbb{P}(N_1 = n) \sim \kappa_1^{(1)} \left(\frac{1}{x^*}\right)^n$$

II. If $y^* \neq Y^*(x^*)$ *and* $r_2 > 1$ *(and then* $r_1 \leq 1$*),*

$$\mathbb{P}(N_1 = n) \sim \kappa_2^{(1)} (r_1)^n$$

III. If $y^* \neq Y^*(x^*)$ *and* $r_2 \leq 1$*,* $1/r_1$ *is removable from* $P(x, 1)$ *and*

$$\mathbb{P}(N_1 = n) \sim \kappa_3^{(1)} \frac{1}{n\sqrt{n}} \left(\frac{1}{x_3}\right)^n$$

IV. If $y^* = Y^*(x^*)$ *and* $x^* = x_3$*,*

$$\mathbb{P}(N_1 = n) \sim \kappa_4^{(1)} \frac{1}{\sqrt{n}} \left(\frac{1}{x_3}\right)^n$$

where

$$\kappa_1^{(1)} = \frac{\nu_1\nu_2(1-\rho)((1-p)\nu_2 x^* - p\nu_1(y^*)^2)}{(\lambda x^* - p\nu_1)(\nu_2^2 + \lambda\nu_1(y^*)^2)x^*}$$

$$\kappa_2^{(1)} = P(0,1) + \frac{1-p}{p}\left(1-\rho-P(r_1^{-1},0)\right)$$

$$\kappa_3^{(1)} = \frac{(1-\rho)\lambda\nu_1\nu_2}{4\sqrt{\pi}(\lambda x_3 - p\nu_1)} \frac{\frac{\lambda^2(1-p)}{p\nu_2}x_3^2 + 2\lambda x_3 - (p\lambda+\nu_1)}{\mathcal{Q}_y(x_3)\mathcal{Q}_y^*(x_3)}\sqrt{x_3}\tau_x$$

$$\kappa_4^{(1)} = \frac{(1-\rho)\lambda\nu_1\nu_2}{2\sqrt{\pi}(\lambda x_3 - p\nu_1)} \frac{\frac{\lambda^2(1-p)}{p\nu_2}x_3^2 + 2\lambda x_3 - (p\lambda+\nu_1)}{\sqrt{x_3}\mathcal{Q}_y'(x_3)\mathcal{Q}_y^*(x_3)}\tau_x$$

with $\tau_x = \sqrt{(x_3-x_1)(x_3-x_2)(x_4-x_3)}$ and

$$\mathcal{Q}_y^*(x) = \frac{1}{\lambda^2}\left(x - \frac{p\nu_1 y^*}{x^*}\right)\left(x - \frac{p\nu_1 y_*}{x_*}\right).$$

Example Case I

Take as parameter values

$$\lambda = 1.1,\ \nu_1 = 6,\ \nu_2 = 9,\ p = 0.5,\ r_1 = 0.37,\ r_2 = 0.24,\ \rho = 0.31$$

for which

$$x^* = 4.3303,\ y^* = Y^*(x^*) = 1.6864,\ \kappa_1^{(1)} = 0.5392$$

and

n	$\mathbb{P}(N_1 = n)$	$\kappa_1^{(1)}(x^*)^{-n}$
10	2.3921e-007	2.3261e-007
20	1.0087e-013	1.0034e-013
50	8.0560e-033	8.0552e-033
100	1.2033e-064	1.2033e-064
200	2.6854e-128	2.6854e-128
300	5.9927e-192	5.9927e-192

Example Case II

Take as parameter values

$$\lambda = 1.1,\ \nu_1 = 6,\ \nu_2 = 2,\ p = 0.7,\ r_1 = 0.26,\ r_2 = 1.83,\ \rho = 0.73$$

for which

$$x^* = 1.4545,\ y^* = 1.3333 \neq Y^*(x^*) = 1.5584,\ \kappa_2^{(1)} = 0.4620$$

and

n	$\mathbb{P}(N_1 = n)$	$\kappa_2^{(1)}(r_1)^n$
10	7.9471e-007	7.0154e-007
20	1.1343e-012	1.0653e-012
50	3.7864e-030	3.7307e-030
100	3.0200e-059	3.0127e-059
200	1.9649e-117	1.9647e-117
300	1.2813e-175	1.2813e-175

- Similar results can be obtained for N_2
- The same technique applies to the general class of two-dimensional one-step random walks in the quarter plane
- Determining the dominant singularities could be done without resorting to the boundary value technique
- Many interesting and classical special cases