2IV60 Computer Graphics

Examination, April 11 2016, 18:00 – 21:00

This examination consist of **four** questions with in total 16 subquestions. Each subquestion weighs equally. In all cases: **EXPLAIN YOUR ANSWER. Use sketches where needed to clarify your answer. Read first all questions completely.** If a function is asked, then a description in steps or pseudo-code is expected, which is clear enough to be easily transferred to real code. Aim at compactness and clarity. The use of the book, copies of slides, notes and other material is not allowed.

1 We consider some basic concepts of computer graphics.

a) What is the difference between flat surface rendering and Gouraud surface rendering?

In flat surface rendering, each triangle is given a uniform color; in Gouraud surface rendering colors are computed for vertices and linearly interpolated.

b) Give a characteristic property of a *convex* polygon.

Some typical properties are:

- All interior angles at vertices are smaller than 180 degrees;
- All line segments between two interior points are completely inside the polygon;
- If we take an infinite line through an edge, connecting two vertices, all other vertices are at the same side of this line;
- From any point inside the polygon the complete boundary is visible.

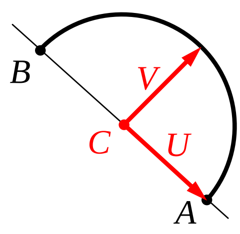
c) What is a viewport in computer graphics terminology?

A viewport is an area (typically a rectangle on the screen or canvas) where the graphics output has to be shown.

d) Describe the simplest version of the depth-sort or painter's algorithm.

In the depth-sort algorithm, first all elements (triangles) are sorted for depth (using the closest vertex or the average depth), next they are rendered from back to front, such that triangles that are drawn later (closer to the viewer) overwrite ones that are drawn earlier (farther from the viewer). At least, that's what one hopes, there are situations where this does not give the correct result.

2 We are given two points $A = (A_x, A_y)$ and $B = (B_x, B_y)$. We want to draw half a circle through these points, at the right side of the line from A to B (see the figure). Answer the following questions, use sketches to illustrate your answers:



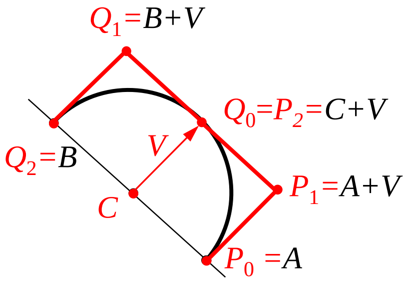
a) The half circle can be modeled as $R(t) = C + U\cos(\pi t) + V\sin(\pi t)$, $0 \le t \le 1$. Give the point C and the vectors U and V.
C is a point half-way A and B: $C = (A+B)/2$.
For U we use a vector from C to A: $U = A - C = (A-B)/2$.
V has the same length as U and is orthogonal to U. We apply a rotation of 90 degrees counterclockwise: $V = (-U_y, U_x)$

b) Suppose we use two quadratic Bézier segments P and Q given by

$P(t) = (1-t)^2 P_0 + 2t(1-t)P_1 + t^2 P_2$, and

$Q(t) = (1-t)^2 Q_0 + 2t(1-t)Q_1 + t^2 Q_2$.

to approximate the half circle. The resulting curve must be smooth, and perpendicular to the line *AB* at the start and end. Furthermore, the curve must pass through R(1/2) and must be parallel to AB, halfway the curve. Draw a control polygon and give the positions of the control points.



We use the vectors derived in question a):

$P_0 = A$; $P_1 = A+V$; $P_2 = C+V$;

$Q_0 = C+V$; $Q_1 = B+V$; $Q_2 = B$.

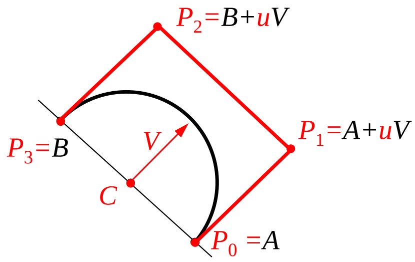
This gives a curve, consisting of two segments, that meet all requirements:

- It starts at *A* and ends at *B;*
- At the start ($P(0)$), the direction for a Bézier curve is from P_0 to P_1 , which is $P_1 - P_0 = V$, perpendicular to *AB;* the same holds for the end ($Q(0)$);
- The curve is smooth, because the endpoint of the first segment matches with the starting point of the second segment ($P_2 = Q_0$), also, here the directions match: $P_2 - P_1 = Q_1 - Q_0 = U$;
- The point halfway lies at the circle: $P_2 = Q_0 = C+V=R(1/2)$.

c) Explain why in question b) two segments were needed to meet the requirements.
If we use a single segment, the first and last point are obvious: match with *A* and *B.* The intermediate control point has to lie on a line perpendicular to *AB* through *A*, as well as on a line perpendicular to *AB* through *B*, to meet the requirements for perpendicularity. However, these lines are parallel, so we cannot find a point that lies on both lines.

d) Suppose we use one cubic Bézier segment to approximate the half circle, where a segment is given by $P(t) = (1-t)^3 P_0 + 3t(1-t)^2 P_1 + 3t^2(1-t)P_2 + t^3 P_3$.

