Degree Bounds for Constrained Pseudo-Triangulations

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Abstract

We introduce the concept of a constrained pointed pseudo-triangulation $\mathcal{T}_G$ of a point set S with respect to a pointed planar straight line graph $G = (S, E)$. For the case that G forms a simple polygon P with vertex set S we give tight bounds on the vertex degree of $\mathcal{T}_G$.

1 Introduction

Pseudo-triangulations, also called geodesic triangulations, are planar partitions that have received considerable attention during the last years due to their applications to visibility [8, 9], ray shooting [3, 4], kinetic collision detection [1, 6, 7], rigidity [12], and guarding [11].

A *pseudo-triangle* is a planar polygon that has exactly three convex vertices, called *corners*, with internal angles less than π. Three concave chains, called *side chains*, join the three corners. A *pseudo-triangulation* for a set S of n points in the plane is a partition of the convex hull of S into pseudo-triangles whose vertex set is exactly S. A vertex is called *pointed* if it has an adjacent angle greater than π. A planar straight line graph is pointed if every vertex is pointed.

Since a pseudo-triangulation is a planar graph, we can borrow graph terminology: The *degree of a vertex* is the number of edges incident to it. Even though a standard triangulation has average vertex degree $O(1)$, there are sets of n points in the plane for which each possible triangulation has a vertex of degree $n-1$. In contrast, Kettner et al. [5] recently established that every point set in the plane has a pointed pseudo-triangulation of maximum vertex degree five, and this bound is tight. A natural question to ask is whether a bound on the maximum vertex degree is also attainable if certain edges are constrained to be part of the pseudo-triangulation.

Assume that we are given a pointed planar straight line graph $G = (S, E)$. A *constrained pointed pseudo-triangulation* $\mathcal{T}_G$ of S with respect to G is a partition of the convex hull of S into pseudo-triangles such that each edge from E is part of the pseudo-triangulation and each vertex in S remains pointed[1]. The edges in E are called *constrained edges*. An edge $e \notin E$ connecting two points of S is called *admissible* with respect to G, if it does not properly cross any edge $e' \in E$ and if $E \cup \{e\}$ is pointed. From now on we will refer to a pointed pseudo-triangulation simply as pseudo-triangulation.

Streinu showed [12] that any pointed set of edges can be extended to a pseudo-triangulation by greedily inserting admissible edges. This implies that a constrained pseudo-triangulation $\mathcal{T}_G$ of S exists for every pointed planar straight line graph G and immediately leads to the following question: For which classes of graphs is the maximum vertex degree of a constrained pseudo-triangulation bounded by a constant?

Here we start to answer this question by giving tight degree bounds for constraining graphs that form a simple polygon P with vertex set S. We begin by describing a recursive construction of a pseudo-triangulation of the closed interior of P with maximum vertex degree five. In contrast, we show that the degree of the dual graph of any pseudo-triangulation of the interior of a simple polygon cannot be bounded by a constant (Section 2.4). Finally, in Section 3 the results are extended to construct a constrained pseudo-triangulation of S with maximum vertex degree seven.

2 The interior of a simple polygon

In this section we present a recursive construction to pseudo-triangulate the closed interior $\mathrm{Int}(P)$ of a simple polygon P. At each step we apply one of two operations to P to obtain polygons of smaller size. The first operation *prunes* a convex vertex u_c from P and either reduces the size of P by at least one vertex or splits P in two or more subpolygons. The second operation *wraps* a pseudo-triangle around a reflex vertex u_r and splits P into at least two subpolygons. The vertex u_r can reappear only as a convex vertex and in at most one of the resulting subpolygons. Throughout this paper we denote a convex vertex with the subscript c and a reflex vertex with the subscript r.

We define the load of a vertex v, denoted by $l(v)$, to be the degree of v minus its *constrained degree* where the constrained degree $d_G(v)$ of a vertex v denotes its degree in the constraining graph G. Note that degree always refers to the degree of a vertex with respect to the current version of the pseudo-triangulation that we are building. If G is a simple polygon then $d_G(v)$ equals two for all $v \in S$. We say that an edge (u, v) of P is *loaded* if both u and v have a load greater than zero.

We maintain the following invariants for all simple polygons P that arise as recursive subproblems:

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[1] Rote et al. [10] use the term constrained pseudo-triangulation to denote a pseudo-triangulation that is a subset of a given triangulation of a point set.

