

# Plenty of Franklin Magic Squares, but none of order 12

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## Abstract

We show that a genuine Franklin Magic Square of order 12 does not exist. This is done by choosing a representation of Franklin Magic Squares that allows for an exhaustive search of all order 12 candidate squares. We further use this new representation (in terms of polynomials) to generate large classes of true Franklin Magic Squares of orders 8 and multiples of 16. Next we show how Franklin Magic Squares of orders  $n = 20 + 8k$  can be constructed. Finally we indicate how almost-Franklin Magic Squares of order 20 can be constructed in a general way.

## 1 Franklin Magic Squares

According to various descriptions a natural *Franklin Magic Square* of even size  $n$  is a square matrix  $M$  with  $n$  rows and columns with the properties

1. the entries of  $M$  are  $1, 2, \dots, n^2$ ;
2. each row and each column has a fixed entry sum  $n(1 + n^2)/2$ ;
3. each two by two sub-square  $\begin{bmatrix} M_{i,j} & M_{i,j+1} \\ M_{i+1,j} & M_{i+1,j+1} \end{bmatrix}$  has sum  $2(1 + n^2)$ ;
4. each half row starting in column 1 or  $n/2 + 1$  has sum of entries equal to  $n(1 + n^2)/4$ , and similar for half columns starting in row 1 or  $n/2 + 1$ ;
5. each half of the main diagonal (starting in column 1 or  $n/2 + 1$ ) together with each half of the back diagonal has total sum (such as  $\sum_{i=1}^{n/2} (M_{i,i} + M_{i,n+1-i})$ ) equal to  $n(1 + n^2)/2$ . This construction is called a *bent diagonal*. The sum requirements also hold for so-called *bent rows*, which are translates of the two half-diagonals, possibly wrapping over the matrix sides.

These squares are called after the former US president and scientist Benjamin Franklin who constructed a few of such matrices, two of order eight, one of order 16. Note that the fourth property implies that  $n$  is a multiple of four. It turns out that the condition on 2x2 subsquares is the most prominent one and generates a lot of shapes with a constant-sum-property. In particular we have that

2x2 squares wrapping along one side of the matrix also have fixed sum  $2(1 + n^2)$ ,

and further that

for arbitrary  $i, j, k$  the entries in  $\begin{bmatrix} M_{i,j} & M_{i,j+1+2k} \\ M_{i+1,j} & M_{i+1,j+1+2k} \end{bmatrix}$  have sum  $2(1 + n^2)$ .

Combining this on two consecutive rows  $i, i + 1$  we find that for arbitrary  $i, j, k$ ,

$$M_{i,j} + M_{i,j+1+2k} \text{ and } M_{i+2,j} + M_{i+2,j+1+2k} \text{ have equal values.}$$

In the end this leads to the observation that for arbitrary  $i, j, k, m$  we have that

$$\text{each four-tuple } M_{i,j}, M_{i+2m+1,j}, M_{i,j+2k+1}, M_{i+2m+1,j+2k+1} \text{ has sum } 2(1 + n^2).$$

This property of Franklin Magic Squares is often referred to as the mirroring property because its consequence is that on any shape that is symmetric horizontally and vertically along a line separating rows or columns, respectively, the entries of square add up to a number that is independent of the choice of the intersection of the axes of symmetry. Here we allow moving over the border of the square by embedding it on a torus. Note that this property is merely based on the 2x2 sub-square property.

Applying the above insights on the top halves of the main and back diagonal we find that the sum-of-half-diagonals property is equivalent (given the 2x2 square with fixed sum) to the statement that  $M_{1,1}, M_{2,2}, M_{1,3}, M_{2,4}, \dots, M_{1,n/2-1}, M_{2,n/2}$  and  $M_{2,n/2+1}, M_{1,n/2+2}, \dots, M_{2,n-1}, M_{1,n}$  together sum up to  $n(1 + n^2)/2$ .

Subtracting  $n/4$  ‘subsquares’  $[M_{1,2k}, M_{2,2k}, M_{1,n+1-2k}, M_{2,n+1-2k}]$  of constant sum we find that  $M_{1,1} - M_{1,2} + \dots + M_{1,n/2-1} - M_{1,n/2}$  plus its mirror image  $M_{1,n+1-1} - M_{1,n+1-2} + \dots + M_{1,n+1-n/2-1} - M_{1,n+1-n/2}$  equals zero.

Adding a full row sum leads to a pattern of  $n/2$  entries, with

$$(M_{1,1} + M_{1,3} + \dots + M_{1,n/2-1}) + (M_{1,n/2+2} + M_{1,n/2+4} + \dots + M_{1,n}) = n(1 + n^2)/4.$$

which holds for ordinary magic squares with the 2x2 square property and the bent-diagonals-property.

Subtracting a *fixed half row sum* starting in column  $n/2 + 1$  we finally obtain the property that

$$M_{1,1} + M_{1,3} + \dots + M_{1,n/2-1} \text{ equals } M_{1,n/2+1} + M_{1,n/2+3} + \dots + M_{1,n-1}.$$

This *alternate sum property* is hence equivalent with the bent-diagonal property (in presence of the other franklin conditions), but much more easily checked. Obviously a similar reasoning is possible for vertical bent-diagonals, leading to columns having the alternate sum property.

All this leads to the following more compact definition of a Franklin Magic Square of arbitrary order  $4k$ , which is a matrix with properties:

1. entries are  $1, \dots, n^2$ ;
2. each 2x2 sub-square has entries summing up to  $2(1 + n^2)$ ;
3. the first half of the first row, the second half of the first row, the first half of the first column, and the second half of the first column, each have entries that sum up to  $n(1 + n^2)/4$ ;

4. entries on odd positions in the first half of the first row add up to the same value as entries on odd positions in the second half of the first row; similarly, entries on odd positions in the first half of the first column add up to the same value as entries on odd positions in the second half of the first column.

## 2 More compact representation

### 2.1 Isomorphisms

It turns out that any Franklin Magic Square maintains its magic properties under a number of matrix transformations, namely:

1. reflection along the horizontal, or vertical axis of symmetry;
2. permutation of row (column) indices within the sets  $S_1 = \{2k + 1 \mid 0 \leq k < n/4\}$ ,  $S_2 = \{2k \mid 1 \leq k \leq n/4\}$ ,  $S_3 = \{2k + 1 \mid n/4 \leq k < n/2\}$  and  $S_4 = \{2k \mid n/4 < k \leq n/2\}$ ;
3. exchanging the  $n/4$  rows (columns) indexed by  $S_1$  with those indexed by  $S_3$ ; similarly, exchanging the  $n/4$  rows (columns) indexed by  $S_2$  with those indexed by  $S_4$ ;
4. reflection along the diagonal;
5. replacing each entry  $M_{ij}$  by  $n^2 + 1 - M_{ij}$ .

The first three properties suffice to prove that we can assume without loss of generality that the first entry  $M_{1,1} = 1$ . It is evident that the transformations above leave the compact definition of Franklin Magic Squares intact.

### 2.2 Bookkeeping

Based on the 2x2 sub-square property the square can be fixed by determining the entries on the first row and first column. For computational reasons it is more convenient to index rows and columns by  $0, \dots, n-1$ , and to subtract 1 from each entry in the Franklin square, so that the entries become  $0, \dots, n^2 - 1$ . Note that now the average entry value is  $\nu = (n^2 - 1)/2$ , instead of  $(n^2 + 1)/2$ . We now assume that the upper leftmost element is zero. We call this Franklin square *basic* instead of natural.

Next consider the following transformation  $C(F)$  on any Franklin Magic Square  $F$ :

$$V_{ij} = C(F)_{ij} := \begin{cases} F_{ij} & \text{if } i + j \equiv 0 \text{ modulo } 2 \\ n^2 - 1 - F_{ij} & \text{if } i + j \equiv 1 \text{ modulo } 2 \end{cases}$$

which can be viewed as *complementing* entries on black positions (of the underlying chess board). Note that  $F = C(V)$ .

The 2x2 sub-square property of  $F$  translates into a favorable property for  $V$ , namely:  $V_{i,j} + V_{i+1,j+1} - V_{i,j+1} - V_{i+1,j} = 0$ , for all  $i, j$ . Based on this property, having the zero in  $F_{00}$  gives that  $V$  has the nice property that  $V_{ij} = V_{i0} + V_{0j}$ . Hence to generate candidate Franklin Magic Squares  $F$  we enumerate all vectors  $x = (x_0, \dots, x_{n-1})$  and  $y = (y_0, \dots, y_{n-1})$ , with properties:

1.  $x_0 = y_0 = 0$ ;
2.  $x_0 < x_2 < \dots < x_{n/2-2}$ ,  
 $x_1 < x_3 < \dots < x_{n/2-1}$ ,  
 $x_{n/2} < x_{n/2+2} < \dots < x_{n-2}$ ,  
 $x_{n/2+1} < x_{n/2+3} < \dots < x_{n-1}$ ;
3.  $y_0 < y_2 < \dots < y_{n/2-2}$ ,  
 $y_1 < y_3 < \dots < y_{n/2-1}$ ,  
 $y_{n/2} < y_{n/2+2} < \dots < y_{n-2}$ ,  
 $y_{n/2+1} < y_{n/2+3} < \dots < y_{n-1}$ ;
4.  $x_0 + x_2 + \dots + x_{n/2-2} =$   
 $x_1 + x_3 + \dots + x_{n/2-1} =$   
 $x_{n/2} + x_{n/2+2} + \dots + x_{n-2} =$   
 $x_{n/2+1} + x_{n/2+3} + \dots + x_{n-1}$ ;
5.  $y_0 + y_2 + \dots + y_{n/2-2} =$   
 $y_1 + y_3 + \dots + y_{n/2-1} =$   
 $y_{n/2} + y_{n/2+2} + \dots + y_{n-2} =$   
 $y_{n/2+1} + y_{n/2+3} + \dots + y_{n-1}$ ;
6.  $x_1 < x_{n/2+1}$ ;
7.  $y_1 < y_{n/2+1}$ ;
8.  $\max_i y_{2i} > \max_j x_{2j}$ ;
9.  $0 \leq y_i + x_j \leq n^2 - 1$ , for all  $i, j$ ;
10. the set  $\{y_i + x_j | i + j \equiv 0\} \cup \{n^2 - 1 - y_i - x_j | i + j \equiv 1\}$  equals  $\{0, \dots, n^2 - 1\}$ .

### 3 Not Finding the 12x12 Franklin Magic Square

Evidently the enumeration should be kept to a minimum by pruning the search for candidate Franklin Magic Squares as early as possible. For the 12 by 12 Franklin Square the following strategy turns out to lead to a manageable enumeration scheme.

1. generate a 3x6 sub-matrix on the 6 columns with even index and on rows indexed 0, 2, 4;
2. extend this to a 6x6 sub-matrix on the even columns and the even rows;
3. extend to a 9x9 sub-matrix adding rows and columns indexed 1, 3 and 5;
4. finally extend to a full 12x12 matrix

At each stage, before adding (three) more rows or (three) more columns, we update a list of candidate  $x_j$  or  $y_i$  values, given the partially filled  $F$ . Note that for instance, after the first step, possible values for  $y_6, y_8, y_{10}$  come from a limited common domain, consistent with the 3x6 sub-matrix already filled.

Proceeding in this way we generate:

831083	tuples $x_2, x_4, x_6, x_8, x_{10}$ ;
40467771	extensions $y_4$ ,
1473501105	extensions $y_2, y_4$ , i.e. 3x6 sub-matrices
25663243622	extensions $y_2, y_4, y_{10}$ ,
24473864360	extensions $y_2, y_4, y_6, y_8, y_{10}$ , i.e. 6x6 squares,
22532519520	of which cannot be ruled out immediately;
121404978	9x9 extensions,
93083	of which might be extended to a 12x12 square.

