Parity games

Today: verification problem represented by parity games:

- graph game
- total graph
- two players: ♦ and □ (Even and Odd)
- vertices:
  - integer priority \( p(v) \)
  - owned by one player

Playing the game:

- token on vertex
- player does step to successor vertex if token is on vertex owned by him
- play: infinite sequence of steps
Definition (Parity game)
A parity game $\Gamma$ is a four tuple $(V, E, p, (V_\Diamond, V_\Box))$ where

- $(V, E)$ is a directed graph
- $V$ a set of vertices
- $E$ a total edge relation
- $p : V \to \mathbb{N}$ a priority function
- $(V_\Diamond, V_\Box)$ a partitioning of $V$, i.e. $V = V_\Diamond \cup V_\Box$ and $V_\Diamond \cap V_\Box = \emptyset$

Parity game (example)

$$V_\Diamond = \{s_2, s_3\}$$
$$V_\Box = \{s_1\}$$
$$p = \{s_1 \mapsto 1, s_2 \mapsto 2, s_3 \mapsto 3\}$$
Motivation

▶ BES can blow up exponentially (see e.g. Mader, section 6.4.2)
▶ Semantics of BES hard to understand
▶ Alternative model:
  • additional insights
  • other algorithms
  • graph model more intuitive and easier to understand
  • strong link with BES
▶ Algorithms still exponential

Winning a parity game

Definition (Winner)
Let:
▶ \( \pi = v_1, v_2, v_3, \ldots \) be a play
▶ \( \inf(\pi) \) be the set of priorities occurring infinitely often in \( \pi \)
Play \( \pi \) is winning for player \( \diamond \) iff \( \min(\inf(\pi)) \) is even
Winning a parity game (example)

- Play $s_1 s_2^\omega$ won by player ♦;
- Play $(s_1 s_2 s_3)^\omega$ won by player □.

Strategies

Definition (Strategy)
A strategy for ♦ is a partial function $\varrho_\bigcirc : V^* \times V_\bigcirc \to V$ that decides the vertex the token is played to based on the history of the vertices that has been visited.

- A play $\pi = v_1, v_2, v_3, \ldots$ is consistent with strategy $\varrho_\bigcirc$ for ♦ iff every $v_i \in \pi$ such that $v_i \in V_\bigcirc$ is immediately followed by $v_{i+1} = \varrho_\bigcirc(v_1, \ldots, v_i)$. 
Strategy (example)

Consider strategy $\varphi$ that, from $s_1$ plays token to $s_2$ if $s_1$ has been visited an even number of times, and to $s_3$ otherwise.

What if $\varphi$ always plays token from $s_2$ to $s_2$?

Winning

Definition (Winning strategy)

Strategy $\varphi$ is a winning strategy for $\bigcirc$ from set $W \subseteq V$ if every play starting from a vertex in $W$, consistent with $\varphi$ is winning for $\bigcirc$.

Goal of solving parity games: determine unique partitioning $(W_\bigcirc, W_\Box)$ of $V$ such that:

- Player $\bigcirc$ has winning strategy from $W_\bigcirc$
- Player $\Box$ has winning strategy from $W_\Box$
Winning strategy (example)

- \( \varphi(\ldots, s_2) = s_2 \)
- \( \varphi(\ldots, s_1) = s_3 \)
- \( \varphi(\ldots, s_3) = \begin{cases} s_1 & \text{if number of occurrences of } s_3 \text{ is prime} \\ s_3 & \text{otherwise} \end{cases} \)

Memoryless strategies

Theorem
For finding winning strategies it suffices to look at history free (also memoryless) strategies.

History free strategy is function \( \varphi: V \rightarrow V \), such that:
- vertex \( v_i \) always gets the same successor \( v_{i+1} \)
- independent of path by which \( v_i \) is reached.
Memoryless strategy (example)

Let $\varrho(s_2) = s_2$, $\varrho(s_3) = s_1$, and $\varrho(\Box s_1) = s_3$.