Invariant 1 *Exactly one edge of P is loaded.*

Invariant 2 *The loaded edge is of one (or more) of four types:*

- **I** (u_c, v_c) *with* $l(u_c) \leq 3$ *and* $l(v_c) \leq 2$.
- **II** (u_c, v_r) *with* $l(u_c) \leq 3$ *and* $l(v_r) \leq 1$.
- **III** (u_c, v_r) *with* $l(u_c) \leq 2$ *and* $l(v_r) \leq 2$.
- **IV** (u_r, v_r) *with* $l(u_r) \leq 2$ *and* $l(v_r) \leq 1$.

In order to establish the invariants for the initial polygon P, we pick an arbitrary edge of P and declare it loaded.

Let us now assume that we are given a simple polygon P that satisfies both invariants. The appropriate operation to choose depends on the type of the loaded edge e. We distinguish two cases: If e is of type I or II then we prune, if e is of type III or IV then we wrap.

The following sections explain the two operations in detail and illustrate how the invariants are maintained. First, we introduce some additional notation. Recall that a *geodesic path* between two points inside a simple polygon is the shortest path that connects them and stays completely within the polygon. The region bounded by the three geodesic paths connecting three vertices a, b, and c is called a *geodesic triangle* [3]. It consists of a central pseudo-triangle $T(a, b, c)$ and three (possibly empty) *tails* connecting the corners $t(a)$, $t(b)$, and $t(c)$ of $T(a, b, c)$ with the vertices a, b, and c (see Figure 1).

Each operation selects a specific pseudo-triangle $T(a, b, c) \subset P$ such that E and the edges of $T(a, b, c)$ form a pointed edge set. The set $P \setminus T(a, b, c)$ is possibly split into several (smaller) simple polygons which we process recursively. The side chains of $T(a, b, c)$ consist of constrained edges and diagonals. The load of every vertex that appears on a side chain is raised by either one or two. Furthermore, these vertices also belong to either one or two subproblems. Since we are interested in an upper bound on the maximal load of any vertex, we assume in the following that every vertex that appears on a side chain picks up a load of two and belongs to two subproblems.

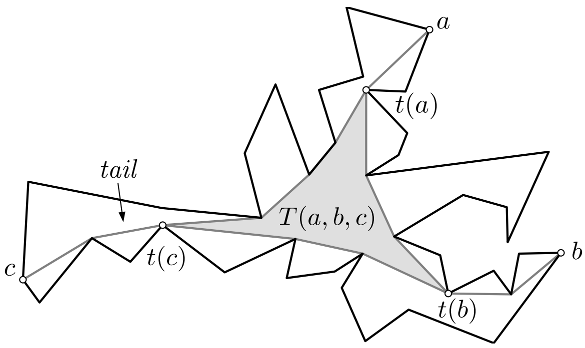
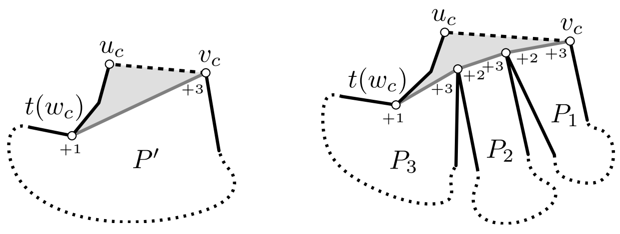


Figure 1: A geodesic triangle inside a simple polygon.

2.1 Prune. Assume that we are given a simple polygon P that satisfies both invariants and has a loaded edge of either type I or II.

Type I. P has a loaded edge (u_c, v_c) connecting two convex vertices with $l(u_c) \leq 3$ and $l(v_c) \leq 2$. Let w_c denote the first convex vertex we encounter when walking away from u_c on P not passing over v_c. We prune u_c by adding the pseudo-triangle $T(w_c, u_c, v_c)$ to our pseudo-triangulation. Note that in this case we always have $u_c = t(u_c)$ and $v_c = t(v_c)$.



(a) (b)

Figure 2: A loaded edge of type I: pruning u_c.

If the side chain connecting $t(w_c)$ to v_c consists of a single diagonal (see Figure 2(a)), then the polygon P' formed by pruning u_c from P has exactly one loaded edge of either type I (if $t(w_c)$ is convex) or type II (if $t(w_c)$ is reflex).

If the side chain connecting $t(w_c)$ to v_c consists of several diagonals, then each polygon P_i formed by pruning u_c from P has exactly one loaded edge (see Figure 2(b)). Each vertex that newly appears on the side chain belongs to two subpolygons and is convex in both of them. To ensure that their total load will not exceed three, we consider each of these vertices to have load two in one subpolygon and load three in the other one. We alternate the loads assigned to each of these vertices as depicted in Figure 2(b) (the numbers inside each subpolygon indicate the load). This guarantees that P_1 (the subpolygon containing v_c) has a loaded edge of type I, P_k (the subpolygon containing $t(w_c)$) has a loaded edge of either type I (if $t(w_c)$ is convex) or type II (if $t(w_c)$ is reflex), and each P_i, $1 < i < k$, has a loaded edge of type I.