In the end, none of these would lead to the desired Franklin Magic Square. Computation of these cases was carried out by a network of 50 computers. For this we split the work into 70 cases, corresponding with the possible settings for  $x_2 \in \{1, \dots, 70\}$ . (For a higher value for  $x_2$  we would have that  $x_4 + y_{\max} \geq 72 + 73 = 145 > \max\{0, \dots, 12^2 - 1\}$ . The total computation time was approximately 160 hours.

## 4 A generic scheme for building Magic Squares

The above described formulation in terms of vectors  $x$  and  $y$  can also be used in a generic way to generate (Franklin) Magic Squares with the 2x2 sub-square property for arbitrary even order, with or without additional properties. To this purpose we formulate the magic square properties in terms of an equation in polynomials.

### 4.1 An encoding in polynomials

The polynomials we consider have coefficients 0 and 1. Let  $\delta$  or  $\delta(P)$  denote the degree of a polynomial  $P$ . Then for a polynomial in  $z$ ,  $P(z)$ , of degree  $\delta$ , and any number  $\nu \geq \delta$ , let  $\overline{P}^\nu$  be defined by  $\overline{P}^\nu(z) = z^\nu P(1/z)$ . We are more or less writing  $P$  backwards, or better, we are reflecting its exponents with respect to the value  $\nu/2$ . If we do not mention  $\nu$  we take by convention the degree of the polynomial.

Let us now associate with each (Franklin) Magic Square, given in terms of  $x$  and  $y$ , the polynomials  $A(z) := \sum_{j=0}^n z^{x_j}$ , and  $B(z) := \sum_{i=0}^n z^{y_i}$ . Let us further split these summations over odd and even indices:  $A(z) = A_0(z) + A_1(z)$ , with  $A_k(z) := \sum_{j=0, j \equiv k}^n z^{x_j}$ , for  $k = 0, 1$ ; and  $B(z) = B_0(z) + B_1(z)$ , with  $B_k(z) := \sum_{i=0, i \equiv k}^n z^{y_i}$ , for  $k = 0, 1$ . A square with numbers  $\{0, \dots, n^2 - 1\}$  with the 2x2 sub-square property then satisfies the condition:

$$(A_0 B_0 + A_1 B_1)(z) + \overline{A_0 B_1 + A_1 B_0}^{n^2-1}(z) = \frac{z^{n^2} - 1}{z - 1} \quad (1)$$

which simply stipulates that all numbers are present once in the matrix. Here  $A_0, A_1, B_0, B_1$  are polynomials with  $n/2$  terms each. Solving the above system (to find the square) is possible if one restricts to certain types of solutions. One such restriction (**Type 1a**) could be to choose

$$B_0 = B_1 = \overline{B_0}^{\delta(B_0)} \quad (2)$$

which leads to the following simplification of the equation above:

$$(A B_0)(z) + \overline{A B_0}^{n^2-1}(z) = (A + \overline{A}^{n^2-1-\delta(B_0)})(z) B_0(z) = \frac{z^{n^2} - 1}{z - 1} \quad (3)$$

A variant of this approach (**Type 1b**) would be to choose

$$B_0 = B_1 \neq \overline{B_0}^{\delta(B_0)} \text{ and } A = \overline{A}^{\delta(A)} \quad (4)$$

which leads to the simplification:

$$(AB_0)(z) + \overline{AB_0}^{n^2-1}(z) = A(z)(B_0 + \overline{B_0}^{n^2-1-\delta(A)})(z) = \frac{z^{n^2} - 1}{z - 1} \quad (5)$$

A second approach (**Type 2**) could be to assume the existence of a number  $\nu$  such that

$$A_0 = \overline{A_0}^{n^2-1-\nu}, \quad A_1 = \overline{A_1}^{n^2-1-\nu}, \quad B_0 = \overline{B_0}^\nu, \quad B_1 = \overline{B_1}^\nu \quad (6)$$

A third approach (**Type 3**) is to assume an integer  $\nu$  exists such that

$$A_0 = \overline{A_1}^{n^2-1-\nu}, \quad A_1 = \overline{A_0}^{n^2-1-\nu}, \quad B_0 = \overline{B_1}^\nu, \quad B_1 = \overline{B_0}^\nu \quad (7)$$

Both (6) and (7) translate equation (1) into the simple

$$A(z)B(z) = \frac{z^{n^2} - 1}{z - 1} \quad (8)$$

Now, for  $n = 2^q k$ , with odd  $k$ , the right hand side in equation (1) can be rewritten as

$$\frac{z^{n^2} - 1}{z - 1} = \frac{z^{k^2 2^{2q}} - 1}{z^{k 2^{2q}} - 1} \frac{z^{k 2^{2q}} - 1}{z^{2^{2q}} - 1} \prod_{j=0}^{2q-1} (1 + z^{2^j})$$

Note that the first two factors in this decomposition are polynomials in  $z$  with  $k$  terms each, whereas the other factors are two-term polynomials. In case  $k = 1$  this decomposition is unique, but for other values there are many possible decompositions.

Using the first method to solve (1), we look for a candidate polynomial  $B_0$ , with  $n/2$  terms, by selecting one of the two first factors, and  $q - 1$  factors from the other ones. Their product is indeed a polynomial in  $n/2$  terms, and is symmetric (meaning  $B_0 = \overline{B_0}^\nu$ , for some  $\nu$ ). The factors not selected form a product  $\Sigma(z)$  that is in fact a symmetric polynomial in  $2n$  terms, and that we have to set equal to the sum  $A + \overline{A}^{n^2-1-\delta(B_0)}$  by an appropriate choice for  $A$ .

For the variant we select one factor from the first two and next  $q$  factors from the second part so as to build  $A$ . The co-factor (with  $n$  terms) must then match  $B_0 + \overline{B_0}^{n^2-1-\delta(A)}$  for an appropriate choice of  $B_0$ .

When using the second or third method, we may define  $B$  say, by taking one of the two first factors, and adding  $q$  factors from the other ones. Their product is then a symmetric polynomial in  $n$  terms, which can further be split into  $B_0$  and  $B_1$ . The remaining factors are used to build  $A_0$  and  $A_1$ .

In the remainder of the paper we show how to construct various types of Franklin Magic Squares. We first formulate how additional requirements on the constructed squares translate into conditions on the vectors  $x$  and  $y$ , and hence on the polynomials  $A_0, A_1, B_0, B_1$ . Define  $X_{ik} = \sum_{j \equiv i, [2j/n]=k} x_j$  for  $i, k \in \{0, 1\}$ . Similarly, let  $Y_{ik} = \sum_{j \equiv i, [2j/n]=k} y_j$ .

**magic row sum**  $X_{00} + X_{01} = X_{10} + X_{11}$  or, equivalently, the sum of exponents in  $A_0$  equals the sum of exponents in  $A_1$ ;

**magic column sum**  $Y_{00} + Y_{01} = Y_{10} + Y_{11}$  or, equivalently, the sum of exponents in  $B_0$  equals the sum of exponents in  $B_1$ ;

**magic sum on horizontal bent diagonals**  $X_{00} = X_{11}$  and  $X_{01} = X_{10}$ , or, equivalently,  $A_0 = A_{00} + A_{01}$ ,  $A_1 = A_{10} + A_{11}$  is a split into four polynomials of  $n/4$  terms each with exponents in  $A_{00}$  ( $A_{10}$ ) adding up to the same as those in  $A_{11}$  ( $A_{01}$ , respectively);

**magic sum on vertical bent diagonals**  $Y_{00} = Y_{11}$  and  $Y_{01} = Y_{10}$ , or, equivalently,  $B_0 = B_{00} + B_{01}$ ,  $B_1 = B_{10} + B_{11}$  is a split into four polynomials of  $n/4$  terms each with exponents in  $B_{00}$  ( $B_{10}$ ) adding up to the same as those in  $B_{11}$  ( $B_{01}$ , respectively);

**half the magic sum in first and second half row**  $X_{00} = X_{10}$  and  $X_{01} = X_{11}$ , or, equivalently, there is a split of  $A$ , as above, with exponents in  $A_{00}$  summing to the same as those in  $A_{10}$ , and exponents in  $A_{01}$  summing to the same as those in  $A_{11}$ ;

**half the magic sum in first and second half column**  $Y_{00} = Y_{10}$  and  $Y_{01} = Y_{11}$ , or, equivalently, there is a split of  $B$ , as above, with exponents in  $B_{00}$  summing to the same as those in  $B_{10}$ , and exponents in  $B_{01}$  summing to the same as those in  $B_{11}$ ;

**pan-diagonal magic sum**  $X_{00} + X_{01} + X_{10} + X_{11} + Y_{00} + Y_{01} + Y_{10} + Y_{11} = n(n^2 - 1)/2$ , or equivalently, exponents in  $A$  and  $B$  add up to the magic sum;

**most-perfect** This means: complementary entries lie on the same diagonal,  $n/2$  positions apart. That is,  $M_{i,j} + M_{i+n/2,j+n/2} = (n^2 + 1)$ , for all  $i, j$ . We then have  $x_j + x_{j+n/2} + y_i + y_{i+n/2} = n^2 - 1$ , for all  $i, j$ , implying that  $x_j + x_{j+n/2} = \delta(A)$ , for all  $j < n/2$ , and  $y_i + y_{i+n/2} = \delta(B)$ , for all  $i < n/2$ , and each of  $A_0, A_1, B_0, B_1$  must be symmetric;

**four-on-a-row** This means: blocks of 4 consecutive entries partitioning a row (or column) each have magic entry sum. In other words,  $M_{i,4k+1} + M_{i,4k+2} + M_{i,4k+3} + M_{i,4k+4} = 2(n^2 + 1)$ , for all  $i, k$ . Then  $x_{4j} + x_{4j+2} = x_{4j+1} + x_{4j+3}$ , for all  $j$ , implying that pairs of exponents in  $A_{00}$  match with pairs of exponents in  $A_{10}$  having the same sum, etcetera.

## 4.2 Simple Magic Squares

### 4.2.1 Method 1a

If we pose no further restrictions, then for each symmetric polynomial  $\Sigma(z)$  of  $2n$  terms we can easily find  $2^n$  different solutions  $A(z)$  as follows: for the  $n$  smallest powers  $z^j$  in  $\Sigma$  we have that  $z^{N-j}$  is in  $\Sigma$  as well, where  $N = \delta(\Sigma)$ . For each  $j$ , select one of  $\{j, N - j\}$  to be in the set of powers of  $A$ . For instance

$$1 + z^2 + z^3 + z^6 + z^9 + z^{12} + z^{13} + z^{15} = (1 + z^3 + z^6 + z^{13}) + \overline{(1 + z^3 + z^6 + z^{13})}^{15}$$

Now  $A$  and  $B$  will lead to a square matrix of numbers  $\{0, \dots, n^2 - 1\}$  satisfying the  $2 \times 2$  sub-square property. Each column will have fixed sum  $n(n^2 - 1)/2$ , for the simple reason that  $B_0$  and  $B_1$  are equal.

