Ralph E. Gomory and Integer Linear Programming

Dr. Ralph E. Gomory's research interests are linear and integer programming, network flows, nonlinear differential equations and computers. In his research he considers practical problems and modeling as well as scientific, particularly mathematical, theory building. He is a mathematical engineer in the true sense of the word: mathematician and engineer. Accordingly he is member of both the National Academy of Sciences and the National Academy of Engineering. His most influential result is the cutting plane method for integer linear programming; it bares his name.

His career. In 1954 Ralph Gomory received a PhD in Mathematics from Princeton University. After that he served three years in the US Navy and worked two years at Princeton University. In 1959 he went to the research department of IBM, where he became IBM Fellow in 1964 and Director of the Mathematical Sciences Department in 1965. Between 1970 and 1986, Gomory was responsible for the research department of IBM, first as Director of Research, later as Vice President and Senior Vice President. In 1986 he became IBM Senior Vice President for Science and Technology. In 1989 he left IBM and became President of the Alfred P. Sloan Foundation.

Ralph Gomory is board member of various companies, including The Washington Post Company and the Polaroid Corporation. He has been active in many academic, industrial and government organizations.

He received many prizes, among others the Lanchester Prize from ORSA in 1963, the John von Neumann Theory Prize form ORSA in 1984, the IEEE Engineering Leadership Recognition Award in 1988, the National Medal of Science, awarded by the President of the United States in 1988, the Arthur M. Bueche Award of the National Academy of Engineering in 1993, the Heinz Award for Technology, the Economy and Employment in 1998, and the Madison Medal of Princeton University in 1999. From 1990 until 1993 he served at the President's Council of Advisors on Science and Technology.

Linear and integer programming. Crew scheduling, railway timetabling, chip design, frequency assignment: just a few of the numerous day to day tasks in our modern technological society. Many of these can be modeled as so-called linear or integer programming problems. *Linear programming* concerns optimizing a linear function subject to linear side constraints. When in addition the variables are only allowed to take integer values, we speak of *integer programming.*

In 1947, George B. Dantzig developed his *simplex method* for linear programming. This very effective method is still broadly used. During the past decades a combination of mathematical insight, programming skills and hardware developments has made it possible to develop computer programs that routinely solve linear programming problems with hundreds of thousands, and even millions, of variables and constraints.

The birth of this successful approach to linear programming immediately stimulated the modeling of many practical problems in this framework. Soon, however, the shortcomings of the model appeared. In linear programming problems, variables may take nonintegral values, whereas the quantities to be modeled are often intrinsically integer. Surely, when the model advises to produce 100.5 cars, also 100 or 101—or even 102—will be good enough. But what to do when the model advises to send 3.65 ships across the ocean? Even worse, when the model contains a variable x_i expressing whether to take some action ($x_i = 1$) or not ($x_i = 0$), and then assigns a value 0.5 to x_i, the model is not very helpful.

In these cases there is no way out: integrality has to be included in the model. This leads to integer (linear) programming problems, that is, problems that ask to optimize a linear function cx where x is a vector with *integral* coefficients that satisfies a system $a_1x \leq b_1, \ldots, a_mx \leq b_m$ of linear inequalities.

The strength of integer programming, its virtually unlimited modeling potential, is also its weakness. Integer programming is so general that so far no efficient method has been developed that solves all its instances. Even worse, it is generally believed that this is impossible. This in contrast to linear programming, for which efficient methods do exist. The complexity of integer programming, the resemblance between linear and integer programming and the availability of good linear programming methods already early on raised the question if, and to what extent, linear programming can be of any help in solving instances of its more complicated integer counterpart.

One of the founding fathers of that approach is Ralph Gomory.

Gomory's cutting plane method for integer programming.[1] The idea behind Gomory's method is to initially neglect the integrality requirements and solve the corresponding linear programming problem, for instance with the simplex method. This will give an optimal vector $\widehat{x}$. If all coefficients of $\widehat{x}$ happen to be integer, it is also optimal for the integer programming problem, which thereby is solved. What if this is not the case?

