Geometry in architecture and building

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The picture on the cover was kindly provided by W. Huisman, Department of Architecture and Building. The picture shows the Olympic Stadium for the 1972 Olympic Games in Munich, Germany, designed by Frei Otto.

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Chapter 1

Shapes in architecture

1.1 A brief tour of shapes

1.1.1 Take a look at modern architecture and you will soon realize that the last decades have produced an increasing number of buildings with exotic shapes. Of course, also in earlier times the design of buildings has been influenced by mathematical ideas regarding, for instance, symmetry. Both historical and modern developments show that mathematics can play an important role, ranging from appropriate descriptions of designs to guiding the designer's intuition. This course aims at providing the mathematical tools to describe various types of shapes in a mathematical way and to manipulate them. In handling them in more involved situations, mathematical computer software such as *Maple* is very useful. This section discusses a few examples of architectural shapes and hints at the relevant mathematics. Related to these shapes you can think of questions like the following. How can I describe this object with equations? How can I down-size the object, or make it more curved? How does the surface area or volume change if the designer changes the position of a wall?

1.1.2 Some words on coordinates
Geometry deals with shapes, but in actually handling these shapes, it is profitable to bring them within the mathematical realm of numbers and equations. The usual way to get numbers in relation to shapes in your hands is through the use of coordinates. There are many coordinate systems, but the most common coordinate system is the familiar cartesian coordinate system, where you choose an origin in 3–space and three mutually perpendicular axes through the origin (often, but not necessarily, labelled as x–axis, y–axis and z–axis), etc. Each point in space is then characterized by its three coordinates, for instance $(2, -\sqrt{3}, 0)$. (In 2–space, only two axes are needed and points are described by a pair of coordinates.) We usually refer to coordinatized 2–space and 3–space as $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. Equations, like $x + 2y + 3z = -5$, describe shapes in 3–space: A point (x, y, z) in 3–space is on the plane precisely if its coordinates satisfy the equation. In this case the resulting shape is a plane. All sorts of geometric operations have their algebraic counterparts. For example, the result of reflecting the point $P = (x, y, z)$ in the x, y–

plane is the point with coordinates $(x, y, -z)$. Rotating the point around the z–axis over $90°$ yields the point $(-y, x, z)$ or $(y, -x, z)$, depending on the orientation of the rotation. Coordinates of some sort and the corresponding algebraic machinery are at the basis of computations and of useful implementations in computer software. This mix of shapes and numbers is central in this course.

Of course, it is up to the user to choose a convenient origin and to fix the direction of the axes. Two designers may have decided to use different coordinate systems. To be able to deal with each other's data they are confronted with the question how to transform one system of coordinates into the second one. For instance, if you view a building from two different points, then how are the two viewpoints related exactly?

1.1.3 A brief word on lines and planes

Flat objects are easier to describe than curved ones. So in Chapter 2 flat objects like lines and planes will be discussed before we turn to a more detailed study of curved objects in later chapters. Here is a tiny preview.

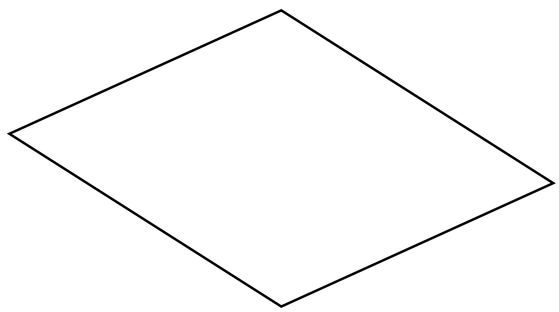

Figure 1.1: *A plane in 3–space, usually described by an equation of the form $ax+by+cz = d$. Of course, only part of the plane is drawn, since the plane extends indefinitely. If we describe walls by planes, we need to be aware of the fact that only part of the plane corresponds to the wall.*

Suppose we work with ordinary cartesian coordinates x and y in the plane. A line in the plane is described by an equation in the variables x and y of the form

$$ax + by = c,$$

such as $2x - 7y = 9$. A plane in space involves a linear equation in one more variable, z:

$$ax + by + cz = d.$$

For example, $3x + 2y - 7z = 9$ is a plane in 3–space. These equations provide *implicit* descriptions: you know the condition the coordinates have to satisfy in order to be the coordinates of a point of the line or plane. There are also *explicit* descriptions for lines and planes, so–called *parametric descriptions*. This chapter is not the place to discuss these matters in detail. Instead we give a sketch.

