Fast Computation of Clebsch–Gordan Coefficients

Jesús A. De Loera, Tyrrell McAllister

1Mathematics, One Shields Avenue, Davis, CA 95616, USA

Abstract

We present an algorithm in the field of computational algebra, a field to which Weispfenning has made many contributions. Our algorithm computes Clebsch–Gordan coefficients for complex semisimple Lie algebras. Computing these numbers is \#P-hard in general. However, by combining recent developments in lattice point enumeration and in polyhedral realizations of algebraic constants, we can compute Clebsch–Gordan coefficients in polynomial time when the rank is fixed. Our experiments show that this algorithm is superior in practice to the standard techniques for computing these coefficients when the weights have large size but small rank. Using our software, we find experimental evidence for a conjectured generalization of the Saturation Theorem.

1 Introduction

Weispfenning’s contributions to the field of algorithmic algebra are extensive and well known. He has also supported algorithmic investigations in all parts of algebra. See, for example, [GKW03]. Here we present new computational techniques in the representation theory of complex semisimple Lie algebras. It is a pleasure to congratulate Prof. Weispfenning on his birthday.

Complex semisimple Lie algebras are classified by their associated root systems. Our work concerns the so-called classical root systems \( A_r, B_r, C_r, D_r, r = 1, 2, \ldots \). By a classical result due to Weyl, an irreducible representation of a complex semisimple Lie algebra is determined by a particular element of the associated root lattice called the “highest weight” of the representation. We denote the irreducible representation with highest weight \( \lambda \) by \( V_\lambda \).

Given highest weights \( \lambda, \mu, \) and \( \nu \) for a complex semisimple Lie algebra \( g \), we denote by \( C_{\lambda,\mu}^\nu \) the multiplicity of the irreducible representation \( V_\nu \) in the tensor product of \( V_\lambda \) and \( V_\mu \); that is, we write

\[
V_\lambda \otimes V_\mu = \bigoplus_\nu C_{\lambda,\mu}^\nu V_\nu.
\]
The values $C_{\lambda,\mu}^\nu$ are known as Clebsch–Gordan coefficients. In the particular case in which $g$ is of type $A_r$, they are also called Littlewood–Richardson numbers.

The concrete computation of Clebsch–Gordan coefficients has attracted a lot of attention from not only representation theorists, but also from physicists, who employ them in the study of quantum mechanics (e.g. [CdG96, Wyb90]). The importance of these coefficients is also evidenced by their widespread appearance in other fields of mathematics besides representation theory. For example, the Littlewood–Richardson coefficients appear in combinatorics via symmetric functions and in enumerative algebraic geometry via Schubert varieties and Grassmannians (see for instance [Sta77, Ful97]). More recently, Clebsch–Gordan coefficients are playing an important role on the study of $P$ vs. $NP$ (see [MS03]). Very recently, Narayanan has proved that the computations of Clebsch–Gordan coefficients is in general $\#P$-complete [Nar]. Here are our contributions:

Our contribution is to combine recent developments in the polyhedral realization of Clebsch–Gordan coefficients with the lattice point enumeration algorithm of Barvinok ([Bar94]) to produce an algorithm for computing Clebsch–Gordan coefficients in polynomial time in the sizes of the weights when the rank $r$ is fixed.

The remainder of this abstract is organized as follows. In Section 2, we briefly describe the results of Knutson and Tao ([KT99]) and of Berenstein and Zelevinsky ([BZ01]) which provide the polytopes needed by the algorithm. In Section 3, we describe some of the advantages of using lattice point enumeration to compute Clebsch–Gordan coefficients, and we make a conjecture that generalizes the Saturation Theorem of Knutson and Tao. Finally, we make some concluding remarks.

## 2 The Algorithm

In 1992 ([BZ92]), Berenstein and Zelevinsky presented a combinatorial interpretation of the Littlewood–Richardson numbers as the number of lattice points in members of a certain family of polytopes. In 1999, Knutson and Tao introduced another family, the hive polytopes, which they used to prove the Saturation Theorem. Each of the polytopes presented by Berenstein and Zelevinsky in 1992 is the image under an injective lattice-preserving linear map of a hive polytope ([PV]). Therefore $C_{\lambda,\mu}^\nu$ equals the number of integer lattice points in a corresponding hive polytope. Finally, in 2001, Berenstein and Zelevinsky ([BZ01]) introduced polytopes which enumerate Clebsch–Gordan coefficients for any complex semisimple Lie algebra. We refer to this last family of polytopes as the $BZ$-polytopes.