Type II. The prune operation performed in this case (cf. Figure 3) is similar to the one described above for loaded edges of type I. More details can be found in the full paper.

2.2 Wrap. Assume that we are given a simple polygon P that satisfies both invariants and has a loaded edge e of either type III or IV. The operation described in this section wraps a pseudo-triangle around the loaded edge. In doing so it might happen that both endpoints of e are completely covered and do not appear in any subpolygon. Since we are interested in an upper bound on the vertex degree, we will consider only the case where both endpoints do appear.

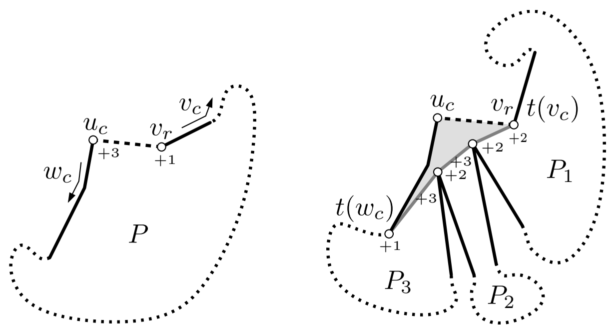
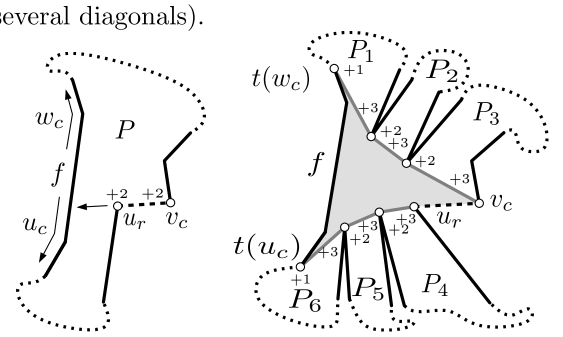


Figure 3: A loaded edge of type II: pruning u_c.

Type III. P has a loaded edge $e = (u_c, v_r)$ connecting a convex vertex with a reflex vertex. The load of both u_c and v_r is at most two. We shoot a ray from v_c along e passing over u_r and denote by f the first polygon edge hit by the ray (Figure 4(a)). Let w_c denote the first convex vertex we encounter when walking from f along P in clockwise direction, and let u_c denote the first convex vertex we encounter when walking from f along P in counterclockwise direction. We wrap the edge e by adding the pseudo-triangle $T(u_c, v_c, w_c)$ to our pseudo-triangulation (Figure 4(b)).

The load distribution on the side chains connecting $t(w_c)$ to v_c and $t(u_c)$ to u_r is again very similar to the one described above for loaded edges of type I. We kindly ask the reader to refer to Figure 4(b) for details (we only depict the case where the side chains consist of several diagonals).



(a) (b)

Figure 4: A loaded edge of type III: wrapping a pseudo-triangle around $e = (u_r, v_c)$.

Type IV. The wrap operation performed in this case is similar to the one described above for loaded edges of type III. The details can be found in the full paper.

Theorem 1 *For any simple polygon P there is a pointed pseudo-triangulation of $Int(P)$ such that every convex vertex has maximum load two and every reflex vertex has maximum load three.*

Proof: In each step of our recursive construction we add a pseudo-triangle T to our pseudo-triangulation and add to the load of all vertices of T. If T contains a convex vertex w_c that has not previously been part of the pseudo-triangulation, then w_c has to be a corner of T. Sections 2.1 and 2.2 show that in this case the load of w_c after the insertion of T is one. But the three types of loaded edges that contain convex vertices allow each of them to have load two. We can therefore protect each convex vertex with an additional virtual load of one. This way no convex vertex receives an actual load greater than two. □

2.3 Lower bound construction.

Theorem 2 *There is a simple polygon P on twelve vertices such that in any pointed pseudo-triangulation of $Int(P)$ there is a vertex of degree five (see Figure 5).*

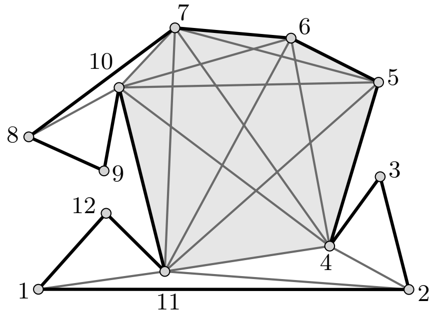


Figure 5: A polygon that requires a vertex of degree 5. Admissible edges are shown in grey.