The requirements are furthermore the same as in question b). Draw a control polygon and give the positions of the control points.



We use again the vectors derived in question a):

$P_0 = A$; $P_1 = A+uV$; $P_2 = B+uV$; $P_3 = B$.

This makes sure that

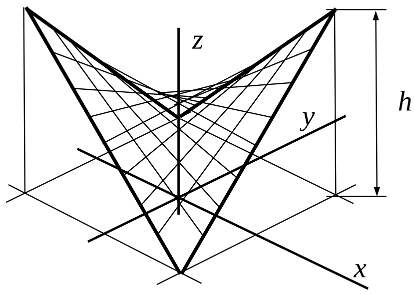
- The curve starts at *A* and ends at *B;*
- At the start ($P(0)$), the direction is $P_1 - P_0 = uV$, and hence perpendicular to *AB;* the same holds for the end ($Q(0)$). We do not know yet where to put P_1 exactly, but it has to be on the half line $A+uV$, $u > 0$. The same holds for P_2, which has to be on the half line $B+uV$, $u > 0$.
- To enforce symmetry across the line $C+wV$, we use the parameter u twice, both for P_1 and P_2. As a result, the tangent line at $P(1/2)$ will be parallel to the line *AB.*

We finally have to fix the value of u, such that the curve passes through the point $R(1/2) = C+V$. We find:

$$\begin{aligned} P(1/2) &= (1/8)\, P_0 + (3/8)\, P_1 + (3/8)\, P_2 + (1/8)\, P_3 \\ &= (1/8)\, A + (3/8)\, (A+uV) + (3/8)\, (B+uV) + (1/8)\, B \\ &= (1/8 + 3/8)\, A + (3/8 + 3/8)\, uV + (3/8 + 1/8)\, B \\ &= (A + B)/2 + (3/4)uV \\ &= C + (3/4)uV \end{aligned}$$

To make sure that $R(1/2) = C+V$, we have to satisfy $(3/4)u = 1$, hence $u = 4/3$.

3 We consider a surface $z = axy + b$, with $-1 \le x, y \le 1$, see the figure. Answer the following questions.



a) What values should a and b have such that the minimum value of z is 0 and that the maximum value of z is equal to h?
The minima for z are reached for $xy = -1$: at opposite corners of the domain, at points $(1, -1, -a + b)$ and $(-1, 1, -a + b)$. This gives the constraint that $-a + b = 0$, hence $b = a$.
The maxima for z are reached for $xy = 1$, again at opposite corners of the domain, at point $(1, 1, a + b)$ and $(-1, -1, a + b)$. This gives the constraint that $a + b = h$. Substituting $b = a$ gives

$$a = b = h/2.$$

b) Give a parametric description of the surface: define the surface as $P(u,v)$, $0 \le u,v \le 1$.
A parametric description using x and y as parameters is simply $P(x, y) = (x, y, hxy/2 + h/2)$. However, x and y do not have the proper range: $[-1, 1]$ instead of $[0, 1]$.
We align the u parameter with the x direction and apply a linear transformation $x = pu + q$. We require $x(0) = -1$ and $x(1) = 1$, which gives $q = -1$ and $p + q = 1$. Elaboration gives $x = 2u - 1$. Similarly, we find $y = 2v - 1$. Substitution in the given surface equation gives $z = h(2u - 1)(2v - 1)/2 + h/2$. Combining all this gives the required parametric description of the surface:

$$P(u,v) = (2u - 1, 2v - 1, h(2u - 1)(2v - 1)/2 + h/2),$$ or, alternatively,

$$P(u,v) = (2u - 1, 2v - 1, h(2uv - u - v + 1)).$$

c) Given a point on the surface (by coordinates or by parameter values, choose yourself), give an expression for a normal vector N on the surface for that point.
Using an implicit definition of the surface, $f(x, y, z) = z - hxy/2 - h/2$, is the simplest here:

$$N(x, y, z) = \nabla f(x, y, z) = (-hy/2, -hx/2, 1).$$

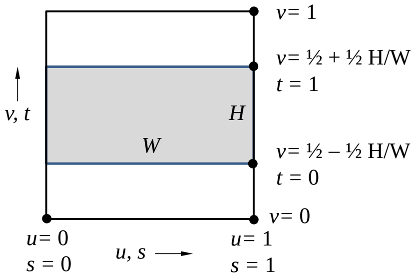
Using the parametric description gives:

$$\begin{aligned} N(u, v) &= (\partial P(u,v) / \partial u) \times (\partial P(u,v) / \partial v) \\ &= (2, 0, 2hv - h) \times (0, 2, 2hu - h) \\ &= (2h - 4hv, 2h - 4hu, 4). \end{aligned}$$

d) We want to apply texture mapping to this surface. We are given an image with width W and height H, $W > H$. The image is given as a texture in a unit square $0 \le s, t \le 1$. When using isometric projection with camera position $(0, 0, 10h)$, center point $(0, 0, 0)$, and view-up vector $(0, 1, 0)$, we want to see the image horizontally and centered on the screen, as large as possible, without distortion. Give the texture coordinates $s(u,v)$ and $t(u, v)$ to achieve this.

In the sketch we show the top view of the surface, and marked some points, with the associated values for u, v, s, and t. We align u and s. This is straightforward: if $u = 0$, then $s = 0$; if $u = 1$, then $s = 1$: the left side of the image is mapped to the left side of the surface, and the same for the right sides. Hence we can use:

$$s(u, v) = u.$$



In the vertical direction, things are more involved. We align v and t, but directly using $t = v$ would stretch the image. We see that to prevent distortion, the bottom side of the image (with $t = 0$) has to be positioned at $v = ½ − ½\, H/W$: the relative height of the image is H/W, and to center it, we have to start at half this size below the centreline $v = ½$. At the upper side of the image, we find $t = 1$ and $v = ½ + ½\, H/W$. Next, we need a linear mapping from v to t to accomplish this. We set $t(u, v) = pv + q$, and the constraints are:

$0 = p\,(½ − ½\, H/W) + q$ and
$1 = p\,(½ + ½\, H/W) + q$.

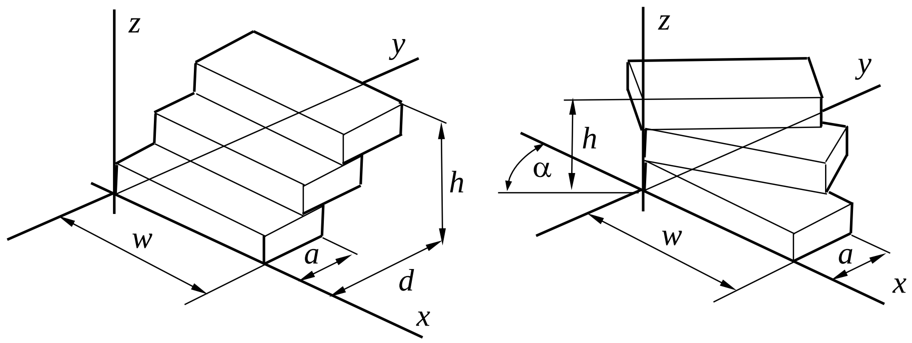
Solving this gives $p = W/H$ and $q = ½\,(1 − W/H)$, hence

$t(u, v) = (W/H)(v − ½) + ½$.

4 Suppose, a stair has N steps, each step has width w and depth a. The total height is h; for a straightforward stair, the total depth is d. See the left figure, where a stair with $N = 3$ is shown. Assume that a function *DrawStep*(M) is available to draw a step, subject to a homogenous transformation M. A step in local coordinates is a block $−1 \le x, y, z \le 1$, centered around the origin. Points X' in world coordinates are derived from local coordinates X by $X' = MX$, where M is a homogenous transformation matrix.

Furthermore, suppose that a standard set of functions to define transformation matrices is available:

- $T(d_x, d_y, d_z)$ gives a translation matrix over a vector (d_x, d_y, d_z);
- $S(s_x, s_y, s_z)$ gives a scaling matrix with scale factors s_x , s_y, and s_z for the axes; and
- $R_k(a)$ gives a counterclockwise rotation over a radians around axis k (= x, y, or z).



a) Define a matrix M_i to draw step i , $i = 1, \ldots, N$, for the straightforward stair.