Let us consider the line in the plane with equation $2x + 3y = 6$. We can solve this equation for y in terms of x: $y = (6 - 2x)/3$. If we assign the value λ to x, then the pair can be described as

$$\begin{aligned} x &= \lambda \\ y &= 2 - 2\lambda/3. \end{aligned}$$

We rewrite this as

$$(x, y) = (0, 2) + \lambda(1, -2/3),$$

so that the relation with points in $\mathbb{R}^2$ comes out more clearly. Substituting *any* value for λ in the expression on the right-hand side (no condition on λ) produces the *explicit* coordinates of a point on the line. For instance, for $\lambda = 9$, the corresponding point on the line is $(9, 2 - 9 \cdot 2/3) = (9, -4)$. The parametric description $(x, y) = (0, 2) + \lambda(1, -2/3)$ also has a clear geometric interpretation: draw a line through $(0, 2)$ whose slope is $-2/3$.

There are more ways of writing down the solutions of the equation $2x+3y = 6$ explicitly. For instance,

$$(x, y) = (3, 0) + \mu(3, -2)$$

describes the same line! (In fact, we have found this parametric description by solving x in terms of y.) To check that these points are on the line, just plug the corresponding values of x and y into the equation, i.e., substitute $x = 3 + 3\mu$ and $y = -2\mu$ into $2x + 3y$ and verify that the resulting expression simplifies to 6:

$$2(3 + 3\mu) + 3(-2\mu) = 6 + 6\mu - 6\mu = 6.$$

This last representation has the slight advantage, at least for humans — computers don't mind that much, that there are no fractions in the expression. Again, the parametric description $(x, y) = (3, 0) + \mu(3, -2)$ is easy to represent graphically: just start at the point $(3, 0)$ and then draw the line through $(3, 0)$ with slope $-2/3$ (or: for every 3 steps to the right go 2 steps down).

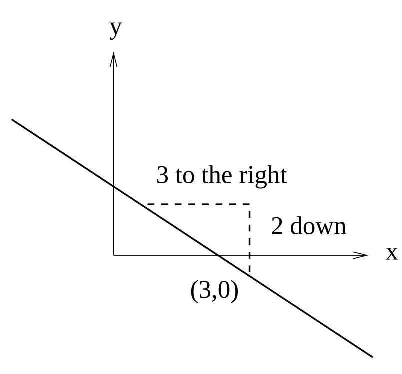

Figure 1.2: *The graphical representation of* $(x, y) = (3, 0) + \mu(3, -2)$.

With fairly elementary techniques you can switch from parametric descriptions to equations and vice versa. For instance, starting with $(x, y) = (3, 0) + \mu(3, -2)$, you first extract

$x = 3 + 3\mu$ and $y = -2\mu$, then add two times the first equality and three times the second one to find that x and y satisfy

$$2x + 3y = 2(3 + 3\mu) + 3 \cdot (-2\mu) = 6$$

(the addition was set up in order to make μ drop out), i.e., $2x + 3y = 6$.

More aspects of lines and equations will be dealt with in the exercises and in the following chapters.

1.1.4 Buildings with flat walls

Here begins our trip along various architectural objects. Take a look at the picture of the Van Abbe museum in Eindhoven (Fig 1.3)[1].

Figure 1.3: *The Van Abbemuseum in Eindhoven (Photo: Peter Cox)*

The extension was designed by the Dutch architect Abel Cahen. The walls of the museum are flat or planar, but some of them are sloping walls. Obvious questions are: How much are they inclined? Where do two walls meet exactly? What angle subtend two of these planes? What would change if you change such an angle a bit (just think of the surface area, the position of the roof, etc.)? Of course, mathematics is intended here to support the designing process of the architect. It is no substitute for the architect's creativity. Let us take a closer look at two of these questions: the intersection of two walls and the angle between two walls.

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1, \\ a_2x + b_2y + c_2z &= d_2, \end{aligned}$$

(where a_1, a_2, etc., are the coefficients of the equations).

[1] http://www.vanabbemuseum.nl/nederlands/gebouw/

Intersecting two planes

The two walls meet along a line, but which one? And how does this line change if the architect decides to change one or both of the planes in the design? Here is a concrete example of dealing with the intersection of two planes (but this chapter is not the place to discuss the techniques used in detail):

$$\begin{aligned} x + y + z &= 3 \\ 2x + y - z &= 5 \end{aligned}$$

We manipulate the equations in such a way that both x and y can be expressed in terms of z. To do so, we need two steps:

a) We first try to eliminate x from the second equation. By subtracting the first equation

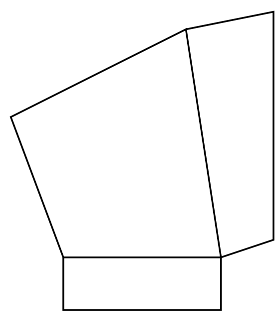

Figure 1.4: *Sloping walls.*

two times from the second, we obtain the system

$$\begin{array}{ccccccc} x & + & y & + & z & = 3 \\ & & -y & - & 3z & = -1 \end{array}$$

b) Next, we add the new second equation to the first one in order to get rid of the variable y in the first equation. We find

$$\begin{aligned} x - 2z &= 2 \\ -y - 3z &= -1 \end{aligned}$$

Now, both x and y can be expressed in terms of z as follows: $x = 2 + 2z$ and $y = 1 - 3z$. Introduce a parameter λ by $z = \lambda$. Then we get:

$$\begin{aligned} x &= 2 + 2\lambda \\ y &= 1 - 3\lambda \\ z &= \lambda. \end{aligned}$$

Separating the 'constant part' and the 'variable part', we usually rewrite this as

$$(x, y, z) = (2, 1, 0) + \lambda(2, -3, 1).$$

This notation suggests clearly that, not surprisingly, we are dealing with a line: start at the point $(2, 1, 0)$ and move from there in the direction of $(2, -3, 1)$ by varying λ.