Our algorithm uses the hive polytopes to compute Littlewood–Richardson numbers, that is, to compute Clebsch–Gordan coefficients in the type $A_r$ case. To compute the coefficients in the other classical root systems, we turn the the $BZ$-polytopes.
Now that the computation has been reduced to a lattice point enumeration problem, we use an algorithm of Barvinok, which allows us to compute the number of integer lattice points in a rational polytope in polynomial time when the dimension is fixed. Barvinok’s algorithm has been implemented in the software \texttt{LattE} by a team led by the first author.

### 3 Analysis and a Conjecture

Based on the explicit definitions for the hive and BZ-polytopes as the sets of solutions to systems of linear equalities and inequalities, we wrote a \texttt{Maple} notebook that, when given a triple of highest weights, produces the corresponding hive or BZ-polytope in \texttt{LattE}-readable input format.

The polyhedral method of computing Clebsch–Gordan coefficients complements the methods employed in standard software such as \texttt{LiE}. \texttt{LiE} is effective for relatively large ranks, but the sizes of the weights must be kept small. This is because \texttt{LiE} uses the Klymik formula to generate the entire direct sum decomposition of the tensor product, after which it dispenses with all but the single desired term. However, computing all of the terms in the direct sum decomposition is not feasible when the sizes of the entries in the weights grow into the 100s.

On the other hand, lattice point enumeration is often very effective for large weights, so long as the rank is relatively low. This is because lattice point enumeration is polynomial time in the input size, but it is exponential time in the dimension of the polytope, and the dimensions of these polytopes grow quadratically with the rank of the Lie algebra. In practice, this means that for rank $r \leq 5$, some Clebsch–Gordan problems will cause memory overfill errors when using \texttt{LiE}, but these same problems can be computed in seconds using lattice point enumeration with \texttt{LattE}. This is the sense in which our software provides a complement to the standard techniques.

In 1999, Knutson and Tau used the hive polytopes to prove the Saturation Theorem, which states, in polyhedral language, that every hive polytope contains an integral lattice point. Since lattice point enumeration works effectively with large weights, we can compute the Ehrhart quasi-polynomials $f(n) = C_{\nu n \lambda \mu}$ of the BZ-polytopes. These computations motivate a conjectured generalization of the Saturation Theorem.

**Conjecture 1** Given an irreducible representation of a complex semisimple Lie algebra of type $A_r, B_r, C_r,$ or $D_r$, the coefficients of the Ehrhart quasi-polynomial of the associated BZ-polytope are all nonnegative.

The type $A_r$ case of this conjecture was conjectured by King, Tollu, and Toumazet in [KTT04]. That our conjecture implies the Saturation Theorem follows from a result of Derk-
sen and Weyman ([DW02]) showing that the Ehrhart quasi-polynomials of Hive polytopes are in fact just polynomials.

## 4 Conclusion

The intersection of representation theory and polyhedral geometry is proving to be fertile ground from which both fields are reaping benefits. The algorithm we described above is an example of this. An algorithm designed to enumerate lattice points in polyhedra can be directly applied to compute Clebsch–Gordan coefficients of complex semisimple Lie algebras. Moreover, this algorithm makes possible experiments which motivate a new conjecture about the behavior of these coefficients under the operation of stretching the weights.

## References

Clebsch–Gordan Coefficients


Jesús A. De Loera received a Bachelor in Science degree from the National University of Mexico in 1989. In 1995 he completed his Ph.D work at the Center for Applied Mathematics of Cornell under the direction of Bernd Sturmfels. He held postdoctoral positions at ETH-Zürich Switzerland and the University of Minnesota. He is currently an associate professor of mathematics at the University of California Davis. His main research interests are algorithmic and discrete mathematics. In 2003 he was the chair of the MSRI Berkeley program in “Discrete and Computational Geometry”. In 2004 he was awarded an Alexander von Humboldt fellowship.

deloera@math.ucdavis.edu

Tyrrell McAllister received his B.S. in 2001 from the University of California at Davis. He is currently working towards his Ph.D. in the mathematics department at UC Davis under the direction of Jesús A. De Loera. His research interests are in combinatorics and the use of polyhedral geometry in representation theory.

tmcal@math.ucdavis.edu