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On minimum weight pseudo-triangulations

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ABSTRACT

In this note we discuss some structural properties of minimum weight pseudo-triangulations of point sets.

1. Introduction

Optimal triangulations for a set of points in the plane have been, and still are, extensively studied within Computational Geometry. There are many possible optimality criteria, often based on edge weights or angles. One of the most prominent criteria is the *weight* of a triangulation, that is, the total Euclidean edge length. Computing a *minimum weight triangulation* (MWT) for a point set has been a challenging open problem for many years [4] and various approximation algorithms were proposed over time; see e.g. [3] for a short survey. Mulzer and Rote [9] showed only very recently that the MWT problem is NP-hard.

Pseudo-triangulations are related to triangulations and use *pseudo-triangles* in addition to triangles. A pseudo-triangle is a simple polygon with exactly three interior angles smaller than π. Also for pseudo-triangulations several optimality criteria have been studied, for example, concerning the maximum face or vertex degree [5]. Optimal pseudo-triangulations can also be found via certain polytope representations [10] or via a realization as locally convex surfaces in three-space [1]. Not all of these optimality criteria have natural counterparts for triangulations. Here we consider the classic minimum weight criterion for pseudo-trian gulations.

Rote et al. [11] were the first to ask for an algorithm to compute a minimum weight pseudo-triangulation (MWPT). The complexity of the MWPT problem is unknown, but Levcopoulos and Gudmundsson [7] show that a 12-approximation of an MWPT can be computed in $\mathcal{O}(n^3)$ time. Moreover, they give an $\mathcal{O}(\log n \cdot w(\mathrm{MST}))$ approximation of an MWPT, in $\mathcal{O}(n \log n)$ time. Here $w(\mathrm{MST})$ is the weight of the minimum Euclidean spanning tree, which is a subset of the obtained structure.

A pseudo-triangulation is called *pointed* (or minimum) if every vertex p has one incident region (either a pseudo-triangle or the exterior face) whose angle at p is greater than π. A pointed pseudo-triangulation minimizes the number of edges among all pseudo-triangulations of a given point set. Since a spanning tree is not necessarily pointed (see [2]) the pseudo-triangulation constructed by the approximation algorithm of [7] is also not necessarily pointed. It is logical to conjecture

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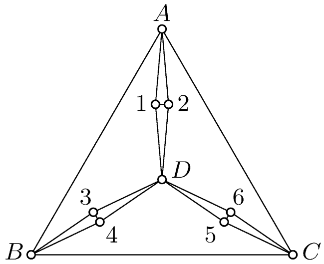

Fig. 1. Non-pointed MWPT of a point set.

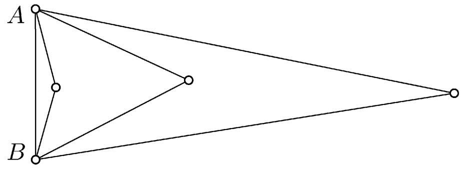

Fig. 2. An MWPPT with linear vertex degree.

that the MWPT should be pointed. However, we show that this does not need to be the case. As a consequence, the MWPT and the minimum weight pointed pseudo-triangulation (MWPPT) of a point set are different concepts. We also discuss the relation of MWP(P)Ts to greedy pseudo-triangulations and we give conditions on point sets under which the MWPT is lighter than the MWT.

2. Does an MWPT have to be pointed?

Lemma 1. *The minimum weight pseudo-triangulation of a point set is not necessarily pointed.*

Proof. See Fig. 1. This pseudo-triangulation, $\mathcal{PT}$, contains exactly one non-pointed vertex. To make $\mathcal{PT}$ pointed, we have to reduce the number of its edges by exactly one. However, no single edge can be removed to achieve this. Observe further that all used inner edges adjacent to A, B, C, or D have equal length, and that the edges $\overline{12}$, $\overline{34}$, and $\overline{56}$ are arbitrarily short. It is easy to see that the shortest non-used edge ($\overline{13}$ or equivalent) is at least $3/2$ times longer than the longest used edge (apart from convex hull edges, of course). Therefore, it suffices to show that no two edges of $\mathcal{PT}$ can be replaced by a single new edge without increasing the weight of the pseudo-triangulation.