The angle between two planes
From the equations $x + y + z = 3$ and $2x + y - z = 5$, the angle ϕ between the planes can be computed. The relevant information is contained in the coefficients of x, y and z of both equations (the coefficients 3 and 5 on the right–hand side are irrelevant). The three coefficients of the first equation lead to $(1, 1, 1)$. It turns out that the direction from $(0, 0, 0)$ to $(1, 1, 1)$ is perpendicular to the first plane (more on this in Chapter 2). Likewise, the three coefficients of the second equation lead to $(2, 1, -1)$, and the direction from $(0, 0, 0)$ to $(2, 1, -1)$ is perpendicular to the second plane. In this setting, where directions come into play, we usually speak of *vectors*. The angle between the two planes equals the angle between these two 'vectors' (make a picture to convince yourselves). It turns out (and this will be dealt with more extensively in Chapter 2) that the cosine of the angle is computed as follows from the two vectors $(1, 1, 1)$ and $(2, 1, -1)$:

$$\cos\phi = \frac{1\cdot 2 + 1\cdot 1 + 1\cdot(-1)}{\sqrt{1^2+1^2+1^2}\cdot\sqrt{2^2+1^2+(-1)^2}} = \frac{2}{\sqrt{3}\cdot\sqrt{6}} = \frac{2}{3\sqrt{2}} = \frac{\sqrt{2}}{3}\,.$$

From this expression, the angle is easily found to be approximately 1.08 radians or 61.9 degrees.

Varying the plane
If we replace one or more of the coefficients in the equation of one of the planes by an (expression in an) auxiliary parameter, we obtain a varying *family of planes*: for each value of the parameter a plane is defined. Such a family may be useful in the design of a building where you have to specify the position of a wall, say, given that it passes through certain points, intersects the ground level along a certain line, etc. Here are a few examples.

For every value of a the equation $x + y + z = a$ describes a plane. All these planes are parallel to one another (they are all perpendicular to the vector $(1, 1, 1)$). The plane with equation $x+y+z = 0$ (i.e., $a = 0$) contains the origin $(0, 0, 0)$, but the plane with equation $x + y + z = 1$ evidently does not. You might look for a plane in this family which touches a sphere with center in the origin and given radius.

Here is a family with other properties. The family of planes $x + y + az = 3$ all pass through the point $(3, 0, 0)$, but no two of them are parallel. They all have the line of intersection of the two planes $z = 0$ and $x + y = 3$ in common. Among these planes you might be looking for one which makes an angle of 60° with a horizontal plane.

Rotations and translations are also important ways of varying a plane; these operations will be discussed in later chapters.

1.1.5 Buildings with curved exteriors

Modern buildings show a variety of curved shapes, like the Gherkin in London, see Fig. (1.5)[2]. To handle these, nonlinear equations and nonlinear parametric descriptions are

[2] http://www.bconstructive.co.uk/onsite/projects.asp

needed, i.e., equations and parametric descriptions involving, say, combinations of squares, square roots, exponentials, sines and cosines, etc. Curved shapes are central in Chapters 3 and 4.

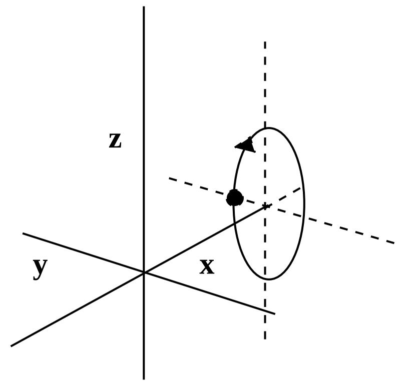

Figure 1.5: *The Gherkin in London (Photo Nigel Young/Foster and Partners). Rotating the point* $(x, f(x), 0)$ *around the* x*–axis (right). For each value of* x *this produces a circle with equation* $y^2 + z^2 = f(x)^2$*. Apart from details, the shape of the gherkin comes close to a surface of revolution.*

For instance, let us look at the shape that is obtained by rotating the graph of a function in the x, y–plane around the x–axis. The graph is represented by the equation

$$y = f(x) \text{ and } z = 0,$$

where we assume $f(x) > 0$ for all x. So every point of the curve in 3–space is of the form $(x, f(x), 0)$. If we rotate such a point around the x-axis, then the first coordinate remains the same, but the second and third coordinate satisfy the equation of a circle with radius $f(x)$, i.e.,

$$y^2 + z^2 = f(x)^2$$

(see Fig. 1.5). For variable x, then, the equation $y^2 + z^2 = f(x)^2$ describes a so–called *surface of revolution.*