2.4 Logarithmic face degree. Consider the family P_k, $k \geq 2$, of polygons with $n = 2^k + 1$ vertices. These vertices are arranged in $k+1 = \log_2(n-1)+1$ horizontal layers, as shown in Figure 6 for P_5. The pseudo-triangle

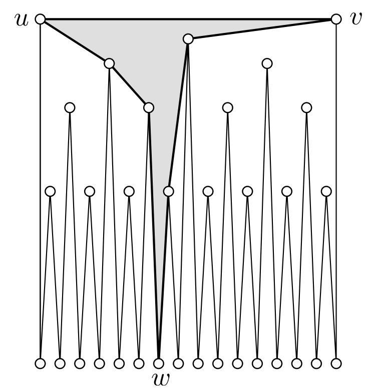


Figure 6: Any pseudo-triangle based on (u, v) has logarithmic face degree.

Δ which contains the edge (u, v) uses $k - 1 \in \Theta(\log n)$ diagonals of P. On the other hand, for any simple polygon there always exists a pseudo-triangulation T such that any pseudo-triangle of T uses at most a logarithmic number of diagonals [2]. Therefore our bound is tight.

3 A simple polygon

In this section we show how to construct a constrained pseudo-triangulation for S if the constraining graph G

forms a simple polygon P on S. Each constrained pseudo-triangulation of S has to contain the convex hull $CH(S)$ of S, which at the same time is the convex hull of P (see Figure 7). P divides its convex hull in several regions, each of which is again a simple polygon. We will pseudo-triangulate the interior of each of these regions separately.

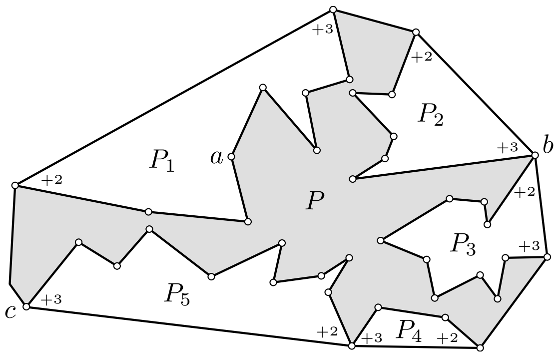


Figure 7: The constraining graph G forms a simple polygon P.

Theorem 3 *For any pointed planar straight line graph $G = (S, E)$ that forms a simple polygon on S there is a constrained pointed pseudo-triangulations $\mathcal{T}_G$ of S with maximum vertex degree seven.*

Proof: S contains two different types of vertices: convex hull vertices and interior vertices. Interior vertices (for example Vertex a in Figure 7) are part of exactly two polygons, namely P and one of the induced polygons P_i. These vertices are reflex with respect to one polygon and convex with respect to the other. Theorem 1 then implies, that there is a pseudo-triangulation of P and P_i such that the degree $d(a)$ is at most seven, namely $d(a) \leq 2 + 3 + d_G(a) = 2 + 3 + 2 = 7$.

The convex hull vertices (for example Vertex b in Figure 7) receive additional degrees, $d_H(b)$, from the convex hull edges. They can also be part of up to three polygons, P and two of the induced polygons. In oder to protect these vertices from receiving too high a degree in each of the polygons they are part of, we add virtual loads to all convex hull edges as it is indicated with the numbers in Figure 7. Each of the induced polygons then starts the construction of its pseudo-triangulation with exactly one loaded edge of type I and therefore satisfies both invariants. The total degree of a convex hull vertex is then $d(b) \leq 2 + 1 + 0 + d_G(b) + d_H(b) \leq 2 + 1 + 2 + 2 = 7$. □

3.1 Lower bound construction.

Theorem 4 *There is a simple polygon P on 21 vertices such that in any pointed pseudo-triangulation of P there is a vertex of degree at least seven (see Figure 8).*

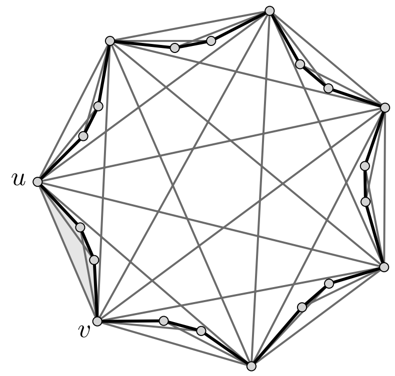


Figure 8: A polygon that requires a vertex of degree 7. Admissible edges are shown in grey.

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