We now exclude the unused edges one by one. Edge $\overline{A3}$ is longer than the edges $\overline{A1}$ and $\overline{1D}$ together, which are the longest interior edges of $\mathcal{PT}$. Therefore, $\overline{A3}$ may not be inserted instead of two used interior edges because this would raise the weight of $\mathcal{PT}$. Edge $\overline{A4}$ is inapplicable because it is even longer than edge $\overline{A3}$. If we insert edge $\overline{13}$ then we also have to insert edge $\overline{A3}$ (or edge $\overline{B1}$, which is of the same length) to maintain a pseudo-triangulation. But we already argued that the insertion of edge $\overline{A3}$ is not allowed, and therefore edge $\overline{13}$ cannot be inserted, either. Edges $\overline{14}$ and $\overline{24}$ are inapplicable for similar reasons: Insertion of edge $\overline{14}$ forces either edge $\overline{B1}$ or edge $\overline{A4}$, and inserting edge $\overline{24}$ makes it necessary to add edge $\overline{A4}$ (or edge $\overline{B2}$, resp.) or two of the previously mentioned edges to maintain a pseudo-triangulation. The last possible edge is $\overline{AD}$, which either involves the insertion of an already excluded edge, or can be exchanged with edge $\overline{12}$, which is the shortest edge of $\mathcal{PT}$. □

3. Vertex degrees in an MW(P)PT

Lemma 2. *A minimum weight* (*pointed*) *pseudo-triangulation can have vertices with arbitrarily high vertex degree.*

Proof. See Fig. 2. For each two consecutive triangles based on $\overline{AB}$, the distance between their tips is larger than the longest edge of the smaller triangle. This implies that the shown pseudo-triangulation is indeed minimum weight. The degree of the vertices A and B is $n-1$ if the example is drawn on n points. □

4. Greedy (pointed) pseudo-triangulations

The greedy pseudo-triangulation of a point set S is obtained by inserting edges spanned by S in increasing length order, such that no crossings are caused and until a pseudo-triangulation of S is obtained. Though such a greedy pseudo-triangulation clearly exists, the concept is not meaningful, as we are going to show below.

Lemma 3. *Let* ∇ *be any pseudo-triangle that is not a triangle. Then* ∇ *contains some diagonal that is shorter than the longest edge of* ∇.

Proof. As the sum of angles in a triangle is π, it is immediate that the three (interior) angles at the corners of ∇ sum up to less than π. Hence there exists a corner c of ∇ where the interior angle is less than $\frac{\pi}{3}$. Let s be the line segment connecting the two vertices of ∇ neighbored to c. Moreover, denote with Δ the triangle spanned by s and c. Clearly, the longest edge of Δ is not s but rather an edge of ∇, say e. So, if s is a diagonal of ∇ then we are done. Otherwise, there have to exist vertices in the interior of Δ. Corner c sees at least one of them, u, and $\overline{cu}$ is a diagonal of ∇ that is shorter than e. The lemma follows. □

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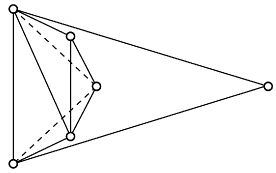

Fig. 3. The greedy pointed pseudo-triangulation (solid) differs from the MWPPT (dashed).

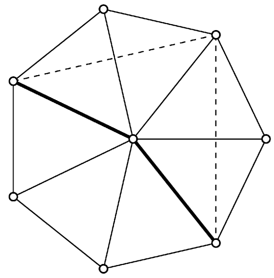

Fig. 4. This regular wheel of degree 7 is both the MWT and the MWPPT of the underlying point set.

Corollary 4. *For every point set S, the greedy pseudo-triangulation equals the greedy triangulation.*

Proof. Assume that the greedy pseudo-triangulation of S contains a pseudo-triangle, ∇, that is not a triangle. Then, by Lemma 3, ∇ contains some diagonal, d, being shorter than its longest edge. So, during the greedy process of constructing the pseudo-triangulation, d would have been inserted before completing the insertion of the edges that form ∇ – a contradiction. □

Requiring pointedness of a greedy pseudo-triangulation changes the situation. This concept is well defined, too, as each face of the pointed graph produced so far – if not a pseudo-triangle – can be split into two faces using any geodesic and without violating pointedness. Not surprisingly, the greedy pointed pseudo-triangulation can differ from the MWPPT. Fig. 3 gives a simple example. There are sets of n points for which the greedy triangulation has length $\Omega(\sqrt{n})$ times the length of the minimum weight triangulation [6]. This bound is tight [8]. Unfortunately it seems that neither of these constructions is applicable to pseudo-triangulations, which raises the question how well the greedy pointed pseudo-triangulation approximates the MWPPT.