To describe such a surface explicitly, we bring in the standard parametric description of a circle with radius r and center $(0, 0)$ in the plane: for varying θ, the point $(r\cos\theta, r\sin\theta)$ runs through a circle with radius r. In our situation the radius is varying with x. At level x, the radius equals $f(x)$. Therefore, a possible parametric description is

$$\begin{aligned} x &= \lambda \\ y &= f(\lambda)\cos\theta \\ z &= f(\lambda)\sin\theta, \end{aligned}$$

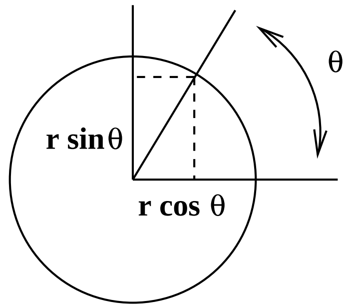

Figure 1.6: *Any point on a circle with center* $(0,0)$ *and radius* r *can be described in the form* $(r\cos\theta, r\sin\theta)$.

where θ can be chosen in the interval $[0, 2\pi]$ or $[-\pi, \pi]$. The 2–dimensionality of the surface corresponds to the presence of two parameters, λ and θ.

Here are some questions related to such a curved object:

- How do you describe mathematically a certain pattern on the surface, like the one on the Gherkin?
- How curved is the object? Is there a measure for it?
- How do we describe a surface which is rotated around another line?

1.1.6 Example. (*Positioning windows in a spherical building*)
Part of a building has a spherical shape, say a hemisphere with equation

$$x^2 + y^2 + z^2 = 4,$$

with $z \geq 0$ as drawn in Fig. (1.7). The radius of this hemisphere is 2.

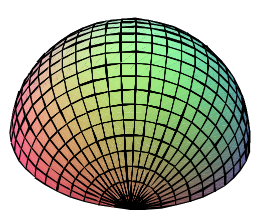

Figure 1.7: *Design for a hemispherical building.*

You decide to put windows in the building by intersecting the building with a number of planes in a symmetric fashion. So you take one plane, for instance $x + y + z = a$, and you determine a in such a way that the intersection with the hemisphere has a reasonable size. We will not explain all details here, but restrict to a sketch of the computation of the size of the intersection. Let us take $a = 3$.

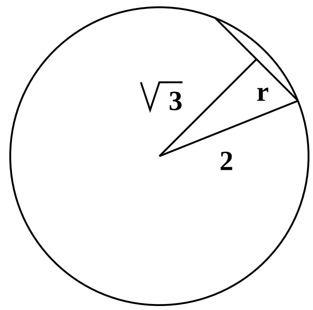

Figure 1.8: *A suitable cross–section of the hemisphere to illustrate the computation of the radius of the circle:* $(\sqrt{3})^2 + r^2 = 2^2$.

a) First, one determines the distance of the origin $(0, 0, 0)$ to the plane: this distance turns out to be $a/\sqrt{3}$, which equals $\sqrt{3}$ because of our choice $a = 3$ (details of such computations will be provided in Chapter 2).

b) The intersection of the plane with the (hemi)sphere is a circle (no proof here, but you'll probably agree), say with radius r. Now we apply Pythagoras' theorem to a triangle with vertices $(0, 0, 0)$, the center of the circle, and a point on the circle (any point will do):
$$\sqrt{3}^2 + r^2 = 2^2.$$
So we conclude that $r = 1$.

If you decide to have four windows in the hemisphere, positioned symmetrically around the z–axis, then the remaining three windows are obtained by rotating the plane over 90°, 180° and 270°, and intersecting with the hemisphere. The result of rotating the plane $x + y + z = 3$ over 90° is the plane with equation $-x + y + z = 3$ (never mind how we found this).

In the designing process, you may decide to move the plane further away from or closer to the origin. This can be accomplished by replacing the right-hand side of the equation $x+y+z = 3$ by a parameter c, say, and experiment with different values of c. Introducing parameters in other places in the equation allows one to experiment even more drastically with the position of the windows.