In order to have fixed row sums as well, we should be able to split  $A(z) = A_0(z) + A_1(z)$ , with exponents in  $A_0$  adding to the same sum as the exponents in  $A_1$ . This can be done in

general as follows. Let  $1 + z^{2^j}$  be a factor of  $\Sigma$ , that is  $\Sigma(z) = (1 + z^{2^j})\Omega(z)$ , where  $\Omega$  is symmetric as well, and with (even)  $n$  terms. Pair the terms in  $\Omega(z)$  with matching exponents  $(z^t + z^{\delta(\Omega)-t})$ . Taking  $A(z) = \Omega(z)$  gives  $\overline{A}^{\delta(\Omega)+2^j}(z) = z^{2^j}A(z)$ . Now split  $A$  into  $A_0$  and  $A_1$  by taking for  $A_0$  half of the  $n/2$  pairs in  $\Omega(z)$ , and for  $A_1$  the remaining  $n/4$  pairs. As each pair contributes  $\delta(\Omega)$  to the sum of exponents, the split will be balanced. So the exponents in  $A_0$  will add up to the same sum as those in  $A_1$ , and hence the resulting matrix will have constant row sums.

#### 4.2.2 Method 1b

In order to find Type 1b squares with fixed row and column sum, we have to be able to split the  $n$ -term polynomial  $A$  into two parts of equal exponent sum. This is easily achieved by the method described above: form pairs of matching terms  $z^j, z^{\delta(A)-j}$ , and divide these pairs over two groups of equal size. If  $n$  is a multiple of four, such a split is possible in  $\binom{n/2}{n/4}$  ways, for any given  $A$ .

In order to find a matching  $B_0$  it suffices to pair matching terms  $z^j, z^{n^2-1-\delta(A)-j}$  and select one of each pair as an element of  $B_0$ . There are  $2^{n/2}$  possible  $B_0$ 's, for a given  $A$ . By definition  $B_0$  and  $B_1$  have the same sum of exponents.

#### 4.2.3 Method 2

After decomposing  $\frac{z^{n^2}-1}{z-1} = A(z)B(z)$  where both  $A$  and  $B$  have  $n$  terms, we have lots of ways to split  $A$  into parts  $A_0, A_1$  with  $A_i = \overline{A}_i^{\delta(A)}$  by pairing the  $j$ th lowest term with the  $j$ th highest term in  $A$ , and then assign half of these pairs to  $A_0$  and the other half to  $A_1$ . Similar for  $B$ . There are  $\binom{n/2}{n/4}$  ways to generate  $A_0$  and there are  $\binom{n/2}{n/4}$  ways to generate  $B_0$  (for fixed  $A$  and  $B$ ). We only need  $n$  to be a multiple of 4.

Note that by keeping matching exponents close together method 2 as described above generates 4x4 blocks that have the property that each row, each column and each (broken) diagonal has magic sum  $2(n^2 + 1)$ . So if we apply this method to generate squares of order  $4k$ , only caring about 4 fields on-a-row having fixed sum  $2(n^2 + 1)$ , we get squares that have the property that each 8x8 sub-square has all the Franklin Magic Square properties as far as row, column and bent-diagonal sums are concerned, with average entry value equal to  $(n^2 + 1)/2$ . For instance take  $n = 12$ ,  $A(z) = (1 + z^{48} + z^{96})(1 + z^8)(1 + z^4)$ , and  $B(z) = (1 + z^{16} + z^{32})(1 + z^2)(1 + z)$ . With  $x = (0, 4, 108, 104, 8, 12, 100, 96, 48, 52, 60, 56)$  and  $y = (0, 1, 35, 34, 2, 3, 33, 32, 16, 17, 19, 18)$  we obtain the square  $M_{12.2}$  given in Figure 1. It contains four 8x8 subsquares, aligned with the 4x4 block structure, with all the Franklin Magic Square properties (except for containing 64 consecutive numbers). Notice that each 4x4 block in the structure is most-perfect in the sense that its 2x2 subsquares are one another's complement. Further observe that in this example every 4x4 block with upper left entry in an odd row and an odd column has magic row and column sum! This can be enforced in general by building the  $x$ -vector in strips of four with values  $j, j + \alpha, N - j, N - j - \alpha$ , for some fixed  $\alpha$ , and similarly build the  $y$ -vector in strips of four of value  $j, j + \beta, N' - j, N' - j - \beta$ , for some fixed  $\beta$ . Here  $N = \delta(A)$  and  $N' = \delta(B)$ . The features are highlighted in bold font.

$$M_{12.2} =$$

1	140	109	40	9	132	101	48	49	92	61	88
143	6	35	106	135	14	43	98	95	54	83	58
36	105	144	5	44	97	136	13	84	57	96	53
110	39	2	139	102	47	10	131	62	87	50	91
3	138	111	38	11	130	103	46	51	90	63	86
141	8	33	108	133	16	41	100	93	56	81	60
34	107	142	7	42	99	134	15	82	59	94	55
112	37	4	137	104	45	12	129	64	85	52	89
17	124	125	24	25	116	117	32	65	76	77	72
127	22	19	122	119	30	27	114	79	70	67	74
20	121	128	21	28	113	120	29	68	73	80	69
126	23	18	123	118	31	26	115	78	71	66	75

Figure 1: Block structure with 4x4 most-perfect magic subsquares

#### 4.2.4 Method 3

After decomposing  $\frac{z^{n^2}-1}{z-1} = A(z)B(z)$  where both  $A$  and  $B$  have  $n$  terms, we have lots of ways to split  $A$  into parts  $A_0, A_1$  with  $A_0 = \overline{A_1}$ . However we need to enforce equal sums of exponents. By extracting a factor  $(1+z^\alpha)(1+z^\beta)$  from  $A(z)$ :  $A(z) = (1+z^\alpha)(1+z^\beta)\Omega(z)$ , we can take care for this. Match terms  $z^j$  and  $z^{N-j}$  in  $\Omega(z)$ , where  $N = \delta(\Omega)$ , and write  $(1+z^\alpha)(1+z^\beta)(z^j + z^{N-j}) = (z^j + z^{\alpha+N-j} + z^{\beta+N-j} + z^{\alpha+\beta+j}) + (z^{N-j} + z^{\alpha+j} + z^{\beta+j} + z^{\alpha+\beta+N-j})$ . Both four-tuples have exponent sum  $2(N+\alpha+\beta)$ . Assign one 4-tuple to  $A_0$ , and the other to  $A_1$ . There are  $2^{n/8}$  such assignments, for fixed  $\alpha, \beta$ , and there are many ways to choose  $\alpha$  and  $\beta$ , for fixed  $A$ .

Similarly, for  $B$  we find several ways to come up with a proper partition into  $B_0$  and  $B_1$ .

### 4.3 Magic Squares with bent-diagonals with magic sum

As indicated before, the properties of 2x2 subsquares having fixed sum, and rows and columns having fixed magic sum, lead to the equivalence of the bent-diagonal property with the condition that odd positions in the first half and even positions in the second half of the first row add up to half the magic sum. This in terms of  $x$  means  $x_0 + x_2 + \dots + x_{n/2-2} = x_{n/2+1} + \dots + x_{n-1}$ , and in terms of our polynomials this means that  $A_0$  and  $A_1$  must have a subset of  $n/4$  terms each with the same exponent sum.

If  $n$  is a multiple of 8, the above construction of  $A$  by method 1a or 1b already provides such a decomposition of  $A$ . And for  $B = B_0 + B_1$  it is easy to distribute the terms of  $B_0$  and  $B_1$  in a symmetric way. Simply take  $y_i = y_{n-1-i}$ , for all  $i$  (we had  $B_0 = B_1$ ).

If  $n$  is a multiple of 4, method 2 applied in the previous section yields pairs of terms each with the same exponent sum. Keeping these pairs adjacent (i.e. on positions  $i$  and  $i+2$ ) and

in the same half (i.e.  $i + 2 < n/2$  or  $i \geq n/2$ ) yields  $x$  and  $y$  vectors with the right properties. This requires  $n$  to be a multiple of 8.

As method 3 generates 4-tuples of equal exponent sum we can nicely distribute such 4-tuples provided  $n$  is a multiple of 16. Simply keep 4-tuples adjacent (on positions  $i, i + 2, i + 4, i + 6$ ) and on the same half.

#### 4.4 Magic Squares with half rows having half the magic sum

If we insist on the property of having half rows with half the magic sum, and not necessarily having the bent-row property, we can do the same as in the previous subsection. Indeed, in order to have half the magic sum in the first half of the first row it suffices to have a subset of  $n/4$  terms in  $A_0$  and a subset of  $n/4$  terms in  $A_1$  having the same sum of exponents. But this was exactly the same condition we needed for having bent-diagonals with magic sum.

By interchanging the columns  $1, 3, \dots, n/2 - 1$  with the set of columns  $n/2 + 1, n/2 + 3, \dots, n - 1$  a magic square with magic sum on horizontal bent diagonals transforms into one with half the magic sum on half rows, and vice versa. Similarly for vertical bent diagonals and half columns with half the magic sum. Hence the construction for magic squares with bent diagonals having magic sum, can be used to generate magic squares with half the magic sum on half rows and half columns.

### 5 Full Franklin Magic Squares of order $8k$

We can have both half rows and columns with half the magic sum, and bent-diagonals with magic sum, if both  $A$  and  $B$  can be split into four parts each with  $n/4$  terms, such that the subsets of  $A$  have the same sum of exponents, and the subsets of  $B$  have the same exponent sum. As an exponent cannot appear four times a requirement is that  $n$  is at least 8.

In this section we discuss general construction methods for Franklin Magic Squares given that the order is a multiple of 8, with and without special features such as pan-diagonality and perfectness.

#### 5.1 Regular constructions of Franklin Magic Squares

##### 5.1.1 Method 1a

By construction along method 1a  $A$  already admits the partition into four parts of equal size and equal exponent sum. For  $B$  we merely have to pair each  $z^j$  in  $B_0(z)$  with  $z^{\delta(B_0)-j}$ . Again if  $n$  is a multiple of 8, it is then possible to split  $B_0$  into two sets of terms with  $n/8$  pairs each.

For example, for  $n = 8$  one can take

$$\frac{z^{64} - 1}{z - 1} = \underbrace{(z^{32} + 1)(z^{16} + 1)(z^8 + 1)(z^4 + 1)}_{A(z)} \underbrace{(z^2 + 1)(z + 1)}_{B_0(z)}$$

which yields  $A(z) = (1 + z^{56}) + (z^8 + z^{48}) + (z^{16} + z^{40}) + (z^{24} + z^{32})$ , and  $B_0(z) = (1 + z^3) + (z^1 + z^2)$ . Via vectors  $x = (0, 16, 56, 40, 8, 24, 48, 32)$ , and  $y = (0, 2, 3, 1, 1, 3, 2, 0)$  we obtain the 8x8 squares

$$M_{1a} = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 48 & 57 & 24 & 9 & 40 & 49 & 32 \\ \hline 62 & 19 & 6 & 43 & 54 & 27 & 14 & 35 \\ \hline 4 & 45 & 60 & 21 & 12 & 37 & 52 & 29 \\ \hline 63 & 18 & 7 & 42 & 55 & 26 & 15 & 34 \\ \hline 2 & 47 & 58 & 23 & 10 & 39 & 50 & 31 \\ \hline 61 & 20 & 5 & 44 & 53 & 28 & 13 & 36 \\ \hline 3 & 46 & 59 & 22 & 11 & 38 & 51 & 30 \\ \hline 64 & 17 & 8 & 41 & 56 & 25 & 16 & 33 \\ \hline \end{array} \qquad M_{1b} = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 48 & 57 & 24 & 9 & 40 & 49 & 32 \\ \hline 60 & 21 & 4 & 45 & 52 & 29 & 12 & 37 \\ \hline 6 & 43 & 62 & 19 & 14 & 35 & 54 & 27 \\ \hline 63 & 18 & 7 & 42 & 55 & 26 & 15 & 34 \\ \hline 2 & 47 & 58 & 23 & 10 & 39 & 50 & 31 \\ \hline 59 & 22 & 3 & 46 & 51 & 30 & 11 & 38 \\ \hline 5 & 44 & 61 & 20 & 13 & 36 & 53 & 28 \\ \hline 64 & 17 & 8 & 41 & 56 & 25 & 16 & 33 \\ \hline \end{array}$$