5. Comparing MWPT and MWT

We now compare the minimum weight pseudo-triangulation to the minimum weight triangulation of a point set. A useful structure for this comparison is the so-called *wheel*. A wheel is the star-like triangulation of a convex polygon with exactly one interior vertex, the *hub* of the wheel. We call the vertex degree of the hub (i.e., the size of the convex polygon) the *degree* of a wheel. The *spokes* of a wheel are the edges of the wheel incident to the hub. Let us call a *big angle* an angle that is larger than π.

Lemma 5. *There are sets of $n \geqslant 8$ points, that do not lie in convex position, for which the minimum weight pseudo-triangulation is a triangulation.*

Proof. Consider the regular wheel in Fig. 4. It is easy to see that this wheel is the MWT of the underlying point set. To construct a pseudo-triangulation that is not a triangulation we have to make the hub pointed, as it is the only interior vertex. This involves removing at least 3 spokes, and inserting 2 non-spoke edges (dashed edges in Fig. 4). Let δ be the length of the shortest non-spoke edge and let R be the length of a spoke. We have $\delta = 2\sin(2\pi/7)R \approx 1.564R$ and hence any 2 non-spoke edges are longer than 3 times R. □

The point set used in the proof of Lemma 5 contains only one vertex in the interior. Hence the question arises whether requiring a certain number of interior vertices in a point set always ensures the existence of pointed vertices in its MWPT. We settle this question in the affirmative in the remainder of this section.

Observation 1. *If the MWPT of a point set is a triangulation then, for each interior vertex, its incident triangles form a wheel.*

This holds because, otherwise, some edge incident to such a vertex could be removed, creating a proper pseudo-triangle. We continue with a series of properties of wheels that imply the property MWPT $\neq$ MWT.

Observation 2. *An MWPT cannot have an edge whose removal changes a vertex from non-pointed to pointed.*

Removing an interior edge of a pseudo-triangulation either results in a pseudo-triangulation or creates a pseudo-quadrilateral. The latter happens if and only if the number of pointed vertices does not increase. Since every non-pointed vertex of degree 3 or 4 has one adjacent edge whose removal makes this vertex pointed, we have the following:

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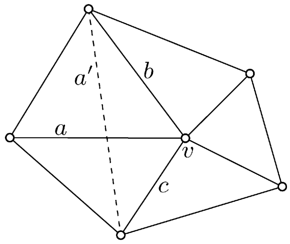

Fig. 5. A non-regular wheel of degree 5.

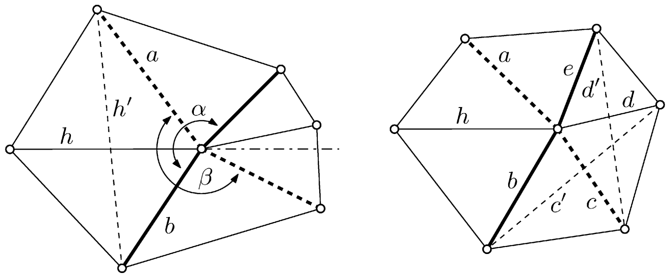

Fig. 6. Two non-regular wheels of degree 6.

Observation 3. *An MWPT cannot have non-pointed interior vertices of degree less than* 5.

Observation 4. *Let a, b, and c be three spokes of a wheel and let a' be the edge joining the non-hub endpoints of b and c. If $|a| > |c|$ then $|a| + |b| > |a'|$.*

This holds since $|a| + |b| > |c| + |b| > |a'|$. In the following we use Observation 4 only in configurations where a crosses a'.