1.2 Different perspectives

Pictures and movies of buildings are 2–dimensional ways of representing them. Given the 3–d structure of a building, how do you generate projections suitable for various purposes, such as 'artist's impressions' or technical drawings? By setting up appropriate coordinate

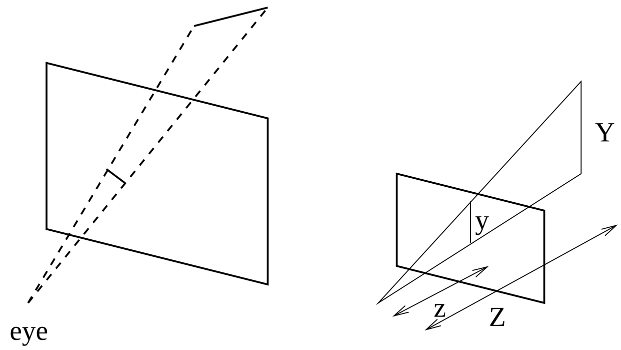

Figure 1.9: *Taking a picture is like a central projection, apart from details concerning lenses (left–hand side). Similar triangles are at the basis of computations involving projections:* $Y/y = Z/z$ *(right–hand side).*

systems (cartesian or otherwise) and using some geometry, the projection of an object can be described in coordinates if the situation is not too complicated (Fig. 1.9 shows an example). Again, this approach opens up the way of handling projections by computer. The above discussion focused on central projection, but other projections may be relevant

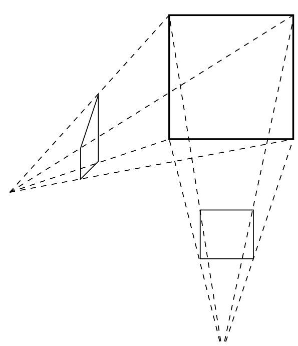

Figure 1.10: *Starting from the position of the square at the back, various central projections can be determined. The difficulty is to catch these projections in coordinates.*

for a specific purpose, such as parallel projection in which lines connecting a point of the original object and its image point are all parallel.

1.2.1 Example. Suppose our eye is in the origin of an ordinary cartesian coordinate system, and suppose we want to find the image in the plane $z = 1$ of a given triangle T in space. If $P = (3, 2, 4)$ is a corner of T, then P is in the plane $z = 4$, a plane at distance 3 from the plane $z = 1$ on which we are projecting (central projection with center the origin and onto the plane $z = 1$). This implies that the coordinates of the projection P' of P are obtained by shrinking those of P by a factor 4; Fig. (1.11) illustrates the relevant triangles that can

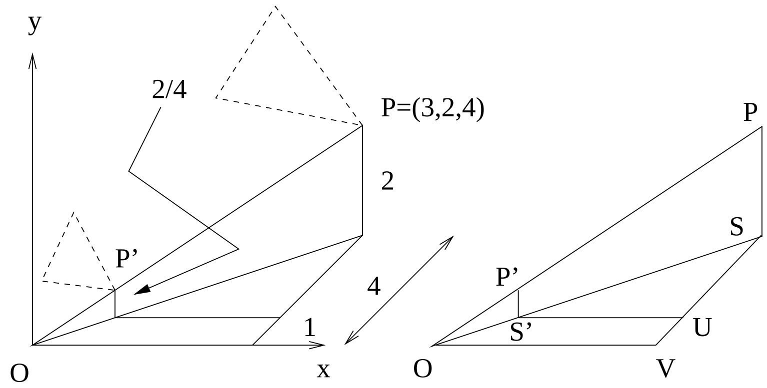

Figure 1.11: *Central projection with center O on the $z = 1$ plane. The z–coordinate is shortened by a factor 4, so the y–coordinate is shortened by a factor 4 as well as can be deduced from* $|PS| : |P'S'| = |OS| : |OS'| = |VS| : |VU| = 4 : 1$. *Here we use the two pairs of similar triangles* $\triangle OPS \sim \triangle OP'S'$ *and* $\triangle OSV \sim \triangle S'SU$ *(right–hand side).*

be used to prove this claim for the y–coordinate. So the image point P' has coordinates $(3/4, 1/2, 1)$.

But what happens if we put our eye at position $(11, 0, 0)$, for instance, but leave the plane on which we are projecting at $z = 1$? In this case, the image point P'' still has z–coordinate 1 and y–coordinate 1/2 (why?). But the x–coordinate of P'' turns out to be $11 - 8/4 = 9$. So $P'' = (9, 1/2, 1)$. It is again an exercise in similar triangles to find this x–coordinate.

The geometry in terms of coordinates of various projections will be discussed in Chapter 5.

1.3 Numbers in architecture: The golden ratio

The golden ratio is a number that has fascinated humans throughout the centuries. This fascination finds its origin in the interpretation of the number as a ratio that has often been viewed as ideal or especially pleasing for paintings, sculptures and (parts of) buildings. The ratio of the width and height of the Parthenon in Greece is approximately equal to the golden ratio, see Fig. (1.12)[3]. The golden ratio is approximately equal to 1.618, but the exact expression to be discussed below is more interesting since it reveals more of its properties.

Here follows a short digression on some geometric and number theoretic aspects of the golden ratio.