Figure 2: Franklin Magic Squares obtained by methods 1a and 1b

$$V = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 0 & 16 & 56 & 40 & 8 & 24 & 48 & 32 \\ \hline 2 & 18 & 58 & 42 & 10 & 26 & 50 & 34 \\ \hline 3 & 19 & 59 & 43 & 11 & 27 & 51 & 35 \\ \hline 1 & 17 & 57 & 41 & 9 & 25 & 49 & 33 \\ \hline 1 & 17 & 57 & 41 & 9 & 25 & 49 & 33 \\ \hline 3 & 19 & 59 & 43 & 11 & 27 & 51 & 35 \\ \hline 2 & 18 & 58 & 42 & 10 & 26 & 50 & 34 \\ \hline 0 & 16 & 56 & 40 & 8 & 24 & 48 & 32 \\ \hline \end{array} \longrightarrow F = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 0 & 47 & 56 & 23 & 8 & 39 & 48 & 31 \\ \hline 61 & 18 & 5 & 42 & 53 & 26 & 13 & 34 \\ \hline 3 & 44 & 59 & 20 & 11 & 36 & 51 & 28 \\ \hline 62 & 17 & 6 & 41 & 54 & 25 & 14 & 33 \\ \hline 1 & 46 & 57 & 22 & 9 & 38 & 49 & 30 \\ \hline 60 & 19 & 4 & 43 & 52 & 27 & 12 & 35 \\ \hline 2 & 45 & 58 & 21 & 10 & 37 & 50 & 29 \\ \hline 63 & 16 & 7 & 40 & 55 & 24 & 15 & 32 \\ \hline \end{array}$$

and finally we obtain a square  $M_{1a}$  given in Figure 2.

### 5.1.2 Method 1b

When we apply method 1b we again have to be able to split  $A$  into four parts with equal exponent sum, and  $B_0$  into two parts with equal exponent sum. The first part is easy because  $A$  is symmetric and if  $n$  is a multiple of 8 we can easily create  $n/2$  pairs and partition them over 4 groups. As  $B_0$  is not symmetric in all cases we have to enforce this by defining  $(B_0 + \overline{B_0}^{n^2-1-\delta(A)})(z) = (1 + z^\alpha)\Omega(z)$ , and take  $B_0 = \Omega$ . Match complementary terms and split the set of pairs in two.

For an example of method 1b let us consider  $A(z) = (z^{32} + 1)(z^{16} + 1)(z^8 + 1)$  as above, and  $(B_0 + \overline{B_0}^7)(z) = (z^4 + 1)(z^2 + 1)(z + 1) = (z^2 + 1)B_0(z)$ , with  $B_0(z) = B_1(z) = (1 + z^5) + (z^1 + z^4)$ . With vectors  $x = (0, 16, 56, 40, 8, 24, 48, 32)$  and  $y = (0, 4, 5, 1, 1, 5, 4, 0)$  this leads to matrix  $M_{1b}$  given in Figure 2.

Note that both method 1a and 1b lead to symmetry along the horizontal axis: each entry  $f$  mirrors its complement  $n^2 + 1 - f$ .

### 5.1.3 Method 2

Application of method 2 immediately generates  $A$  and  $B$  consisting of pairs of terms with sums of exponents equal to  $\delta(A)$  and  $\delta(B)$ , respectively. As a side-result, all matrices obtained

$$M_2 = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 48 & 57 & 24 & 9 & 40 & 49 & 32 \\ \hline 62 & 19 & 6 & 43 & 54 & 27 & 14 & 35 \\ \hline 8 & 41 & 64 & 17 & 16 & 33 & 56 & 25 \\ \hline 59 & 22 & 3 & 46 & 51 & 30 & 11 & 38 \\ \hline 2 & 47 & 58 & 23 & 10 & 39 & 50 & 31 \\ \hline 61 & 20 & 5 & 44 & 53 & 28 & 13 & 36 \\ \hline 7 & 42 & 63 & 18 & 15 & 34 & 55 & 26 \\ \hline 60 & 21 & 4 & 45 & 52 & 29 & 12 & 37 \\ \hline \end{array}$$

Figure 3: Franklin Magic Square obtained by method 2

in this way will be magic squares that are pan-diagonal. Further, keeping the pairs adjacent (i.e. on positions  $i$  and  $i+2$ ) and on the same half (either  $i+2 < n/2$  or  $i \geq n/2$ ) yields  $x$  and  $y$  vectors with the right properties, in particular they yield matrices with the bent-diagonal property. For the latter to be true,  $n$  must be a multiple of 8.

For an example of method 2 let us consider  $A(z) = (z^{32} + 1)(z^{16} + 1)(z^8 + 1)$  as above, and  $B(z) = (z^4 + 1)(z^2 + 1)(z + 1)$ , with  $B_0(z) = (1 + z^7) + (z^1 + z^6)$ , and  $B_1(z) = (z^2 + z^5) + (z^3 + z^4)$ . With vectors  $x = (0, 16, 56, 40, 8, 24, 48, 32)$  and  $y = (0, 2, 7, 5, 1, 3, 6, 4)$  this leads to matrix  $M_2$  given in Figure 3. Again, notice the most-perfectness of the 4x4 blocks.

### 5.1.4 Method 3

Application of method 3 yields 4-tuples of the same exponent sums equal to  $2\delta(A)$  and  $2\delta(B)$ , hence for multiples of 16 it works. As an example let us take  $A(z) = (z^{128} + 1)(z^{32} + 1)(z^{16} + 1)(z^8 + 1)$ , and  $B = (z^{64} + 1)(z^4 + 1)(z^2 + 1)(z + 1)$ . For splitting  $A$  we take  $\alpha = 128$ ,  $\beta = 32$ ,  $N = 24$  and we get  $A_{00}(z) = z^0 + z^{152} + z^{56} + z^{160}$ ,  $A_{10}(z) = z^{24} + z^{128} + z^{32} + z^{184}$ ,  $A_{01}(z) = z^8 + z^{144} + z^{48} + z^{168}$ ,  $A_{11}(z) = z^{16} + z^{136} + z^{40} + z^{176}$ . For splitting  $B$  we have  $\alpha = 64$ ,  $\beta = 4$ ,  $N = 3$  yielding  $B_{00}(z) = z^0 + z^{67} + z^7 + z^{68}$ ,  $B_{10}(z) = z^3 + z^{64} + z^4 + z^{71}$ ,  $B_{01}(z) = z^1 + z^{66} + z^6 + z^{69}$ ,  $B_{11}(z) = z^2 + z^{65} + z^5 + z^{70}$ . With vectors  $x = (0, 24, 152, 128, 56, 32, 160, 184, 8, 16, 144, 136, 48, 40, 168, 176)$ , and  $y = (0, 3, 67, 64, 7, 4, 68, 71, 1, 2, 66, 65, 6, 5, 69, 70)$  this yields  $M_3$  as given in Figure 4.

Notice that each 8x8 quadrant is *rotationally anti-symmetric*: rotating the quadrant by 180 degrees maps each entry on its complement.

## 5.2 Pan-diagonal Franklin Magic Squares

We may also want to enforce squares with diagonals having the magic sum. Then in addition to the previous conditions we have to restrict ourselves to polynomials  $A$  and  $B$  each splittable in four subsets of equal exponent sum, such that the sum of exponents of  $A_0$  and  $B_0$  add up to the desired value  $n(n^2 - 1)/4$ . Application of methods 2 and 3 directly leads to pan-diagonal Franklin Magic Squares, as by construction the average values of the  $x_j$  and  $y_i$  add up to  $(n^2 - 1)/2$ .

$$M_3 =$$

1 232 153 128 57 224 161 72	9 240 145 120 49 216 169 80
253 28 101 132 197 36 93 188	245 20 109 140 205 44 85 180
68 165 220 61 124 157 228 5	76 173 212 53 116 149 236 13
192 89 40 193 136 97 32 249	184 81 48 201 144 105 24 241
8 225 160 121 64 217 168 65	16 233 152 113 56 209 176 73
252 29 100 133 196 37 92 189	244 21 108 141 204 45 84 181
69 164 221 60 125 156 229 4	77 172 213 52 117 148 237 12
185 96 33 200 129 104 25 256	177 88 41 208 137 112 17 248
2 231 154 127 58 223 162 71	10 239 146 119 50 215 170 79
254 27 102 131 198 35 94 187	246 19 110 139 206 43 86 179
67 166 219 62 123 158 227 6	75 174 211 54 115 150 235 14
191 90 39 194 135 98 31 250	183 82 47 202 143 106 23 242
7 226 159 122 63 218 167 66	15 234 151 114 55 210 175 74
251 30 99 134 195 38 91 190	243 22 107 142 203 46 83 182
70 163 222 59 126 155 230 3	78 171 214 51 118 147 238 11
186 95 34 199 130 103 26 255	178 87 42 207 138 111 18 247

Figure 4: Franklin Magic Square by method 3

For methods 1a and 1b we can enforce this feature in various ways.

### 5.2.1 Method 1a

Consider in the decomposition of  $\frac{z^{n^2}-1}{z-1}$  a factor product of the form

$$W_{\alpha,\beta}(z) = (1 + z^\alpha)(1 + z^\beta) = 1 + z^\alpha + z^\beta + z^{\alpha+\beta}$$

We choose  $W_{\alpha,\beta}$  to be a factor of  $A + \overline{A}^{n^2-1-\delta(B_0)}$ . In  $(A + \overline{A}^{n^2-1-\delta(B_0)})(z)/W_{\alpha,\beta}(z)$  let us pair up terms  $z^j$  and  $z^{N-j}$ , where  $N$  is the degree of the co-factor.

Notice that  $W_{\alpha,\beta}(z)(z^j + z^{N-j}) = (z^j + z^{\alpha+N-j} + z^{\beta+N-j} + z^{\alpha+\beta+j}) + (z^{N-j} + z^{\alpha+j} + z^{\beta+j} + z^{\alpha+\beta+N-j})$ . The first four terms have exponent sum  $2N + 2\alpha + 2\beta$ , and the same holds for the last four terms. Hence the average exponent value is  $(N + \alpha + \beta)/2$ , which is half the degree of  $A + \overline{A}^{n^2-1-\delta(B_0)}$ . Note that the two parts are each others complement (with respect to power  $\nu = N + \alpha + \beta$ ).

One further observation is that in both 4-tuples, the first two terms have exponents adding up to  $N + \alpha$ , whereas the second pair has exponent sum  $N + \alpha + 2\beta$ . Assign one of the two parts to  $A$ . This split is actually already possible for  $n$  being a multiple of four. If  $n$  is a multiple of 16, the aforementioned method allows us to generate polynomials  $A$  that can be split up in four groups with  $n/4$  terms each, such that within each group the average exponent equals  $(n^2 - 1 - \delta(B_0))/2$ . Now, together with averaged exponents in  $B_0$  this leads to Franklin Magic Squares that have the additional property that all diagonals have the magic sum, and all half diagonals (i.e. diagonals within each quadrant) have half the magic sum.