- $\varrho$ is winning from $\{s_2\}$
- $\varrho\Box$ is winning from $\{s_1, s_3\}$

Boolean Equation Systems

A Boolean expression is defined by:

$$f ::= X \mid \text{true} \mid \text{false} \mid f \land f \mid f \lor f$$

A Boolean equation is an equation of the form

$$\sigma X = f$$

A Boolean Equation System (BES) is a sequence of Boolean equations:

$$\mathcal{E} ::= \varepsilon \mid (\sigma X = f)\mathcal{E}$$
\[ \begin{align*}
\mu X &= X \land (Y \lor Z) \\
\nu Y &= W \lor (X \land Y) \\
\mu Z &= \text{false} \\
\mu W &= Z \lor (Z \lor W)
\end{align*} \]

**Notation**

**Rank:**
- natural number
- lowest rank always 0 or 1
- \( \text{rank}(X) \) indicates in which block of like-signed equations \( X \) occurs
- \( \text{rank}(X) \) is odd iff \( X \) is defined in a \( \mu \)-equation
- \( \text{rank}(X) \) inductively defined on structure of BES

**Operand:**
- \( \land, \lor \) or \( \perp \)
- \( \text{op}(X) \) indicates top-level boolean operator of equation for \( X \)
A BES is in **Standard Recursive Form** (SRF) if all right hand sides of Boolean Equations adhere to the following syntax:

\[ f := X \mid \lor F \mid \land F \]

- \( X \) is a proposition variable
- \( F \) is a non-empty set of proposition variables

Observe that:
- all BESs can be transformed into an equivalent BES in SRF
- this transformation can be done in polynomial time
Transformation to SRF

- Introduce equations for subformulae with different Boolean operand in last block of BES
- Introduce equation $\mu Xf = Xf$ for false, $\nu Xt = Xt$ for true, in arbitrary existing block if possible; replace all occurrences of false with $Xf$, and true with $Xt$

Consider the following BES:

This corresponds to the following BES in SRF:

\[
\begin{align*}
\mu X &= X \land (Y \lor Z) \\
\nu Y &= W \lor (X \land Y) \\
\mu Z &= \text{false} \\
\mu W &= Z \lor (Z \lor W) \\
\mu Xf &= Xf \\
\nu Y' &= X \land Y \\
\end{align*}
\]

Parity games vs BES

Definition (Parity game to BES)

Let $(V, E, p, (V_\diamond, V_\boxdot))$ be a parity game.

The corresponding closed BES in SRF contains the following equations:

\[
\begin{align*}
\sigma X_v &= \land \{X_{v'} \mid (v, v') \in E\} \quad \text{if } v \in V_\boxdot \\
\sigma X_v &= \lor \{X_{v'} \mid (v, v') \in E\} \quad \text{if } v \in V_\diamond \\
\end{align*}
\]

where:

- $\sigma = \mu$ if $p(v)$ is odd, $\sigma = \nu$ otherwise
- $X_v$ occurs before $X_u$ in the BES if $p(v) < p(u)$

Theorem

Solution to $X_v$ is true $\iff$ player $\diamond$ has winning strategy from $v$
Parity game to BES example

Corresponds to the following BES:

\[
\begin{align*}
\mu_{X_{s_1}} &= X_{s_2} \land X_{s_3} \\
\nu_{X_{s_2}} &= X_{s_2} \lor X_{s_3} \\
\mu_{X_{s_3}} &= X_{s_3} \lor X_{s_3}
\end{align*}
\]

Dependency graph of a BES in SRF

Let \( \mathcal{E} \) be a BES, then

- \( \text{bnd}(\mathcal{E}) \): variables occurring at the lhs of an equation in \( \mathcal{E} \);
- \( \text{occ}(\mathcal{E}) \): variables occurring at the rhs of an equation in \( \mathcal{E} \);
- \( \text{occ}(f) \): all variables occurring in formula \( f \).