1.3.1 What is the golden ratio?

[3]http://ccins.camosun.bc.ca/ jbritton/goldslide/jbgoldslide.htm

Figure 1.12: *The golden ratio and the Parthenon in Athens.*

The golden ratio is the number

$$\frac{1}{2} + \frac{1}{2}\sqrt{5},$$

often represented in this context by the Greek symbol τ. This number acquired its importance as a ratio of lengths in a geometric setting. There are various ways this ratio occurs, one of which we discuss here. Consider the problem of finding a rectangle $ABCD$ (see Fig. (1.13)), such that if you take away a square $AFED$, the remaining rectangle $BCEF$ is proportional to the original rectangle $ABCD$, i.e., the ratios of the two sides of both rectangles are equal. To solve this problem, we rescale so that $|AF| = 1$ and $|AB| = \tau$.

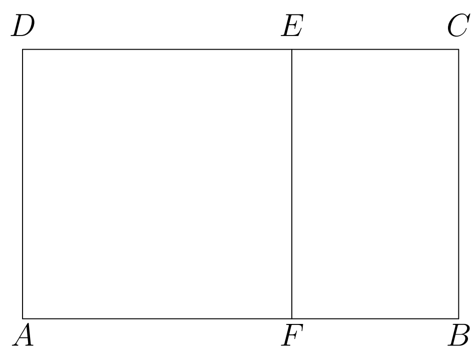

Figure 1.13: *Rectangles $ABCD$ and $BCEF$ are proportional; $AFED$ is a square.*

From

$$\frac{|AB|}{|AD|} = \frac{|BC|}{|FB|}$$

we infer that τ is required to satisfy the equation

$$\frac{\tau}{1} = \frac{1}{\tau - 1}\,.$$

This leads to the quadratic equation $\tau^2 - \tau - 1 = 0$ with only one positive solution: $\tau = \frac{1}{2} + \frac{1}{2}\sqrt{5}$. There are some amusing facts to mention about τ, the first two of which follow immediately from the relation $\tau^2 - \tau - 1 = 0$.

- Rewrite $\tau^2 - \tau - 1 = 0$ in the form $\tau^2 = \tau + 1$ and you see: To square τ you only need to add 1 to τ, so τ^2 is approximately $1.618 + 1 = 2.618$. Similar remarks hold for τ^3, τ^4, etc.

- Divide all terms of $\tau^2 - \tau - 1 = 0$ by τ and rearrange the result as $\dfrac{1}{\tau} = \tau - 1$. So the reciprocal $1/\tau$ of τ can be computed by subtracting 1 from τ. In particular, $\dfrac{1}{\tau}$ is approximately 0.618.

- In a pentagon with sides of length 1, every diagonal turns out to have length τ. Given this fact, some trigonometry shows that $\tau = 2\cos(\pi/5)$.

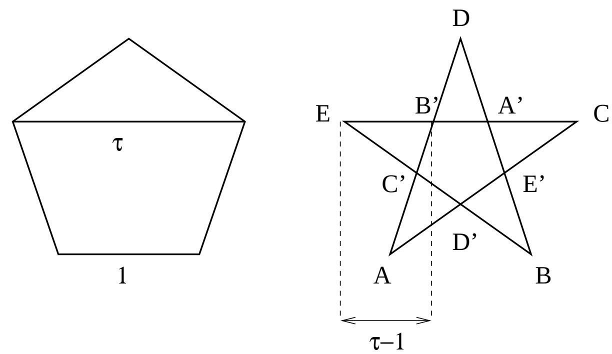

Figure 1.14: *The golden ratio appears in a pentagon and a star with 5–fold symmetry.*

- A surprising form of τ is:

$$\tau = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}}}.$$

- Another surprising shape of τ. The fractions

$$1 + \frac{1}{1+1},\ 1 + \frac{1}{1 + \frac{1}{1+1}},\ 1 + \frac{1}{1 + \frac{1}{1+\frac{1}{1+1}}}, \ldots$$

  constructed from 1's only, approximate τ better and better.

The golden ratio shows up in surprisingly many areas of mathematics and other sciences. In architecture, examples range from the great pyramids in Egypt to the proportional system designed by the Swiss–French architect and painter Le Corbusier (1887–1965).

1.3.2 The golden ratio is irrational[4]

The rational numbers, i.e., numbers of the form a/b (fractions) with a and b integers (and $b \neq 0$), do not fill up the whole real line. Numbers like $\sqrt{2}$, π and e are examples of irrational numbers: numbers that cannot be expressed as the quotient of two integers[5]. The golden ratio turns out to be one of these irrational numbers, and our rectangle provides a geometric way to prove this.