An obvious suggestion would be to try to round $\widehat{x}$ to an integer solution. That, however, is too naive. The rounded solution might not even be feasible, and even if it is, the optimal integer solution is typically not among these rounded vectors. Moreover, the knowledge of those rounded solutions that are feasible is too limited to facilitate efficient selection of an optimal one. Still, rounding is not such a bad idea, that is, as long as we do not round $\widehat{x}$. We will use rounding to find a *cutting plane*.

A *cutting plane* is a new inequality that is satisfied by all integral solutions of the original system, but is violated by $\widehat{x}$. Gomory showed that such a cutting plane can always be found as follows. Suppose that $\widehat{x}$ is a vertex of the set of all solutions to the given system of inequalities—the simplex method does guarantee that that is the case—and that $\widehat{x}$ has some nonintegral coefficients. Then there exists an inequality $a_{m+1}x \leq b$ that is satisfied by all solutions to the system (also the nonintegral ones) such that the following holds: $a_{m+1}\widehat{x} = b$, all coefficients of a_{m+1} are integral, and b is not an integer. Gomory also showed that the output of the simplex method produces such an inequality. Next, round b down to the nearest integer, b_{m+1}. The resulting inequality $a_{m+1}x \leq b_{m+1}$ is the desired cutting plane. *Gomory's cutting plane method* for integer programming adds this cutting plane to the system and iterates the whole procedure. Gomory showed that alternately applying the simplex method and adding cutting planes eventually leads to a system for which the simplex method will give an integer optimum. Until this day basically all software for either general or specific

[1] R.E. Gomory, Outline of an algorithm for integer solutions to linear programs, *Bulletin of the American Mathematical Society 64* (1958) 275–278.

R.E. Gomory, Solving linear programming problems in integers, in: *Combinatorial Analysis*, R. Bellman & M. Hall, Jr., eds., Proceedings of Symposia in Applied Mathematics X, American Mathematical Society, Providence, R.I., 1960, pp. 211–215.

R.E. Gomory, An algorithm for integer solutions to linear programs, in: *Recent Advances in Mathematical Programming*, R.L. Graves & P. Wolfe, eds., McGraw-Hill, New York, 1963, pp. 269–302.

integer programming problems is based on cutting plane methods[2]—not necessarily Gomory's cutting planes, but definitely inspired by them. The application of Gomory's idea to specific problem classes requires good understanding of the applications and the methods; genuine engineering for mathematicians.

Besides algorithmic consequences, Gomory's cutting plane method also led to a better understanding of the geometric aspects of integer solutions to systems of inequalities[3]. These insights still inspire mathematicians working on integer programming and polyhedral combinatorics (which is the application of linear algebra to combinatorial problems).

Other research by Gomory. Other influential research by Gomory concerns multicommodity network flows[4], which led to the famous *Gomory–Hu tree*, and linear programming techniques based on column generation[5] for industrial cutting stock problems as appearing in the paper and glass industry, for which the authors received the Lanchester Prize of ORSA. Inspired by irregularities observed in the solutions of these cutting stock problems[6] Gomory developed yet another approach to integer linear programming, also based on linear programming, by means of solving optimization problems in finite Abelian groups. This led to Gomory's *corner polyhedra*[7], still one of his favorite research topics.

During the last years, Gomory published work on the nature of technology and product development, industrial research, corporate competitiveness, and on economical models for international trade that take *economies of scale* into account.

[2] G.L. Nemhauser & L.A. Wolsey, *Integer and Combinatorial Optimization*, Wiley, New York, 1988.

[3] See Chapter 23 in: A. Schrijver, *Theory of Linear and Integer Programming*, Wiley, Chichester, 1986.

[4] R.E. Gomory & T.C. Hu, Multi-terminal network flows, *Journal of SIAM 9* (1961) 551–570.

[5] P.C. Gilmore & R.E. Gomory, A linear programming approach to the cutting-stock problem, *Operations Research 9* (1961) 849–859.

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[6] P.C. Gilmore & R.E. Gomory, The theory and computation of knapsack functions, *Operations Research 14* (1966) 1045–1074.

[7] R.E. Gomory, On the relations between integer and noninteger solutions to linear programs, *Proceedings of the National Academy of Sciences of the United States of America 53* (1965) 260–265.

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