Working out the above approach for  $n = 16$  yields 40320 different pan-diagonal Franklin Magic Squares the first of which is generated by:

$x$	0	96	225	129	226	130	3	99	160	32	65	193	66	194	163	35
$y$	0	4	28	24	8	12	20	16	16	20	12	8	24	28	4	0
$A_0$	0	225	226	3	160	65	66	163								
$A_1$	96	129	130	99	32	193	194	35								
$B_0$	0	28	8	20	16	12	24	4								
$B_1$	4	24	12	16	20	8	28	0								

which yields a square  $M_{pd1a}$  given in Figure 5.

The square contains all numbers from 1 to 256, with rows, columns and diagonals each summing to 2056; with half rows, half columns and half main and back diagonal summing to 1028; with bent-diagonals summing to 2056, and with each 2x2 square having sum 514. Each four-on-a-row has sum 514. The four sub-matrices are magic themselves, with constant row, column and diagonal sums, including parallels of the diagonals and back diagonals. The matrix is anti-symmetric along the horizontal line of symmetry, opposite entries add up to 257.

### 5.2.2 Method 1b

In this case we have to be able to split  $B_0$  into two parts with equal exponent sum and we like to retain the horizontal axis of symmetry. We borrow from the trick we applied for method 1a, and identify a factorization of  $(B_0 + \overline{B_0}^{n^2-1-\delta(A)})(z) = W_{\alpha,\beta}(z)\Omega(z)$ . We pair up terms

$$M_{pd1a} =$$

1 160 226 127 227 126 4 157	161 224 66 63 67 62 164 221
252 101 27 134 26 135 249 104	92 37 187 198 186 199 89 40
29 132 254 99 255 98 32 129	189 196 94 35 95 34 192 193
232 121 7 154 6 155 229 124	72 57 167 218 166 219 69 60
9 152 234 119 235 118 12 149	169 216 74 55 75 54 172 213
244 109 19 142 18 143 241 112	84 45 179 206 178 207 81 48
21 140 246 107 247 106 24 137	181 204 86 43 87 42 184 201
240 113 15 146 14 147 237 116	80 49 175 210 174 211 77 52
17 144 242 111 243 110 20 141	177 208 82 47 83 46 180 205
236 117 11 150 10 151 233 120	76 53 171 214 170 215 73 56
13 148 238 115 239 114 16 145	173 212 78 51 79 50 176 209
248 105 23 138 22 139 245 108	88 41 183 202 182 203 85 44
25 136 250 103 251 102 28 133	185 200 90 39 91 38 188 197
228 125 3 158 2 159 225 128	68 61 163 222 162 223 65 64
5 156 230 123 231 122 8 153	165 220 70 59 71 58 168 217
256 97 31 130 30 131 253 100	96 33 191 194 190 195 93 36

Figure 5: Pan-diagonal Franklin Magic Square by method 1a

$z^j$  and  $z^{N-j}$  in  $\Omega(z)$ , where  $N = \delta(\Omega)$ .

As before, rewrite  $W_{\alpha,\beta}(z)(z^j + z^{N-j}) = (z^j + z^{\alpha+N-j} + z^{\beta+N-j} + z^{\alpha+\beta+j}) + (z^{N-j} + z^{\alpha+j} + z^{\beta+j} + z^{\alpha+\beta+N-j})$ . The first four terms have exponent sum  $2N + 2\alpha + 2\beta$ , and the same holds for the last four terms. Hence the average exponent value is  $(N + \alpha + \beta)/2$ , which is half the degree of  $B_0 + \overline{B_0}^{n^2-1-\delta(A)}$ . Note that the two parts are each others complement (with respect to power  $\nu = N + \alpha + \beta$ ). Select one of the four-tuples to be a part of  $B_0$ .

If  $n$  is a multiple of 16,  $\Omega(z)$  contains an even number of matched pairs  $z^j, z^{N-j}$ . The four-tuples destined for  $B_{00}$  remain as they are, the four-tuples for  $B_{01}$  should be reversed in order, that is, rewritten as  $(z^{\alpha+\beta+j} + z^{\beta+N-j} + z^{\alpha+N-j} + z^j)$ . By doing so the pairs adjacent terms in  $B_0$  will nicely match with the pairs of adjacent terms in  $B_1$ . Again we define  $y_{n-1-i} = y_i$ , for each even  $i$ .

Application of method 1b is illustrated by the following example. Let us take  $A(z) = (z^{128} + 1)(z^{32} + 1)(z^{16} + 1)(z^8 + 1)$ , and  $B_0 + \overline{B_0}^{71} = (z^{64} + 1)(z^4 + 1)(z^2 + 1)(z + 1)$ . For splitting  $A$  into four parts we simply take matching pairs  $z^j, z^{184-j}$  and distribute these pairs evenly. We may obtain  $A_{00}(z) = z^0 + z^{184} + z^8 + z^{176}$ ,  $A_{10}(z) = z^{16} + z^{168} + z^{24} + z^{160}$ ,  $A_{01}(z) = z^{32} + z^{152} + z^{40} + z^{144}$ ,  $A_{11}(z) = z^{48} + z^{136} + z^{56} + z^{128}$ . To obtain  $B_0$  let us take  $\alpha = 64, \beta = 4, N = 3$ . We may get  $B_{00}(z) = z^0 + z^{67} + z^7 + z^{68}$ ,  $B_{10}(z) = z^1 + z^{66} + z^6 + z^{69}$ ,  $B_{01}(z) = z^{69} + z^6 + z^{66} + z^1$ ,  $B_{11}(z) = z^{68} + z^7 + z^{67} + z^0$ . Now  $A$  has average exponent 92 and  $B$  has average exponent  $71/2$  which sums up to  $255/2 = (n^2 - 1)/2$ . With vectors  $x = (0, 16, 184, 168, 8, 24, 176, 160, 32, 48, 152, 136, 40, 56, 144, 128)$  and  $y = (0, 1, 67, 66, 7, 6, 68, 69, 69, 68, 6, 7, 66, 67, 1, 0)$  we obtain matrix  $M_{pd1b}$  in Figure 6.

### 5.3 Most-perfect Squares

Sometimes we like to have yet another even stronger requirement for symmetry: diagonals should be composed of pairs of complementary integers, at distance  $n/2$ . Complementary integers are pairs of entries with sum  $(n^2 + 1)$ . Being  $n/2$  apart (which is even) they must match with exponents  $x_j, x_{j+n/2}$  both in  $A_0$  or both in  $A_1$ . Hence, methods 1a, 1b and 3 cannot yield such solutions. Method 2 does create solutions that have the right property. It is a matter of ordering the coefficients in  $x$  and  $y$  respectively in the right way so as to have  $x_j + x_{j+n/2} = \delta(A)$ , for all  $j < n/2$  and  $y_i + y_{i+n/2} = \delta(B)$ , for all  $i < n/2$ . For a given  $A$ , as before, write  $A(z) = (1 + z^\alpha)(1 + z^\beta)\Omega(z)$ , and consider matching terms  $z^j$  and  $z^{N-j}$ .

Now we rewrite  $(1 + z^\alpha)(1 + z^\beta)(z^j + z^{N-j}) = (z^j + z^{\alpha+N-j} + z^{\beta+N-j} + z^{\alpha+\beta+j}) + (z^{\alpha+\beta+N-j} + z^{\beta+j} + z^{\alpha+j} + z^{N-j})$ . Now the order in the second 4-term has been rearranged such that complementary terms can be offset in the  $x$ -vector by  $n/2$  positions. The first 4-tuples are used for building the polynomials  $A_{00}$  and  $A_{10}$ , the second 4-tuples are used for  $A_{01}$  and  $A_{11}$ .

For  $n = 16$  this approach leads to 1260 different most-perfect Franklin Magic Squares, with the additional property of four-on-a-row. The first in the series was generated by  $A, B, x$  and  $y$  given by

$x$	0	64	208	144	224	160	48	112	240	176	32	96	16	80	192	128
$y$	0	4	13	9	14	10	3	7	15	11	2	6	1	5	12	8
$A_0$	0	208	224	48	240	32	16	192								
$A_1$	64	144	160	112	176	96	80	128								
$B_0$	0	13	14	3	15	2	1	12								
$B_1$	4	9	10	7	11	6	5	8								

$$M_{pd1b} =$$

1 240 185 88 9 232 177 96	33 208 153 120 41 200 145 128
255 18 71 170 247 26 79 162	223 50 103 138 215 58 111 130
68 173 252 21 76 165 244 29	100 141 220 53 108 133 212 61
190 83 6 235 182 91 14 227	158 115 38 203 150 123 46 195
8 233 192 81 16 225 184 89	40 201 160 113 48 193 152 121
250 23 66 175 242 31 74 167	218 55 98 143 210 63 106 135
69 172 253 20 77 164 245 28	101 140 221 52 109 132 213 60
187 86 3 238 179 94 11 230	155 118 35 206 147 126 43 198
70 171 254 19 78 163 246 27	102 139 222 51 110 131 214 59
188 85 4 237 180 93 12 229	156 117 36 205 148 125 44 197
7 234 191 82 15 226 183 90	39 202 159 114 47 194 151 122
249 24 65 176 241 32 73 168	217 56 97 144 209 64 105 136
67 174 251 22 75 166 243 30	99 142 219 54 107 134 211 62
189 84 5 236 181 92 13 228	157 116 37 204 149 124 45 196
2 239 186 87 10 231 178 95	34 207 154 119 42 199 146 127
256 17 72 169 248 25 80 161	224 49 104 137 216 57 112 129

Figure 6: Pan-diagonal Franklin Magic Square obtained with method 1b

1	192	209	112	225	96	49	144	241	80	33	160	17	176	193	128
252	69	44	149	28	165	204	117	12	181	220	101	236	85	60	133
14	179	222	99	238	83	62	131	254	67	46	147	30	163	206	115
247	74	39	154	23	170	199	122	7	186	215	106	231	90	55	138
15	178	223	98	239	82	63	130	255	66	47	146	31	162	207	114
246	75	38	155	22	171	198	123	6	187	214	107	230	91	54	139
4	189	212	109	228	93	52	141	244	77	36	157	20	173	196	125
249	72	41	152	25	168	201	120	9	184	217	104	233	88	57	136
16	177	224	97	240	81	64	129	256	65	48	145	32	161	208	113
245	76	37	156	21	172	197	124	5	188	213	108	229	92	53	140
3	190	211	110	227	94	51	142	243	78	35	158	19	174	195	126
250	71	42	151	26	167	202	119	10	183	218	103	234	87	58	135
2	191	210	111	226	95	50	143	242	79	34	159	18	175	194	127
251	70	43	150	27	166	203	118	11	182	219	102	235	86	59	134
13	180	221	100	237	84	61	132	253	68	45	148	29	164	205	116
248	73	40	153	24	169	200	121	8	185	216	105	232	89	56	137

Figure 7: Most-perfect Franklin Magic Square, by method 2

and the resulting square  $M_{pf2}$  is given in Figure 7.

## 6 Franklin Magic Squares of order 20 and higher

In section 3 it was shown that no 12 by 12 Franklin Magic Square exists. It turns out that this is a unique exception. Below we show how to construct a Franklin Magic Square of order  $20 + 8k$ , for  $k \geq 0$ . We first construct two squares of order 20.

### 6.1 Franklin Magic Squares of order 20

Using method 1a we aim for a polynomial  $A$  of 20 terms, and a polynomial  $B_0$  of 10 terms, such that  $(A + \overline{A}^{399-\delta(B_0)})(z)B_0(z) = \frac{z^{400}-1}{z-1}$ . We need that  $A$  can be split into four parts of five terms with equal exponent sum, and  $B_0$  must be split into two parts of 5 terms each, again with equal exponent sum.