Suppose that τ is a rational number, say a/b, where a and b are positive integers. Our purpose is to show that using the special property of the rectangle $ABCD$ this assumption leads to something nonsensical and must therefore be rejected. So with this assumption on τ we return to the rectangle: if we choose a unit of length so that $|AF|$ is b units, then $|AB|$ must be a units. The next thing to do is to use the smaller and smaller rectangles in $ABCD$ to rewrite the fraction a/b with a smaller numerator and a smaller denominator (and that will lead to nonsense in the end). The picture has the property that if we

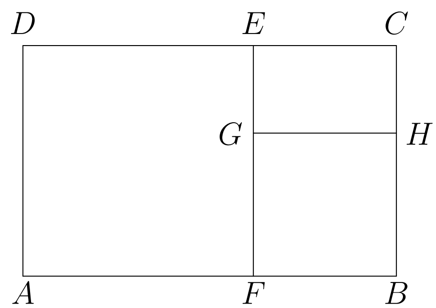

Figure 1.15: *Repeat the process: upon removing square $FBHG$ from rectangle $FBCE$, the remaining rectangle $GHCE$ is similar to $FBCE$ and to the original rectangle $ABCD$.*

leave out the left–hand square $AFED$, the remaining rectangle $FBCE$ is similar to the original rectangle $ABCD$. But if this smaller rectangle has the same shape, then we can apply the process of leaving out a square to this smaller rectangle and obtain an even smaller rectangle $GHCE$, *still similar in shape to the original rectangle*. Now turn to the numbers and ratios again. The length of $|FB|$ is $a-b$ units, and the length of $|HC|$ is $b-(a-b)=2b-a$ units. So comparing the longer side with the shorter side in the similar rectangles $ABCD$, $BCEF$ we obtain:

$$\tau = \frac{a}{b} = \frac{|AB|}{|AD|} = \frac{|BC|}{|FB|} = \frac{b}{a-b},$$

and for the rectangles $BCEF$ and $GHCE$ we find:

$$\frac{b}{a-b} = \frac{|BC|}{|FB|} = \frac{|GH|}{|EG|} = \frac{a-b}{2b-a}\,.$$

[4] Optional reading

[5] Although there are infinitely many rational numbers and infinitely many irrational numbers, it was shown about a century ago that the irrationals outnumber the rationals in a sense. This was the starting point for more subtle investigations into the mysteries of ‘infinity’.

So what? Wel, there are two things to notice here:

a) From $\tau = \frac{a}{b} = \frac{b}{a-b}$ we see that we have replaced numerator and denominator of a/b by smaller numbers ($b < a$ and $a - b < b$ follow from $1 < \tau < 2$). In the second step the fraction $\frac{b}{a-b}$ is replaced by $\frac{a-b}{2b-a}$, a fraction with again a smaller numerator and a smaller denominator as is easily checked.

b) We can repeat the game with the picture over and over again and replace the fraction with fractions numerators and denominators which grow smaller and smaller (as it turns out).

If we do repeat the game over and over again and bookkeep what happens to the fractions (we won't go into the details of this bookkeeping here), we run into trouble: there is no way we can represent a given fraction by smaller and smaller positive numbers. Therefore, we must conclude that our assumption that τ can be represented by a fraction is wrong. And so τ is irrational.

Of course, in practice, a designer works with numerical approximations of the golden ratio and need not worry about the irrationality.

1.4 Exercises

Unless stated otherwise, all coordinates refer to cartesian coordinates. Some exercises are merely intended to train your intuition. They may refer to notions the exact meaning of which has not yet been discussed.

1 The relative position of lines and planes
Test your geometric intuition.

a) As you know, two general lines in the plane intersect in exactly one point. If the lines are in special position with respect to each other, they may meet in a different way. Explain!

b) Take a line and a plane in 3–space. How many points of intersection do you expect in general? Regarding the number of points of intersection, list all possibilities and provide the corresponding picture.

c) Take three general planes in 3–space. How many points of intersection do you expect? Explain! Discuss what happens if the planes are not in general position, for instance if two of the three planes are parallel. Add sketches to support your explanation.

2 Lines through a given point
Take a line ℓ in the plane and let P be a point on the line.

a) How many lines through P are perpendicular to ℓ?

b) How many lines through P make an angle of $45°$ with ℓ?

c) Suppose that ℓ is a line in 3–space. How many lines through P make an angle of $45°$ with ℓ. Make a sketch!

3 Equations and parametric descriptions of lines in the plane
The line ℓ in the plane has parametric description $(x, y) = (1, 2) + \lambda(3, -2)$.

a) Draw the line.

b) Is ℓ the same line as the line with equation $x + y = 3$? Explain your answer algebraically.

c) Start with the parametric description of the line, $x = 1 + 3\lambda$ and $y = 2 - 2\lambda$. Find a and b such that λ drops out from the expression $ax + by$. Use this to find an equation for ℓ.

4 Parametric descriptions of lines in the plane
The equation $2x + 5y = 10$ describes the line ℓ in the plane.

a) Solve for y in terms of x. What is a resulting parametric description?

b) If you solve for x in terms of y, you also get a parametric description. Give one.

c) Is $(10, -2) + \lambda(10, -4)$ a parametric description of ℓ?