Lemma 6. *If the minimum weight triangulation of a point set is a wheel of degree* 5 *then there exists a pseudo-triangulation on these* 6 *points that is lighter.*

Proof. See Fig. 5. If there is a spoke whose removal makes the interior vertex v pointed, then Observation 2 implies the lemma. Otherwise, the removal of any pair of consecutive spokes makes v pointed. Let a be the longest spoke and let b and c be its neighbors. Then according to Observation 4 removing a and b and adding a' results in a lighter pseudo-triangulation. □

Lemma 7. *If the minimum weight triangulation of a point set is a wheel of degree* 6 *then there exists a pseudo-triangulation on these* 7 *points that is lighter.*

Proof. Let h be the longest spoke. If there is a spoke whose removal makes the interior vertex pointed, then Observation 2 implies the lemma. Otherwise, assuming general position, there are at least 2 spokes on either side of the line supporting h. If $\alpha > \pi$ or $\beta > \pi$ in Fig. 6 (left) then Observation 4 implies the lemma. So assume $\alpha < \pi$ and $\beta < \pi$ (Fig. 6 (right)). This implies that removing c and d will make the interior vertex pointed. If $|c| > |d|$ then we add c', otherwise we add d'. Now Observation 4 again implies the lemma. □

Let us summarize: An MWPT can contain a wheel of degree 7 or higher, but not of degree 6 or lower. This directly implies that an MWPT, which is a triangulation, cannot have interior vertices of degree 6 or lower. Further, no MWPT can have non-pointed interior vertices of degree 4 or 3.

Lemma 8. *Let S be a point set with h points on the convex hull and at least* 3 *interior points. For every triangulation of S the sum of degrees of the convex hull vertices, ρ_h, is at least $3 \cdot h + 3$.*

Proof. It is easy to see that ρ_h is minimized if the interior vertices have a triangular convex hull. Thus we only have to consider h points in convex position and a triangle, Δ, inside. The number of triangulation edges then is $2 \cdot h + 6$, exactly $h + 3$ of which are interior to the belt $conv(S) \setminus \Delta$. But each of these interior edges has to be incident to some vertex of $conv(S)$. We conclude $\rho_h \geqslant (h + 3) + 2 \cdot h = 3 \cdot h + 3$. □

Theorem 9. *If a set S of $n \geqslant 15$ points contains more than $\frac{n-9}{2}$ interior points then its minimum weight pseudo-triangulation contains pointed interior vertices.*

Proof. Any triangulation with average *interior* vertex degree $\overline{\rho}_i < 7$ has at least one interior vertex of degree at most 6. From Observations 1 and 3 and Lemmas 6 and 7 we know that, in such a case, we can construct a corresponding pseudo-triangulation which is lighter than this triangulation.

The sum of all vertex degrees in a triangulation of S is exactly $6 \cdot n - 2 \cdot h - 6$ if h points of S are extreme. By Lemma 8, the sum of interior vertex degrees is at most $6 \cdot n - 5 \cdot h - 9$, which gives $n + 5 \cdot i - 9$ if there are $i = n - h \geqslant 3$ interior points. The average interior vertex degree thus is $\overline{\rho}_i \leqslant \frac{n-9}{i} + 5$. If we want $\overline{\rho}_i < 7$ then $i > \frac{n-9}{2}$. □

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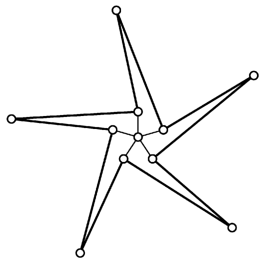

Fig. 7. Non-pointed MWPT of a pointgon.

6. Conclusion

We have given some properties of minimum weight pseudo-triangulations. A main open question is how much weight loss can be guaranteed for any set with sufficiently many interior points when relaxing from triangulations to pseudo-triangulations. Lemma 5 shows that there might be no gain at all and, even worse, the MWPPT may be longer than the minimum weight triangulation. On the other hand, Theorem 9 suggests that the gain might be linear in the number of interior vertices.

In this note we considered MWP(P)Ts of point sets. Gudmundsson and Levcopoulos [7] showed how to compute the MWPT or the MWPPT of a simple polygon in cubic time. However, little is known about MWP(P)Ts of so-called *pointgons* – point sets inside simple polygons. Fig. 7 shows that the MWPT of a pointgon is not necessarily pointed, it can have non-pointed interior vertices of degree 5. For point sets, we know that an MWPT cannot have non-pointed interior vertices of degree 4 and 3 and that there are point sets which have non-pointed interior vertices of degree 6. Hence it would be interesting to compare point sets and pointgons in this respect.

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