A candidate solution for  $B_0$  is of the form  $(1 + z^\gamma + z^{2\gamma} + z^{3\gamma} + z^{4\gamma})(1 + z^{10\gamma})$  which can be split into  $(z^0 + z^\gamma + z^{10\gamma} + z^{11\gamma} + z^{13\gamma}) + (z^{2\gamma} + z^{3\gamma} + z^{4\gamma} + z^{12\gamma} + z^{14\gamma})$ . Each part has exponent sum  $35\gamma$ .

A candidate solution for  $A$  is derived from the general form  $(A + \overline{A}^\nu)(z) = (1 + z^\alpha)(1 + z^\beta + z^{2\beta} + \dots + z^{19\beta})$ . One possible solution is  $A_{\alpha,\beta}(z) := (z^0 + z^\beta + z^{7\beta} + z^{16\beta+\alpha} + z^{17\beta}) + (z^\alpha + z^{4\beta} + z^{8\beta} + z^{11\beta} + z^{18\beta}) + (z^{4\beta+\alpha} + z^{5\beta} + z^{9\beta} + z^{10\beta} + z^{13\beta}) + (z^{2\beta} + z^{5\beta+\alpha} + z^{6\beta} + z^{12\beta} + z^{16\beta})$ . Here each part has sum  $\alpha + 41\beta$ . Take  $\nu = \alpha + 19\beta$ , then  $A$  and  $\overline{A}^\nu$  have no term in common.

1 300	2 396	8 392	117 389	18 382	105 398	6 295	10 394	11 388	14 384
120 381	119 285	113 289	4 292	103 299	16 283	115 386	111 287	110 293	107 297
21 280	22 376	28 372	137 369	38 362	125 378	26 275	30 374	31 368	34 364
160 341	159 245	153 249	44 252	143 259	56 243	155 346	151 247	150 253	147 257
201 100	202 196	208 192	317 189	218 182	305 198	206 95	210 194	211 188	214 184
320 181	319 85	313 89	204 92	303 99	216 83	315 186	311 87	310 93	307 97
221 80	222 176	228 172	337 169	238 162	325 178	226 75	230 174	231 168	234 164
340 161	339 65	333 69	224 72	323 79	236 63	335 166	331 67	330 73	327 77
261 40	262 136	268 132	377 129	278 122	365 138	266 35	270 134	271 128	274 124
360 141	359 45	353 49	244 52	343 59	256 43	355 146	351 47	350 53	347 57
41 260	42 356	48 352	157 349	58 342	145 358	46 255	50 354	51 348	54 344
140 361	139 265	133 269	24 272	123 279	36 263	135 366	131 267	130 273	127 277
61 240	62 336	68 332	177 329	78 322	165 338	66 235	70 334	71 328	74 324
180 321	179 225	173 229	64 232	163 239	76 223	175 326	171 227	170 233	167 237
81 220	82 316	88 312	197 309	98 302	185 318	86 215	90 314	91 308	94 304
200 301	199 205	193 209	84 212	183 219	96 203	195 306	191 207	190 213	187 217
241 60	242 156	248 152	357 149	258 142	345 158	246 55	250 154	251 148	254 144
380 121	379 25	373 29	264 32	363 39	276 23	375 126	371 27	370 33	367 37
281 20	282 116	288 112	397 109	298 102	385 118	286 15	290 114	291 108	294 104
400 101	399 5	393 9	284 12	383 19	296 3	395 106	391 7	390 13	387 17

Figure 8: 20x20 Franklin Magic Square  $M_{20,1}$ , constructed by method 1a

These partial solutions can be combined for  $(\alpha, \beta, \gamma) = (100, 1, 20)$  or  $(\alpha, \beta, \gamma) = (5, 20, 1)$ . We obtain solution  $(x^1, y^1)$  with

$$x^1 = (0, 100, 1, 4, 7, 8, 116, 11, 17, 18, 104, 2, 5, 105, 9, 6, 10, 12, 13, 16), \text{ and}$$

$$y^1 = (0, 280, 20, 240, 200, 80, 220, 60, 260, 40, 40, 260, 60, 220, 80, 200, 240, 20, 280, 0).$$

This yields matrix  $M_{20,1}$ , depicted in Figure 8. The second solution  $(x^2, y^2)$  is given by

$$x^2 = (0, 5, 20, 80, 140, 160, 325, 220, 340, 360, 85, 40, 100, 105, 180, 120, 200, 240, 260, 320),$$

and

$$y^2 = (0, 14, 1, 12, 10, 4, 11, 3, 13, 2, 2, 13, 3, 11, 4, 10, 12, 1, 14, 0),$$

yielding matrix  $M_{20,2}$ . The last square is given in Figure 9.

## 6.2 Franklin Magic Squares of order $20 + 8k$

The construction of 20 by 20 squares given above can be extended to yield an  $n$  by  $n$  Franklin Magic Square for any  $n = 20 + 8k$ , with  $k \geq 0$ . Again we use method 1a.

1 395 21 320 141 240 326 180 341 40	86 360 101 295 181 280 201 160 261 80
386 20 366 95 246 175 61 235 46 375	301 55 286 120 206 135 186 255 126 335
2 394 22 319 142 239 327 179 342 39	87 359 102 294 182 279 202 159 262 79
388 18 368 93 248 173 63 233 48 373	303 53 288 118 208 133 188 253 128 333
11 385 31 310 151 230 336 170 351 30	96 350 111 285 191 270 211 150 271 70
396 10 376 85 256 165 71 225 56 365	311 45 296 110 216 125 196 245 136 325
12 384 32 309 152 229 337 169 352 29	97 349 112 284 192 269 212 149 272 69
397 9 377 84 257 164 72 224 57 364	312 44 297 109 217 124 197 244 137 324
14 382 34 307 154 227 339 167 354 27	99 347 114 282 194 267 214 147 274 67
398 8 378 83 258 163 73 223 58 363	313 43 298 108 218 123 198 243 138 323
3 393 23 318 143 238 328 178 343 38	88 358 103 293 183 278 203 158 263 78
387 19 367 94 247 174 62 234 47 374	302 54 287 119 207 134 187 254 127 334
4 392 24 317 144 237 329 177 344 37	89 357 104 292 184 277 204 157 264 77
389 17 369 92 249 172 64 232 49 372	304 52 289 117 209 132 189 252 129 332
5 391 25 316 145 236 330 176 345 36	90 356 105 291 185 276 205 156 265 76
390 16 370 91 250 171 65 231 50 371	305 51 290 116 210 131 190 251 130 331
13 383 33 308 153 228 338 168 353 28	98 348 113 283 193 268 213 148 273 68
399 7 379 82 259 162 74 222 59 362	314 42 299 107 219 122 199 242 139 322
15 381 35 306 155 226 340 166 355 26	100 346 115 281 195 266 215 146 275 66
400 6 380 81 260 161 75 221 60 361	315 41 300 106 220 121 200 241 140 321

Figure 9: 20x20 Franklin Magic Square  $M_{20,2}$ , constructed by method 1a

For  $B_0$  we need a polynomial with  $10 + 4k$  terms that can be split into two parts with equal exponent sum. We choose  $B_0$  to be of the form  $(1 + z^{(10+4k)\gamma})(1 + z^\gamma + \dots + z^{(5+2k-1)\gamma})$ , where the latter factor has  $5 + 2k$  terms. Now a possible split into two parts may be  $B_{00} = (1 + z^\gamma + [z^{4\gamma} + z^{6\gamma} + \dots + z^{(5+2k-3)\gamma}]) + z^{(10+4k)\gamma}(1 + [z^\gamma + z^{3\gamma} + \dots + z^{(5+2k-2)\gamma}])$ , and  $B_{01} = (z^{2\gamma} + [z^{3\gamma} + z^{5\gamma} + \dots + z^{(5+2k-2)\gamma}]) + z^{(5+2k-1)\gamma} + z^{(10+4k)\gamma}(z^{2\gamma} + [z^{4\gamma} + z^{6\gamma} + \dots + z^{(5+2k-1)\gamma}])$ . Each part has exponent sum  $(5 + 2k)(10 + 4k)\gamma/2 + (5 + 2k)(5 + 2k - 1)\gamma/2$ .

For  $A$  to be derived from  $A + \overline{A}^{n^2-1-\delta(B_0)} = (1 + z^{(10+4k)\gamma})(1 + z^\alpha + z^{2\alpha} + \dots + z^{(n-1)\alpha})$ , we can choose either  $\gamma = 1, \alpha = 15 + 6k$ , or  $\alpha = 1, \gamma = n = 20 + 8k$ . Now write  $(1 + z^\alpha + z^{2\alpha} + \dots + z^{(n-1)\alpha}) = (1 + z^\alpha + z^{2\alpha} + \dots + z^{(4k-1)\alpha}) + z^{4k\alpha}(1 + z^\alpha + z^{2\alpha} + \dots + z^{(n-8k-1)\alpha}) + z^{(n-4k)\alpha}(1 + z^\alpha + z^{2\alpha} + \dots + z^{(4k-1)\alpha})$ .

Now define  $A(z) = 1 \cdot (1 + z^\alpha + z^{2\alpha} + \dots + z^{(4k-1)\alpha}) + z^{(10+4k)\gamma} \cdot z^{(n-4k)\alpha}(1 + z^\alpha + z^{2\alpha} + \dots + z^{(4k-1)\alpha}) + z^{4k\alpha}A_{\gamma,\alpha}(z)$ . Here the last part is taken from the general solution for  $n = 20$  in the previous subsection.

It is not difficult to see that both solutions generate an  $n$  by  $n$  Franklin Magic Square with the symmetry property along the horizontal middle line.

### 6.3 Huub Reijnders' method for a 20 by 20 Franklin Magic Square

The first known 20 by 20 Franklin square was constructed by Huub Reijnders, who did this apparently from scratch. It appears that his solution falls in the scheme set above. The exception is that he has a special way of solving  $A + \overline{A}^{n^2-1-\delta(B_0)} = (1 + z^{n/4})(1 + z^n + z^{2n} + \dots + z^{(n-1)n})$ . His solution for  $n = 20$  is  $A_{20}(z) = (1 + z^5)(1 + z^{20} + \dots + z^{140}) + (z^{160} + z^{180} + z^{200} + z^{220})$  which splits into  $(1 + z^{40} + z^{120} + z^{140}) + z^{180}$ ,  $z^5(1 + z^{40} + z^{120} + z^{140}) + z^{160}$ ,  $(z^{20} + z^{60} + z^{80} + z^{100}) + z^{220}$ , and  $z^5(z^{20} + z^{60} + z^{80} + z^{100}) + z^{200}$ , each with exponent sum 480.

The split for  $B_0$  is the same as in the subsection above.

In terms of vectors Reijnders's solution is given by

$$x = (0, 2, 1, 3, 10, 4, 11, 12, 13, 14, 14, 13, 12, 11, 4, 10, 3, 1, 2, 0), \text{ and}$$

$$y = (0, 5, 60, 65, 80, 85, 220, 200, 120, 125, 140, 145, 180, 160, 100, 105, 40, 45, 20, 25)$$

yielding matrix  $M_{20,r}$  given in Figure 10.

