2IV60 – Exercise 5: Splines

1. A segment of a *quadratic* Bézier-spline is given by

   $$P(t) = (1-t)^2 P_0 + 2t(1-t)P_1 + t^2 P_2, \text{ with } 0 \le t \le 1,$$

   where P_0, P_1 and P_2 are control points (2D: $P_i = (P_{ix}, P_{iy})$ or 3D: $P_i = (P_{ix}, P_{iy}, P_{iz})$), and $P(t)$ is a point on the curve. Give expressions for:

   a) $P(0)$, $P(1/2)$ and $P(1)$;
   $P(0) = P_0$; $P(1/2) = P_0/4 + P_1/2 + P_2/4$; $P(1) = P_2$

   b) $P'(t)$, $P'(0)$, $P(1/2)$ and $P(1)$;
   $P'(t) = -2(1-t)P_0 + (2(1-t) - 2t)P_1 + 2t\, P_2$
   $\quad = 2((1-t)(P_1 - P_0) + t(P_2 - P_1))$; // *what is the geometric interpretation of this?*
   $P'(0) = 2(P_1 - P_0)$;
   $P'(1/2) = P_2 - P_0$;
   $P'(1) = 2(P_2 - P_1)$

   c) a tangent line to $P(t)$;
   $Q(t,s) = P(t) + sP'(t)$

   d) make a sketch of such a segment and describe the most important characteristics.
   The curve starts in P_0 and ends in P_2. The curve is tangent to the line P_0P_1 at the start and tangent to the line P_1P_2 at the end. Halfway the point P_1 is aproached, the tangent line has there the direction of the line P_2P_0.

2. A segment of a *cubic* Bézier-spline is given by

   $$P(t) = (1-t)^3 P_0 + 3t(1-t)^2 P_1 + 3t^2(1-t)P_2 + t^3 P_3, \text{ with } 0 \le t \le 1,$$

   where P_0, P_1, P_2 and P_3 are control points (2D or 3D) and $P(t)$ is a point on the curve. Give expressions for:

   a) $P(0)$, $P(1/2)$ and $P(1)$;
   $P(0) = P_0$; $P(1/2) = P_0/8 + 3P_1/8 + 3P_1/8 + P_3/8$; $P(1) = P_3$

   b) $P'(t)$, $P'(0)$, $P(1/2)$ and $P(1)$;
   $P'(t) = -3(1-t)^2P_0 + 3((1-t)^2 - 2\,t(1-t))\,P_1 + 3(2t(1-t) - t^2)\,P_2 + 3t^2\,P_3$
   $\quad = 3((1-t)^2(P_1 - P_0) + 2t(1-t)(P_2 - P_1) + t^2(P_3 - P_2))$ // *what does this look like?*
   $P'(0) = 3(P_1 - P_0)$;
   $P'(1/2) = (3/4)(P_3 + P_2 - P_1 - P_0)$
   $P'(1) = 3(P_3 - P_2)$

   c) a tangent line to $P(t)$;
   $Q(t,s) = P(t) + sP'(t)$

   and
   d) make a sketch of such a segment and describe the most important characteristics.

The curve starts in P_0 and ends in P_3 . The curve is tangent to the line P_0P_1 at the start and tangent to the line P_2P_3 at the end. Halfway the middle of the line P_1P_2 is approached, and the tangent line has there a direction which is the "average" of the lines P_3P_0 en P_2P_1 .

3. Given the three points *A, B* and *C*. We can connect these with straight line segments *AB* and *BC*, but then we get a sharp corner at B. It is desired to smooth the corner by replacing the two line segments with a Bézier-spline, such that the tangent lines at *A* and *B* do not change. Give all possible solutions. If there are multiple, how to make a "good" choice? Consider this for:
   a) a quadratic Bézier-spline;
   For a quadratic Bézier-spline we only have one option: We use *A, B* en *C* as control points P_0, P_1 and P_2 . The curve then satisfies the requirement (see also 1d).

   b) a cubic Bézier-spline.
   Here we have more freedom. We select *A* for control point P_0 and *C* for control point P_3. The curve hence starts at *A* and ends at *C*. Furthermore, the requirement that the tangent lines at *A* and *B* do not change leads to:
   $P_1 = A + a(B–A)$ and
   $P_2 = C + b(B–C)$.
   Here are *a* and *b* two additional parameters thant can be chosen. Symmetry suggests to choose $a=b$. Furthermore, to make sure that the curve has the right direction, it should hold that $a > 0$. The next puzzle is to choose an "optimal" value for *a*. That depends on what one aims at. A too large value gives a loop, a too small value gives a flattened corner. In practice a value of a=1/2 or a=2/3 will usually give a good result. By forcing that in certain standard situations a standard result is obtained a more precise value can be obtained. We can for instance demand that if *AB* and *BC* have equal lengths, the midpoint of the curves goes through a circle. Another criterion could be to minimize the variation of the length of the tangent vector, which can be obtained by selecting *a* such that $|P'(0)| = |P'(1/2)|$.

4. Given a quadratic Bézier-spline with three 3D control points P, Q and R.
   a) Determine the crossing points of this curve with the plane z=0.
   We are looking for a value of *t* such that $P_z(t) = 0$, or:
   $(1–t)^2P_z + 2t(1-t)\, Q_z + t^2 R_z = 0$. We rewrite this to a standard quadratic equation by equating terms and obtain:
   $At^2 + Bt + C = 0$, with
   $A = P_z – 2Q_z + R_z$
   $B = 2Q_z – 2P_z$
   $C = P_z$.
   We can solve this quadratic equation with the well-known ABC-formula, and get 0, 1 or 2 values for *t*. Next we check if these are in the interval [0, 1].

   b) Suppose that $P_z = R_z = h$, for which value of Q_z do we find exactly 1 crossing point?
   This is the case if the discriminant $D = B^2 – 4AC$ is equal to 0. Call the value of Q_z for the sake of convenience p. Substitution gives: $A = 2(h–p)$; $B = 2(p–h)$; $C = h$ and, after a small calculation $D=4(p^2–h^2)$. It holds that D=0 if $p = h$ or if $p = –h$. In the first case the curve is degenerated (vertically), the height is then constantly

equal to h; in the second case the curve is exactly tangent to the z=0 plane. At least, if $h \neq 0$, because otherwise the complete curve is located in the plane z=0.

5. A generic procedure is desired for drawing the spade symbol: ♠.
   a) Select a number of suitable parameters to describe the shape globally and that enable easy generation of variants.

   We model half of the symbol, the other half is obtained by reflection. We choose the parameters a (height foot), b (height blade), c (width foot) and d (width blade). Further we make two angles (α and β) controllable (see figure).

   b) Define the contour as a sequence of Bézier curves, where the positions of the control points depend on the parameters chosen.

   We model the contour with a straight line segment AB, and three quadratic Bézier-curves (BCD, DEF, FGH). We select for A the origin ($A = (0,0)$), and see that then simply $B = (c, 0)$, $D = (0, a)$ and $H=(0, a+b)$. Furthermore, $E = (d, a - d \tan \beta)$ and $G = (d, a + b - d \tan \alpha)$. We choose F halfway G and E ($F = (G+E)/2$). If more flexibility is desired, an extra parameter could be used. The same holds for the point C, we can choose it for instance halfway B and D, with a little displacement in the direction of A: $C = (4B + 4D + 2A)/10$. The values of the weights are not really critical here, as long as they sum up to 1.

   This is of course just one possible solution, if more detail and control is desired, then more points and parameters can be used.