Definition (Dependency graph)

Let \( \mathcal{E} \) be a BES. Its dependency graph \( G_\mathcal{E} \) is defined as \( (V, E) \), where:

- \( V = \{ V_X \mid X \in \text{bnd}(\mathcal{E}) \} \)
- \( (V_X, V_Y) \in E \) iff there is \( \sigma X = f \) in \( \mathcal{E} \) with \( Y \in \text{occ}(f) \)
Consider the following BES:

\[
\begin{align*}
\mu X &= X \land X' \\
\nu Y &= W \lor Y' \\
\mu Z &= Xf \\
\mu W &= Z \lor (Z \lor W) \\
\mu Xf &= Xf \\
\nu X' &= Y \lor Z \\
\nu Y' &= X \land Y
\end{align*}
\]

Its dependency graph is:

\[
\begin{tikzpicture}
  \node (X) at (0,0) {$V_X$};
  \node (X') at (0,-1) {$V_{X'}$};
  \node (Y) at (1,0) {$V_Y$};
  \node (Y') at (1,-1) {$V_{Y'}$};
  \node (Z) at (2,0) {$V_Z$};
  \node (W) at (2,-1) {$V_W$};
  \node (Xf) at (1,-2) {$V_{Xf}$};
  \node (W) at (2,-2) {$V_W$};

  \path
  (X) edge (Y)
  (X') edge (Y')
  (Z) edge (Y)
  (Z) edge (W)
  (Xf) edge (Z)
  (Xf) edge (W)
  (W) edge (Xf)
  (X) edge (Xf)

\end{tikzpicture}
\]

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**BES vs parity game**

**Definition (BES to parity game)**

Let \( \mathcal{E} \) be a closed BES in SRF. This corresponds to the parity game \( \Gamma = (V, E, p, (V_{\bigotimes}, V_{\boxtimes})) \), where

- \((V, E)\) is the dependency graph \( G_\mathcal{E} \) of \( \mathcal{E} \),
- \( p(V_X) = \text{rank}(X) \) for all variables \( X \in \text{bnd}(E) \),
- \( V_{\boxtimes} = \{ V_X \mid \text{op}(X) = \land \} \), so all conjunctive equations are assigned to \( V_{\boxtimes} \), and
- \( V_{\bigotimes} = V \setminus V_{\boxtimes} \), all other equations are assigned to \( V_{\bigotimes} \).

**Theorem**

*Player \( \bigotimes \) has winning strategy from \( V_X \) \iff the solution of \( X \) is true*
Consider the following BES:
\[
\begin{align*}
\mu X &= X \land X' \\
\nu Y &= W \lor Y' \\
\mu Z &= Xf \\
\mu W &= Z \lor (Z \lor W) \\
\mu Xf &= Xf \\
\nu X' &= Y \lor Z \\
\nu Y' &= X \land Y \\
\end{align*}
\]

Its parity game is:

```
1
\_{\mu X} = X \land f

2
\_{\nu X} = X \land f

3
\_{\nu X} = f

4
\_{\nu X} = X \lor f

5
\_{\nu X} = f

6
\_{\mu X} = X \lor f
```

Transformations on parity games

**Self-loop elimination**

```
1
\_{\mu X} = X \land f

1
\_{\mu X} = X

2
\_{\nu X} = X \lor f

2
\_{\nu X} = X

2
\_{\nu X} = f

2
\_{\nu X} = f

5
\_{\mu X} = X \lor f
```

**Priority compaction**

```
5
\rightarrow

3
```

In case priority 4 does not occur in the parity game. Evenness must be preserved!
Transformations on parity games

Priority propagation

Corresponds to re-ordering of equations in BES, which is generally unsafe!

Bisimulation

Definition (Bisimilarity of vertices)
Let $G = (V, E, p, (V_\Diamond, V_\Box))$ be a parity game. Let $R$ be a symmetric relation. $R$ is a bisimulation relation if $v R v'$ implies

- $v \in V_\Diamond \iff v' \in V_\Diamond$
- $p(v) = p(v')$
- $v \to w$ implies $\exists w'$ such that $v' \to w'$ and $w R w'$

Vertices $v$ and $v'$ are bisimilar ($v \equiv v'$) iff there exists a bisimulation relation $R$ such that $v R v'$.

Theorem
$v \equiv v'$ implies that $v$ and $v'$ are won by the same player
Consider the following modal $\mu$-calculus formula $f$:

$$\phi \equiv \nu X.\mu Y.(([r]X \land [s]X \land (\nu Z.\langle\neg s\rangle Z)) \lor ([r]Y \land [s]Y))$$

- Translate the model checking question $M \models f$ to a BES.
- Transform the resulting BES into a parity game.
- Determine the winner from vertex $V_{X_{s_0}}$.
- Provide winning strategy for this player from $V_{X_{s_0}}$. 