This solution approach can be extended to  $n = 20 + 8k$  by realizing that the above trick works by matching four exponents  $n/4$  against one exponent  $n$ . In the remainder of solution  $A$  one needs only exponents that are multiples of  $n$ . This is easily realized by considering the solution  $A_n(z) = (1 + z^{n/4})(1 + z^n + \dots + z^{7n}) + (z^{(n-4)n/2} + z^{(n-2)n/2} + z^{(n+0)n/2} + z^{(n+2)n/2}) + Q_n(z)$  where  $Q_n(z) = z^{8n} + z^{9n} + \dots + z^{(n-6)n/2} + z^{(n+4)n/2} + \dots + z^{(n-9)n}$ . Note that  $Q(z)$  contains  $n - 20 = 8k$  terms with an average exponent of  $(n - 1)n/2$ . The terms in  $Q$  can be paired up in  $4k$  pairs each with exponent sum  $(n - 1)n$ , and these pairs can be evenly divided over four sets with equal exponent sum.

## 7 Almost-Franklin Magic Squares of order 12

It was proved in section 3 that no true Franklin Magic Squares of order 12 exist. Hence, one may try to construct Magic Squares that are as 'Franklin' as possible. We will stick to the

1 398 2 397 11 396 12 388 14 386	15 387 13 389 5 390 4 399 3 400
395 8 394 9 385 10 384 18 382 20	381 19 383 17 391 16 392 7 393 6
61 338 62 337 71 336 72 328 74 326	75 327 73 329 65 330 64 339 63 340
335 68 334 69 325 70 324 78 322 80	321 79 323 77 331 76 332 67 333 66
81 318 82 317 91 316 92 308 94 306	95 307 93 309 85 310 84 319 83 320
315 88 314 89 305 90 304 98 302 100	301 99 303 97 311 96 312 87 313 86
221 178 222 177 231 176 232 168 234 166	235 167 233 169 225 170 224 179 223 180
200 203 199 204 190 205 189 213 187 215	186 214 188 212 196 211 197 202 198 201
121 278 122 277 131 276 132 268 134 266	135 267 133 269 125 270 124 279 123 280
275 128 274 129 265 130 264 138 262 140	261 139 263 137 271 136 272 127 273 126
141 258 142 257 151 256 152 248 154 246	155 247 153 249 145 250 144 259 143 260
255 148 254 149 245 150 244 158 242 160	241 159 243 157 251 156 252 147 253 146
181 218 182 217 191 216 192 208 194 206	195 207 193 209 185 210 184 219 183 220
240 163 239 164 230 165 229 173 227 175	226 174 228 172 236 171 237 162 238 161
101 298 102 297 111 296 112 288 114 286	115 287 113 289 105 290 104 299 103 300
295 108 294 109 285 110 284 118 282 120	281 119 283 117 291 116 292 107 293 106
41 358 42 357 51 356 52 348 54 346	55 347 53 349 45 350 44 359 43 360
355 48 354 49 345 50 344 58 342 60	341 59 343 57 351 56 352 47 353 46
21 378 22 377 31 376 32 368 34 366	35 367 33 369 25 370 24 379 23 380
375 28 374 29 365 30 364 38 362 40	361 39 363 37 371 36 372 27 373 26

Figure 10: 20x20 Franklin Magic Square  $M_{20,r}$ , constructed by Reijnders

property of 2x2 squares having constant sum. Further we will stick to the typical Franklin feature of having bent diagonals with the magic sum. As order 12 Franklin Magic Squares do not exist we have to give up on having magic half rows and magic half columns. Actually we may stick to having Franklin half rows and Franklin bent-diagonals if we just give up Franklin half columns. Another opportunity is to have Franklin half rows and Franklin half columns and only horizontal Franklin bent-diagonals.

We may abandon the requirement of having magic half rows and magic half columns, and turn to having either the four-on-a-row property or having most-perfectness.

## 7.1 Horizontally correct Franklin Magic Squares

Application of method 1a yields a polynomial  $A$  of 24 terms and a polynomial  $B_0$  of 6 terms. If  $A + \bar{A}$  is of the form  $(1 + z^\alpha)(1 + z^\beta + \dots + z^{11\beta})$ , with  $\alpha < \beta$  or  $\alpha \geq 12\beta$ , then a solution  $A$  exists that can be split into four parts of equal exponents sum. For instance  $A(z) = (z^{3\beta} + z^{9\beta} + z^{5\beta+\alpha}) + (z^\beta + z^{5\beta} + z^{11\beta+\alpha}) + (z^{4\beta} + z^{10\beta} + z^{3\beta+\alpha}) + (z^{2\beta} + z^{11\beta} + z^{4\beta+\alpha})$ . Here each part has exponent sum  $\alpha + 17\beta$ .

A matching  $B_0$  of the form  $(1 + z^\delta)(1 + z^\gamma + z^{2\gamma})$  leads to a vector  $y$ , with  $y_{11-i} = y_i$ , and thus yields a square with magic bent-diagonals (both horizontally and vertically). Furthermore this square has a horizontal line of symmetry reflecting complementary entries. By properly ordering the exponents one even gets columns with the four-on-a-row property: take  $y = (0, \delta, \delta + \gamma, \gamma, 2\gamma, \delta + 2\gamma, \delta + 2\gamma, 2\gamma, \gamma, \delta + \gamma, \delta, 0)$ .

An example, with  $\beta = 1$ ,  $\alpha = 72$ ,  $\gamma = 12$ ,  $\delta = 36$ , yields

$$\begin{aligned} x &= (3, 1, 9, 5, 77, 83, 4, 2, 10, 11, 75, 76) \text{ and} \\ y &= (0, 36, 48, 12, 24, 60, 60, 24, 12, 48, 36, 0). \end{aligned}$$

The result is the square  $M_{12a}$  given in Figure 11.

By interchanging rows 2, 4, 6 with 8, 10, 12 the square changes into one which has magic half columns, instead of having vertical magic bent-diagonals.

## 7.2 Decomposition and basic arrangements

In table 1 we list the possible decompositions of  $\frac{z^{144}-1}{z-1}$  into two- and three term factors with coefficients 1. There are  $\binom{6}{2} = 15$  of such decompositions.

They are labeled by a sequence of 2s and 3s that indicate the place of the factors with three terms.

Table 2 displays all possible permutations of numbers 0 up to 11 that have the properties

$$v_{4i} - v_{4i+1} + v_{4i+2} - v_{4i+3} = 0, \quad \text{for } i = 0, 1, 2, \tag{9}$$

$$v_0 + v_2 + v_4 = v_7 + v_9 + v_{11}, \tag{10}$$

up to isomorphism. These permutations were found by enumeration.

All rows except the ones marked by an asterisk have the property that for each pair  $j, 11-j$  both entries are on an even position, or both are on an odd position.

Evidently, when properties (9) and (10) hold for a certain vector  $v$ , then they also hold for  $w = Cv$ , where  $C$  is an arbitrary scalar. The right-most entries  $s - t$  in the table denote that for some vectors properties (9) and (10) also hold for exponents in the polynomials  $(1 + z^\alpha + \dots + z^{(s-1)\alpha})(1 + z^\beta + \dots + z^{(t-1)\beta})$  according to the conversion table 3.

$$M_{12a} =$$

4 143 10 139 78 61	5 142 11 133 76 68
105 38 99 42 31 120	104 39 98 48 33 113
52 95 58 91 126 13	53 94 59 85 124 20
129 14 123 18 55 96	128 15 122 24 57 89
28 119 34 115 102 37	29 118 35 109 100 44
81 62 75 66 7 144	80 63 74 72 9 137
64 83 70 79 138 1	65 82 71 73 136 8
117 26 111 30 43 108	116 27 110 36 45 101
16 131 22 127 90 49	17 130 23 121 88 56
93 50 87 54 19 132	92 51 86 60 21 125
40 107 46 103 114 25	41 106 47 97 112 32
141 2 135 6 67 84	140 3 134 12 69 77

Figure 11: As Franklin as possible, no magic half columns

222233	$(1+z)(1+z^2)(1+z^4)(1+z^8)(1+z^{16}+z^{32})(1+z^{48}+z^{96})$
222323	$(1+z)(1+z^2)(1+z^4)(1+z^8+z^{16})(1+z^{24})(1+z^{48}+z^{96})$
222332	$(1+z)(1+z^2)(1+z^4)(1+z^8+z^{16})(1+z^{24}+z^{48})(1+z^{72})$
223223	$(1+z)(1+z^2)(1+z^4+z^8)(1+z^{12})(1+z^{24})(1+z^{48}+z^{96})$
223232	$(1+z)(1+z^2)(1+z^4+z^8)(1+z^{12})(1+z^{24}+z^{48})(1+z^{72})$
223322	$(1+z)(1+z^2)(1+z^4+z^8)(1+z^{12}+z^{24})(1+z^{36})(1+z^{72})$
232223	$(1+z)(1+z^2+z^4)(1+z^6)(1+z^{12})(1+z^{24})(1+z^{48}+z^{96})$
232232	$(1+z)(1+z^2+z^4)(1+z^6)(1+z^{12})(1+z^{24}+z^{48})(1+z^{72})$
232322	$(1+z)(1+z^2+z^4)(1+z^6)(1+z^{12}+z^{24})(1+z^{36})(1+z^{72})$
233222	$(1+z)(1+z^2+z^4)(1+z^6+z^{12})(1+z^{24})(1+z^{48})(1+z^{96})$
322223	$(1+z+z^2)(1+z^3)(1+z^6)(1+z^{12})(1+z^{24})(1+z^{48}+z^{96})$
322232	$(1+z+z^2)(1+z^3)(1+z^6)(1+z^{12})(1+z^{24}+z^{48})(1+z^{72})$
322322	$(1+z+z^2)(1+z^3)(1+z^6)(1+z^{12}+z^{24})(1+z^{36})(1+z^{72})$
323222	$(1+z+z^2)(1+z^3)(1+z^6+z^{12})(1+z^{18})(1+z^{36})(1+z^{72})$
332222	$(1+z+z^2)(1+z^3+z^6)(1+z^9)(1+z^{18})(1+z^{36})(1+z^{72})$

Table 1: Possible decompositions of  $\frac{z^{144}-1}{z-1}$

$v_0$	$v_1$	$v_2$	$v_3$	$v_4$	$v_5$	$v_6$	$v_7$	$v_8$	$v_9$	$v_{10}$	$v_{11}$	remark
0	1	7	6	11	8	2	5	4	3	9	10	4-3, 6-2
0	1	7	6	9	10	4	3	2	5	11	8	6-2
0	1	10	9	6	7	4	3	2	5	11	8	*
0	2	6	4	11	10	8	9	3	1	5	7	3-4
0	2	8	6	7	10	4	1	3	5	11	9	4-3
0	2	8	6	10	7	1	4	3	5	11	9	4-3
0	2	10	8	4	9	11	6	1	3	7	5	
0	2	10	8	6	7	5	4	1	3	11	9	3-4
0	2	10	8	7	6	4	5	1	3	11	9	3-4
0	3	9	6	11	4	1	8	2	5	10	7	2-6
0	3	9	6	10	5	2	7	1	4	11	8	2-6
0	3	9	6	7	8	2	1	4	5	11	10	6-2
0	3	9	6	7	8	5	4	1	2	11	10	*
0	3	9	6	7	8	11	10	2	1	4	5	
0	3	10	7	9	4	1	6	2	5	11	8	2-6
2	5	11	8	0	7	10	3	1	4	9	6	2-6
4	5	11	10	0	3	9	6	2	1	7	8	4-3, 6-2
4	6	10	8	0	5	7	2	1	3	11	9	
5	7	11	9	0	1	3	2	6	4	8	10	3-4
6	7	10	9	1	0	3	4	5	2	8	11	

Table 2: Possible arrangements with 4-on-a-row and bent-diagonal properties

$s$	$t$	conversion of $k$
2	6	$\lfloor k/6 \rfloor \alpha + (k\%6)\beta$
3	4	$\lfloor k/4 \rfloor \alpha + (k\%4)\beta$
4	3	$\lfloor k/3 \rfloor \alpha + (k\%3)\beta$
6	2	$\lfloor k/2 \rfloor \alpha + (k\%2)\beta$

Table 3: Conversion table for order 12 sequences

$$M_{12,V} =$$

56	17	129	88	19	90	127	54	92	125	21	52
86	131	13	60	123	58	15	94	50	23	121	96
68	5	141	76	31	78	139	42	104	113	33	40
80	137	7	66	117	64	9	100	44	29	115	102
41	32	114	103	4	105	112	69	77	140	6	67
107	110	34	39	144	37	36	73	71	2	142	75
47	26	120	97	10	99	118	63	83	134	12	61
95	122	22	51	132	49	24	85	59	14	130	87
53	20	126	91	16	93	124	57	89	128	18	55
101	116	28	45	138	43	30	79	65	8	136	81
62	11	135	82	25	84	133	48	98	119	27	46
74	143	1	72	111	70	3	106	38	35	109	108

Figure 12: 12x12 Magic Square with bent-diagonals and 4-on-a-row, by method 2

### 7.3 Method 2 for bent-diagonal and 4-on-a-row properties

Application of method 2 on any vector  $x$  taken from Table 2, together with a vector  $y$  obtained by taking any row of this table and multiplying it by 12 directly leads to a pan-diagonal 12x12 Magic Square with the bent-diagonals property as well as the four-on-a-row property. One should not take any of the rows marked by an asterisk.