We first translate the block, such that one corner is located in the origin, and the rest of the block in an octant where all coordinates are non-negative, using $T(1, 1, 1)$. Next, we scale the block to give it the right dimensions, using $S(w/2, a/2, h/(2N))$: a combination of scaling with ½ in all directions, and scaling with the width w, depth a, and height h/N of the step. This gives us M_1. The next steps have to be translated. We consider the translation to be applied for step N. In the x-direction, no motion has to be applied. In the y-direction, the translation is $d − a$, move along the full length, but go back one step-width. In the z-direction, the translation is $h − h / N$: move the full height, but go back one step-height. For an intermediate step, we find that a fraction $(i−1)/(N−1)$ of this translation has to be applied. Combining all this gives:

$M_i = T(0, (d − a)(i−1)/(N−1), (h − h / N)(i−1)/(N−1))\; S(w/2, a/2, h/(2N))\; T(1, 1, 1)$,

which can be simplified to

$M_i = T(0, (d − a)(i−1)/(N−1), (i−1)h/N)\; S(w/2, a/2, h/(2N))\; T(1, 1, 1)$,

Quite some variations are possible. For instance, the initial translation and scaling can be changed:

$M_i = T(0, (d − a)(i−1)/(N−1), (i−1)h/N)\; T(w/2, a/2, h/(2N))\; S(w/2, a/2, h/(2N))$.

b) Suppose that someone is ascending the stair, and that we want to animate what he sees. Assume the person is initially standing on the center of step 1, and that the camera is located at a height e above the step, and that finally, the person is standing on the center of step N. Assume that the motion of the camera is smooth and follows a linear path. Define the position $C(t)$ of the camera, with $0 \le t \le T$.

The starting point P is located at the center of the first step, shifted with e in z-direction, hence we find $P = (w/2, a/2, h/N + e)$. For the end point Q we find $Q = (w/2, d - a/2, h + e)$. Linear interpolation between these points gives:

$C(t) = P + (Q - P)\, t / T$, in coordinates:

$C(t) = (w/2, a/2, h/N + e) + (0, d - a, h\,(1 - 1/N))$.

Alternatively, we can take advantage of the answer on question a). For instance, we can define P and Q as $P = \mathrm{M}_1\,(1/2, 1/2, 1, 1)^T + (0, 0, e, 0)^T$ and $Q = \mathrm{M}_N\,(1/2, 1/2, 1, 1)^T + (0, 0, e, 0)^T$. We define points and vectors in homogenous coordinates (hence the fourth coordinate) and as column vectors (hence the transposition).

c) We consider a spiral stair (see the right figure), where the total rotation is α. Define the matrix M_i to draw step i, $i = 1, \ldots, N$, for this spiral stair.

The first transformations, working towards M_1, are similar as for question a). Next, we apply a rotation around the zaxis: $R_z\,(\alpha(i-1) / (N-1))$ and a vertical translation $T(0, 0, (h - h / N)\,(i-1)/(N-1))$. These steps can be interchanged freely. Combining everything gives:

$\mathrm{M}_i = T(0, 0, (h - h / N)\,(i-1)/(N-1))\; R_z\,(\alpha(i-1) / (N-1))\;\; S(w/2, a/2, h/(2N))\; T\,(1, 1, 1)$.

d) We want to make a similar animation as in question b) for the spiral stair. The camera follows a smooth, spiral curve. What is the position $C(t)$?

The starting point is the same as in question *b*):

$P = (w/2, a/2, h/N + e)$.

This point has to be rotated around the z-axis and shifted upwards. The latter can be done by adding a vector $(0, 0, (h-h/N)\, t/T)$. The rotation can be done by using the pattern of a standard 2D rotation $((x', y') = (\cos(u)x - \sin(u)y, \sin(u)x + \cos(u)y)$ over an angle $u = \alpha t/T$, giving:

$C(t) = (\cos(u)w/2 - \sin(u)a/2, \sin(u)w/2 + \cos(u)a/2,\; h/N + e + (h-h/N)\, t/T)$.

Alternatively, using the given standard transformation matrices, we get:

$C(t) = R_z(\alpha t/T)\,(w/2, a/2, h/N + e, 1)^T + (0, 0, (h-h/N)\, t/T)^T$.

As yet another alternative, we can take advantage again of the answer of the previous question c). The index i denoted the number of the step, we can also consider it as a real number, in the range [1, N]. We can map t to this range, using

$i = 1 + (t / T)\,(N - 1)$,

and define $C(t)$ as $C(t) = \mathrm{M}_i\,(1/2, 1/2, 1, 1)^T + (0, 0, e, 0)^T$.