5 Intersecting lines in the plane
Lines are usually given by equations (implicit description) or parametric equations (explicit description). In this exercise we consider the problem of intersecting two lines given in various guises.

a) To find the point of intersection of the lines $2x + 5y = 11$ and $x + y = 1$, you look for a suitable combination of the two equations such that, for instance, the variable x is no longer in the resulting equation. In our situation, if we subtract the second equation two times from the first we obtain $3y = 9$, so that $y = 3$. Substitution ('back substitution' it is sometimes called) of $y = 3$ in the equation $2x + 5y = 11$ yields $x = -2$. In this approach we have eliminated x first. What would you do to eliminate y first?

b) Find the intersection of the line ℓ given by $2x + 5y = -1$ and the line m given by $(x, y) = (3, 2) + \lambda(1, 3)$.

c) Suppose two lines, ℓ and m, are both given by a parametric description, say ℓ is given by $(x, y) = (5, 6) + \lambda(2, 1)$ and m is given by $(x, y) = (4, 3) + \mu(-3, 1)$. To compute the intersection of the two lines, first find λ (or μ) from the system

$$\begin{aligned} 5 + 2\lambda &= 4 - 3\mu \\ 6 + \lambda &= 3 + \mu. \end{aligned}$$

Explain why the problem of finding the intersection of the lines leads to this system of equations.

6 Families of lines
This exercise deals with families of lines.

a) For a few values of a, draw the line with equation $2x + 3y = a$. What is the relative position of these lines as a varies?

b) For each value of a, the equation $ax + 2y = 4$ describes a line in the plane. Draw a few of these lines. All these lines have a point in common, which one? Do all lines through this point belong to the family or are there exceptions?

c) Give an example (by giving equations) of a family of lines which all pass through the point $(2, 0)$.

d) Consider the family of lines $a(x - 2) + b(y - 3) = 0$, where a and b are not both 0. Which point in the plane belongs to all these lines? Does this family contain all lines through this point?

7 Various types of equations for lines in the plane
The equation $ax + by = c$ describes a line in the plane. Of course, you are familiar with the equation $y = ax + b$ from previous mathematics courses.

a) Explain why lines parallel to the y-axis cannot be described by equations of the form $y = ax + b$. Relate this to the coefficient a.

b) Which lines in the plane cannot be described by equations of the form $x = ay + b$?

c) Starting with the equation $y = ax + b$, it is easy to give a parametric description of the line. Give a parametric description of the line in terms of λ if we assign the value λ to x.

8 Rotating around the x–axis
This exercise is about rotating around the x–axis in 3–space.

a) Rotate a line parallel to the x–axis around the x–axis (in 3–space). What kind of figure does this give rise to? What is the equation if you start with the line $y = 3$, $z = 0$?

b) Same questions for the line $y = x$, $z = 0$.

9 Rotating around coordinate axes
The graph of $y = \sin(x)$, where $x \in [0, \pi]$, is rotated around the x–axis in 3–space.

a) What is the equation of the resulting surface?

b) Does rotation around the y–axis lead to a reasonable surface?

10 Special members of a family of lines
This exercise is about families of lines in the plane.

a) For a few values of a draw the line with equation $2x + 5y = a$ and convince yourself that they are all parallel.

b) Draw a few members of the family of lines $ax + 3y = 0$. Does this family contain a horizontal member? And a vertical member?

c) The lines $x+y = 0$ and $-x+y = 0$ belong to the family mentioned in b) and are also perpendicular to one another. Find more pairs of perpendicular lines in the family.

11 Switching to another coordinate system
Suppose you have chosen a cartesian coordinate system in a plane. Every point in the plane is then described by a pair (x, y).

a) Now your collegue comes in and prefers to have the origin at 'your' $(2, 3)$. She uses coordinate axes with the same direction as in your coordinate system. What is the relation between the x', y'–coordinates she uses to describe a point and your coordinates?

b) How does she describe 'your' line $y = 4x - 5$?

c) Describe in your coordinates the circle she describes with the equation $(x' - 1)^2 + (y' + 6)^2 = 11$.

12 Projections
Connect the points $(0, 0, 2)$, $(4, 2, 2)$ and $(0, 4, 2)$.

a) Central projection on the plane $z = 1$ with center the origin takes the triangle into its image. Find the images of the three vertices of the triangle.

b) Do you think it is possible to change the position of the plane $z = 1$ so that the image triangle is equilateral (has three equal sides)?

13 Properties of the golden ratio
Refer to Fig. (1.15) related to the golden ratio. Suppose $|AB| = \tau$ and $|AD| = 1$.

a) Show that $|FB| = 1/\tau$ and that $|CH| = 1/\tau^2$. If you continue to split off a square, what will be the lengths of the ever smaller sides of the rectangles that show up?

b) You can also reverse the process: take $ABCD$ and construct a square on side AB (with only side AB in common with rectangle $ABCD$). What are the lengths of the sides of the resulting rectangle? What is the pattern if you repeat this construction?

14 Properties of the golden ratio
The golden ratio τ satisfies the relation $\tau^2 = \tau + 1$.

a) Use this relation to show that $\tau^3 = 2\tau + 1$ and $\tau^4 = 3\tau + 2$.

b) Use this relation to show that $\tau^{-2} = 2 - \tau$.