Computational Group Theory

Soria Summer School 2009
Session 4: The small Mathieu groups

Technische Universiteit
Eindhoven
University of Technology

July 2009, Hans Sterk (sterk@win.tue.nl)

Where innovation starts

On the Mathieu groups

- **Small Mathieu groups:** M_{10}, M_{11}, M_{12}
- **Large Mathieu groups:** M_{22}, M_{23}, M_{24}
- M_{10} has a normal subgroup of index 2 isomorphic to A_6, but:
- The last 5 belong to the 26 sporadic simple groups from the classification of finite simple groups

In fact, computational techniques were developed in order to construct and study sporadic simple groups. We will indicate how previously discussed techniques play a role in the construction of these groups. Main focus on M_{10}.

Constructing groups: designs and geometries

A useful way of studying groups is in relation with other, say geometric, structures.

- Isometries in euclidean geometry
- Isometries in hyperbolic geometry
- Automorphism groups of algebraic curves, surfaces, ...
- Permutation groups acting on certain structured sets, such as (graphs of) cubes,...
- in particular, certain designs and geometries

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The affine plane of order 3

- **Points:** 9 vectors in $GF(3)^2$ or $\mathbf{F}_3^2$
- **Lines:** triples of points in a coset of 1-dim'l subspace

Some observations:

- 12 lines:
  - 4 lines through the origin (8 nonzero vectors etc.)
  - Any of these lines has 2 translates not through the origin (cosets), so 12 lines in total
  - Or: 4 lines through each point: 36, but each line is counted 3 times.
- Every 2 points determine a unique line

This affine plane is an example of a 2-$(9, 3, 1)$**-design:**

- 2-$(\mathbf{9}, 3, 1)$: 9 points
- 2-$(9, \mathbf{3}, 1)$: subsets (lines) containing 3 points
- $\mathbf{2}$-$(9, 3, \mathbf{1})$: every 2 points determine a unique line

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Designs: some basics

Design $\Delta = (P, B)$: pointset P, a collection B of subsets of P, called **blocks.**

- t-(v, k, λ)-design:
  - v points
  - every block consists of k points
  - any set of t points is contained in precisely λ blocks
- Aut(Δ): the automorphism group consists of the permutations of P mapping blocks to blocks

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Examples: projective plane

- Projective plane over $\mathbf{F_q}$: lines through origin in $\mathbf{F}_q^3$
  - $\mathbf{F}_q^3 \setminus \{0\}$
  - Identify points on lines through origin:

$$\mathbf{F}_q^3 \setminus \{0\}/\mathbf{F}_q^*$$

    Points:

$$\frac{q^3 - 1}{q - 1} = q^2 + q + 1$$

  - Yields a 2-$(q^2 + q + 1, q + 1, 1)$ design

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Examples: a construction

Constructing a design out of another one, illustrated for the 2-$(q^2+q+1, q+1, 1)$ design (projective plane)

- **Construction** uses that every 2 blocks meet in 1 point:
  - **Points**: Fix a block B_∞, and remove it from the pointset: q^2 points left
  - **Blocks**: for every block $B \neq B_\infty$ take

    $$B \setminus B_\infty$$

    Every block contains q elements
  - Yields a 2-$(q^2, q, 1)$ design, an **affine plane**

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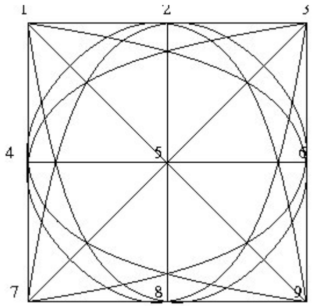
The affine plane of order 3: further details

Labeling of points and lines:

$(0,0)$	1
$(1,0)$	2
$(-1,0)$	3
$(0,1)$	4
$(1,1)$	5
$(-1,1)$	6
$(0,-1)$	7
$(1,-1)$	8
$(-1,-1)$	9

1	2	3	1	6	8
1	5	9	1	4	7
2	6	7	2	4	9
2	5	8	3	4	8
3	5	7	3	6	9
4	5	6	7	8	9

Picture:



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Automorphisms 1: linear algebra

From the geometric set-up:

- Translations, a group of order 9
- The stabilizer of $(0,0)$ contains $GL_2(3)$
  - first column: 3^2-1 possibilities
  - second column: 3^2-3 possibilities
  - Therefore $(3^2-1)(3^2-3)=48$ elements
- The automorphism group contains a group of order $3^2\cdot 48=432$:

$$3^2 : GL_2(3)$$

This group acts transitively on points, lines; also 2-transitively on points:

- Given two pairs (p_1,p_2) and (q_1,q_2)
- Both pairs determine unique lines; after translation: p_1,p_2 on a line through $(0,0)$, but $\neq (0,0)$. Similarly for q_1,q_2.
- Use base transformation (lin algebra) to find a linear map mapping p_i to q_i.

