2IV10/2IV60 Computer Graphics

Intermediate Examination, December 14 2015, 10:45 – 12:30

This examination consist of **three** questions with in total 9 subquestion. Each subquestion weighs equally. In all cases: **EXPLAIN YOUR ANSWER. Use sketches where needed to clarify your answer. Read first all questions completely.** If an algorithm is asked, then a description in steps or pseudo-code is expected, which is clear enough to be easily transferred to real code. Aim at compactness and clarity. Use additional functions and procedures if desired. Give for each function and procedure a short description of input and output. The use of the book, copies of slides, notes and other material is not allowed.

1 We use 2D transformations to transform a square, defined by $0 \le x \le 1$, and $0 \le y \le 1$ in local coordinates. New positions X are derived from local positions X' by $X=MX'$, where M is a homogeneous transformation matrix. Suppose that we have a set of functions to define basic transformation matrices, *i.e.*,

- $T(x,y)$ gives a translation matrix over a vector (x,y);
- $S(s_x, s_y)$ gives a scaling matrix with scale factors s_x and s_y for the axes; and
- $R(a)$ gives a counterclockwise rotation over a radians around the origin.

a) Give a matrix M, expressed in the basic transformation functions, to obtain a rectangle with width w and height h, centered around the origin, and rotated over an angle α, with $0 \le \alpha < \pi/2$.

Using global transforms:we first center the rectangle ($T(-1/2, -1/2)$; next we scale the rectangle ($S(w, h)$); and finally rotate the rectangle ($R(\alpha)$). Multiplying these transformations from left to right gives: $M = R(\alpha)\, S(w, h)\, T(-1/2, -1/2)$.

The scaling and translation can be interchanged, this gives: $M = R(\alpha)\, T(-w/2, -h/2) S(w, h)$.

b) Calculate a bounding box, defined by $x_{min} \le x \le x_{max}$ and $y_{min} \le y \le y_{max}$, around the rectangle of question 1 a). The values x_{min}, x_{max}, y_{min}, and y_{max} must be chosen such that the bounding box fits tightly around the rectangle.

Using a sketch, for the center R of the right boundary of the rectangle we find:

$R = [(w/2) \cos \alpha, (w/2) \sin \alpha]$,
for the center T of the top boundary we find
$T = [-(h/2) \sin \alpha, (h/2) \cos \alpha]$.

Using $0 \le \alpha < \pi/2$, we see:
the point $R + T$ has the largest y-coordinate: $y_{max} = (w/2) \sin \alpha + (h/2) \cos \alpha$;
the point $R - T$ has the largest x-coordinate: $x_{max} = (w/2) \cos \alpha + (h/2) \sin \alpha$.

The rectangle is symmetric around the origin, hence $x_{min} = -x_{max}$ and $y_{min} = -y_{max}$.

c) Suppose that the (rotated) rectangle of question 1 a) must be displayed in a viewport with width a pixels and height b pixels. We use a clipping window, defined by $wx_{min} \le x \le wx_{max}$ and $wy_{min} \le y \le wy_{max}$. Calculate the values of wx_{min}, wx_{max}, wy_{min}, and wy_{max} such that the rectangle is displayed as large as possible, centered in the viewport and without distortion.

The window should have the same aspect ratio as the viewport:
$(wy_{max} - wy_{min}) / (wx_{max} - wx_{min}) = b / a$. (1)

Furthermore, the window should fit around the rectangle. The width w' of the bounding box is $w' = x_{max} - x_{min}$, the height $h' = y_{max} - y_{min}$. Now, two different cases can occur.

First, if $h' / w' < b / a$, the width w' of the bounding box is dominant. The viewport has to be chosen such that it tightly matches with the horizontal dimensions of the bounding box:
Hence:
$wx_{min} = x_{min}$, and
$wx_{max} = x_{max}$.

The height has to be adjusted such that the resulting viewport has the b / a aspect-ratio. We scale relative to the vertical center $(y_{max} + y_{min})/2$ to get:
$wy_{min} = (y_{max} + y_{min})/2 + w'b / (2a)$, and
$wy_{max} = (y_{max} + y_{min})/2 - w'b / (2a)$.

Substitution in equation (1) shows that this gives the result aimed at.

Second, if $h' / w' > b / a$, the height h' of the bounding box is dominant. Using similar reasoning, this gives
$wy_{min} = y_{min}$,
$wy_{max} = y_{max}$.
$wx_{min} = (x_{max} + x_{min})/2 + h'a / (2b)$, and
$wx_{max} = (x_{max} + x_{min})/2 - h'a / (2b)$.

2 Given a triangle PQR in 3D space.

a) Give a formula for $N(P, Q, R)$: the normal vector on a triangle, and use this to calculate the equation $f(X) = 0$ of the plane in which the triangle is embedded, where X is a point in 3D space. Standard vector operations must be used, do not use vector coordinates explicitly.

We use the cross-product of two vectors along succeeding edges to get the normal: $N(P, Q, R) = (Q - P) \times (R - Q)$. The equation of the plane is of the form $f(x) = N.(X - A)$, where A is an arbitrary point of the triangle. Hence, taking $A = P, Q$, or R all give a proper result.

b) Given a quadrilateral $PQRS$, give a formula to determine if the quadrilateral is planar or not.

We take the equation of the plane of the triangle PQR, and substitute S. If $f(S) = 0$, the quadrilateral is planar. More formally:
$Planar(P, Q, R, S)$ = true if $N(P, Q, R)(S - P) = 0$
false otherwise

c) Give a formula for an approximation of the normal vector on the quadrilateral, which gives a useful result even when the quadrilateral is not planar.

This can be done in various ways. For instance by adding the normal of the triangles PQR and RSP:
$N' = N(P, Q, R) + N(R, S, P)$
Or the normal of the triangles QRS and SPQ:
$N' = N(Q, R, S) + N(S, P, Q)$.
If we expand these equations, we find we get the same result:
$N' = P \times Q + Q \times R + R \times S + S \times P$.
Hence, also if the normals of all triangles PQR, QRS, RSP, and SPR are added, still a normal in the same direction is obtained.

3 A standard cone is defined by $z = x^2 + y^2$, with $0 \le z \le 1$.

a) Give a parametric definition $P(u, v) = (P_x(u, v), P_y(u, v), P_z(u, v))$ with $0 \le u, v \le 1$ of this cone.

A cone is rotationally symmetric, a cross-section is a circle. This suggests to use one as the parameters (say u) as an angle along the cone. The radius varies with height z, this gives the second parameter v. Elaborating gives: $P(u, v) = (v \cos 2\pi u, v \sin 2\pi u, v)$.

b) Give a unit normal N for an arbitrary point on the cone. Choose yourself if you assume the point to be given via coordinates (x, y, z) or parameter values (u, v).

The easiest approach is to define the cone implicitly with $f(x, y, z) = x^2 + y^2 - z$, to take the gradient, and normalize the length:
$N = [2x, 2y, -1] / \sqrt{(4x^2 + 4y^2 + 1)}$.

c) We want to define a cone with height h, distorted such that the boundary of the cone at $z = h$ is an ellipse with diameter $2a$ in the x-direction and diameter $2b$ in the y-direction. Give an implicit definition of the form $f(x, y, z) = 0$, with $0 \le z \le h$.

The cone is scaled, with a factor h in z-direction, and with factors a and b in x- and y-direction. We can handle this by taking the implicit definition (see 3 b), and scaling all coordinates appropriately:
$f(x, y, z) = (x/a)^2 + (y/b)^2 - z/h = 0$.

Substitution of for instance the point $[a, 0, h]$ shows that this point is indeed located on the cone.