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Automorphisms 2: permutations

The automorphism group H of the design Θ is a subgroup of S_9, containing:

$$\begin{aligned} a &= (1,2,3)(4,5,6)(7,8,9), && \text{a translation} \\ b &= (1,4,7)(2,5,8)(3,6,9), && \text{a translation} \\ c &= (2,9,3,5)(4,6,7,8), \\ d &= (2,7,3,4)(5,8,9,6), \\ e &= (5,7)(4,9)(6,8), \\ f &= (4,7)(5,8)(6,9). \end{aligned}$$

- c, d, e, f stabilize 1
- e, f stabilize $1, 2$ (and also 3)

$G = \langle a, b, c, d \rangle$ is normal subgroup of H of order 72

Automorphisms 3: permutations

$$\begin{aligned} a &= (1,2,3)(4,5,6)(7,8,9), && \text{a translation} \\ b &= (1,4,7)(2,5,8)(3,6,9), && \text{a translation} \\ c &= (2,9,3,5)(4,6,7,8), \\ d &= (2,7,3,4)(5,8,9,6), \\ e &= (5,7)(4,9)(6,8), \\ f &= (4,7)(5,8)(6,9). \end{aligned}$$

- $[1,2,4]$ is a base:
  - $1, 2$ fixed, so 3 fixed
  - $1, 4$ fixed, so 7 fixed
  - $3, 4$ fixed, so 8 fixed, etc.
- Order: $9 \cdot 8 \cdot 6 = 432$ (use e and f to show that the orbit of 4 under the stabilizer of $1, 2$ has 6 elements
- $\{a, b, c, d, e, f\}$ is a strong generating set:
  - $|H_1|$ has order $432/9 = 48$, just like $\langle c, d, e, f\rangle$
  - $|H_{1,2}|$ has order $48/8 = 6$, just like $\langle e, f\rangle$

The Mathieu group M_{10}

M_{10} is constructed as subgroup of the automorphism group of a 3-$(10, 4, 1)$-design. Outline:

- $\Delta = (P, B)$: a 3-$(10, 4, 1)$-design
  - Blocks:
$$\binom{10}{3}/4 = 30$$
  - Every point is on exactly 12 blocks: $\binom{9}{2}/3$
- If $\Delta = (P, B)$ is a 3-$(10, 4, 1)$-design, then the residue Δ_p at any point p is a 2-$(9, 3, 1)$-design
  - Take $\Theta = \Delta_{10}$. Any Δ-block not containing 10 meets any Θ-block in at most 2 points: for 3 common points would lead to another Δ-block containing 10, and thus 2 blocks on the three points.

The idea is to construct a 3-$(10, 4, 1)$-design from the 2-$(9, 3, 1)$-design, i.e., to determine 18 remaining blocks

Constructing a 3-$(10, 4, 1)$-design

- 4-**arc:** any 4 points of Θ meeting any Θ-block in at most 2 points.
- Number of 4-arcs:

$$\frac{9 \cdot 8 \cdot 6 \cdot 3}{4 \cdot 3 \cdot 2 \cdot 1} = 54$$

- Orbit-algorithm: H is transitive on 4-arcs
- Orbits under G: 3 orbits of size 18

Construction of 3-$(10, 4, 1)$-design:

- Points: $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
- Blocks:
  - 12 sets of the form $b \cup \{10\}$ with b a Θ-block
  - G orbit of $\{1, 2, 4, 5\}$, i.e., eighteen 4-arcs

  (One can show: arcs C_1, C_2 in the same G-orbit iff $|C_1 \cap C_2|$ is even.)

Constructing a 3-$(10, 4, 1)$-design (ctd)

Blocks:

- 12 sets of the form $b \cup \{10\}$ with b a Θ-block
- G orbit of $\{1, 2, 4, 5\}$, i.e., eighteen 4-arcs

Then the following holds (partially checked with GAP):

- Residue Δ_1 is a 2-$(9, 3, 1)$-design, with translation

$$g = (10, 2, 3)(4, 9, 8)(7, 6, 5)$$

  g leaves block set of Δ invariant
- $M_{10} := \langle G, g \rangle$, acting transitively on the 10 points.
- All residual designs of Δ are affine planes
- $|M_{10}| = 10 \cdot 9 \cdot 8 = 720$; G is point stabilizer of 10
- M_{10} is 3-transitive on the 10 points, transitive on the 30 blocks

The groups M_{11} and M_{12}

In a similar way the Mathieu groups M_{11} and M_{12} can be constructed, related to a 4-$(11, 5, 1)$-design and a 5-$(12, 6, 1)$-design.

- Three G-orbits:

$$O_{10} = \{1, 2, 4, 5\}^G, \quad O_{11} = \{1, 2, 4, 8\}^G, \quad O_{12} = \{1, 2, 4, 6\}^G$$

- Extend 2-$(9, 3, 1)$ design to $\Delta^{10} = \Delta$, Δ^{11}, Δ^{12} giving 3-$(10, 4, 1)$ designs
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$$g_2 = (11, 2, 3)(4, 6, 9)(7, 5, 8) \in \mathrm{Aut}(\Delta^{11}), \quad g_3 = (12, 2, 3)(4, 8, 5)(7, 9, 6) \in \mathrm{Aut}(\Delta^{12})$$

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$$M_{11} = \langle M_{10}, g_2 \rangle, \quad M_{12} = \langle M_{11}, g_3 \rangle$$

- $|M_{11}| = 11 \cdot 720 = 7920$ and $|M_{12}| = 12 \cdot |M_{11}| = 95040$

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