Now we show how method 2 can be applied on a less trivial factorization. Consider the decomposition  $\frac{z^{144}-1}{z-1} = A(z)B(z)$ , with  $A(z) = (1+z+z^2)(1+z^{36}+z^{72}+z^{108})$  and  $B(z) = (1+z^3+z^6+\dots+z^{33})$ . For  $B$  any row from the table not marked by an asterisk, multiplied by 3 will do. Let us take the last one:  $y = (18, 21, 30, 27, 3, 0, 9, 12, 15, 6, 24, 33)$ . For  $A$  pick a row marked 3-4 or 4-3, let us say the one but last row. We have  $\alpha = 1$ ,  $\beta = 36$ ,  $s = 3$ ,  $t = 4$ . The row is converted to  $x = (1+36, 1+108, 2+108, 2+36, 0+0, 0+36, 0+108, 0+72, 1+72, 1+0, 2+0, 2+72) = (37, 109, 110, 38, 0, 36, 108, 72, 73, 1, 2, 74)$ . The resulting square  $M_{12,V}$  is depicted in Figure 12.

Similarly, an even more complicated decomposition can be base of a pan-diagonal 12x12 square with 4-on-a-row and bent-diagonal properties. Consider any decomposition of  $\frac{z^{144}-1}{z-1}$  into four factors, each with a geometric series of 2, 3, 4 or 6 terms. For example, take  $A(z) = (1+z)(1+z^6+z^{12}+z^{18}+z^{24}+z^{30})$  and  $B(z) = (1+z^2+z^4)(1+z^{36}+z^{72}+z^{108})$ . For an appropriate vector  $x$  select a row from table 2 marked 2-6 or 6-2, for a vector  $y$  take a row with mark 3-4 or 4-3. Using the conversion table 3 one constructs  $x$  and  $y$  and from these one builds a 12x12 square with the desired properties.

$v_0$	$v_1$	$v_2$	$v_3$	$v_4$	$v_5$	$v_6$	$v_7$	$v_8$	$v_9$	$v_{10}$	$v_{11}$	remark
0	2	8	6	7	10	10	7	6	8	2	0	4-3
0	2	10	8	6	7	7	6	8	10	2	0	3-4

Table 4: Arrangements of  $B$  with 4-on-a-row, bent-diagonal and symmetry properties

## 7.4 Method 1b for bent-diagonal and 4-on-a-row properties

Using method 1b we start again from a decomposition into four factors as above. Given the decomposition select two factors with  $3 + 4$  or  $2 + 6$  terms, whose product will be  $A$ , and use the conversion table 3 to build an appropriate vector  $x$ .

The other two factors will have as product  $B_0 + \overline{B_0}^{143-\delta(A)}$ . Choose  $B_0$  in such a way that the six terms have exponents two pairs of which have the same sum. This is often possible in many ways. Let  $e_0, \dots, e_5$  be the exponents in  $B_0$  and assume  $e_0 + e_1 = e_2 + e_3$ . Define  $y = (e_0, e_2, e_1, e_3, e_4, e_5, e_5, e_4, e_3, e_1, e_2, e_0)$ . This arrangement will yield a square which has bent-diagonal and 4-on-a-row properties, and in addition, it will have the mirroring property, i.e. complementary entries will reflect in the horizontal axis of symmetry.

In general this procedure will not yield a square which is pan-diagonal. If we want to enforce this property we have to be more restrictive in the choice for  $A$  and  $B_0 + \overline{B_0}^{143-\delta(A)}$ . In particular we need that the average exponent in  $B_0$  equals half the degree of  $B_0 + \overline{B_0}^{143-\delta(A)}$ .

The only basic six-term that has the desired property for  $B_0$  (with  $\delta(A) = 132$ ) is  $z^0 + z^2 + z^6 + z^7 + z^8 + z^{10}$ . Note that there are two ways of pairing these exponents up appropriately. Either take  $e^1 = (0, 8, 2, 6, 7, 10)$  or  $e^2 = (0, 10, 2, 8, 6, 7)$ . Now the basic  $y$ -vectors, with their potential conversions are given in table 4.

A general description to generate a pan-diagonal Magic Square of order 12, with bent-diagonal property, with four-on-a-row property and which reflects along the horizontal axis of symmetry is the following:

1. From decomposition table 1 select a row, and pick two consecutive two-term factors. Multiply them to get a factor  $(1 + z^\alpha + z^{2\alpha} + z^{3\alpha})$ ;
2. From the same row select a three-term factor  $(1 + z^\beta + z^{2\beta})$  such that of the three remaining factors at least two are consecutive;
3. These three remaining factors constitute a polynomial  $A(z)$  for which there are several possible arrangements, by use of table 2 and an appropriate conversion. Rows marked with an asterisk should not be considered;
4. The other factors make up the factor  $B_0 + \overline{B_0}(z)$  to be arranged as one of the rows in table 4.

Remark: the *HSA*-square, designed by a group of Dutch high school students and publicized in March 2007, fits in this scheme. As an example, let us select the second row in the decomposition table  $B_0 + \overline{B_0}^\nu = (1 + z^2)(1 + z^4)(1 + z^{48} + z^{96})$  and  $A(z) = (1 + z)(1 + z^8 + z^{16})(1 + z^{24})$ . Writing out the consecutive factors we obtain  $A(z) = (1 + z)(1 + z^8 + z^{16} + z^{24} + z^{32} + z^{40})$  and  $B_0 + \overline{B_0}^\nu = (1 + z^2 + z^4 + z^6)(1 + z^{48} + z^{96})$ , with  $\nu = 143 - 41 = 102$ . From table 2 pick the first row: 0, 1, 7, 6, 11, 8, 2, 5, 4, 3, 9, 10 to arrange the exponents of

$$M_{pd12.4} =$$

1	143	26	120	42	112	9	127	17	135	34	104
48	98	23	121	7	129	40	114	32	106	15	137
101	43	126	20	142	12	109	27	117	35	134	4
140	6	115	29	99	37	132	22	124	14	107	45
53	91	78	68	94	60	61	75	69	83	86	52
90	56	65	79	49	87	82	72	74	64	57	95
55	89	80	66	96	58	63	73	71	81	88	50
92	54	67	77	51	85	84	70	76	62	59	93
5	139	30	116	46	108	13	123	21	131	38	100
44	102	19	125	3	133	36	118	28	110	11	141
97	47	122	24	138	16	105	31	113	39	130	8
144	2	119	25	103	33	136	18	128	10	111	41

Figure 13: Pan-diagonal symmetric 12x12 Magic Square with bent-diagonals, by method 1b

A. It has a 6-2 generalization, with  $\alpha = 8$  and  $\beta = 1$ . We obtain  $x = (0\alpha + 0\beta, 0\alpha + 1\beta, 3\alpha + 1\beta, 3\alpha + 0\beta, 5\alpha + 1\beta, 4\alpha + 0\beta, 1\alpha + 0\beta, 2\alpha + 1\beta, 2\alpha + 0\beta, 1\alpha + 1\beta, 4\alpha + 1\beta, 5\alpha + 0\beta) = (0, 1, 25, 24, 41, 32, 8, 17, 16, 9, 33, 40)$ .

To build  $B_0$  pick the first row from table 4  $0, 2, 8, 6, 7, 10, 10, 7, 6, 8, 2, 0$ . With the 4-3 factorization with  $\alpha = 2$  and  $\beta = 48$  this leads to  $y = (0\alpha + 0\beta, 0\alpha + 2\beta, 2\alpha + 2\beta, 2\alpha + 0\beta, 2\alpha + 1\beta, 3\alpha + 1\beta, 3\alpha + 1\beta, 2\alpha + 1\beta, 2\alpha + 0\beta, 2\alpha + 2\beta, 0\alpha + 2\beta, 0\alpha + 0\beta) = (0, 96, 100, 4, 52, 54, 54, 52, 4, 100, 96, 0)$ .

Plugging in these vectors yields a square  $M_{pd12.4}$  depicted in Figure 13.

## 7.5 Method 1a for bent-diagonal and 4-on-a-row properties

Using method 1a we start from a decomposition into factors  $A + \bar{A}$  and  $B_0$ , where the first has 24 terms and the second only 6. If we take for the first factor  $1 + z^\alpha$  times a factor representable (by conversion) with a row from table 2, we can take for  $A$  this second factor. If  $B_0(z) = (1 + z^\beta)(1 + z^\gamma + z^{2\gamma})$ , a proper reordering gives  $B_0(z) = (1 + z^{\beta+\gamma}) + (z^{2\gamma} + z^{2\gamma+\beta}) + (z^\gamma + z^\beta)$  and  $B_1(z) = (z^\beta + z^\gamma) + (z^{2\gamma+\beta} + z^{2\gamma}) + (z^{\beta+\gamma} + 1)$ . The resulting square will have bent-diagonal properties, four-on-a-row properties and symmetry along the horizontal axis of symmetry. The result will in general not be pan-diagonal.

To enforce pan-diagonality, the choice for  $A + \bar{A}^{143-\delta(B_0)}$  is restricted to be of the form  $(1 + z^\alpha)$  times a 12-term representable by a row from table 4.

## 7.6 Method 2 for constructing most-perfect order 12 Magic Squares

It is possible to impose on the 12 by 12 Magic Square that it has the most-perfectness property. For this to be true one has to have  $x_j + x_{j+6}$  equal to  $\delta(A)$ . Such an arrangement for  $A(z) = 1 + z + \dots + z^{11}$  can explicitly be found by complete enumeration.

$v_0$	$v_1$	$v_2$	$v_3$	$v_4$	$v_5$	$v_6$	$v_7$	$v_8$	$v_9$	$v_{10}$	$v_{11}$	remark
0	2	6	7	8	10	11	9	5	4	3	1	3-4, 4-3
0	2	6	7	10	8	11	9	5	4	1	3	3-4, 4-3
0	2	6	8	7	10	11	9	5	3	4	1	3-4, 4-3
0	2	7	6	8	10	11	9	4	5	3	1	3-4, 4-3
0	2	7	6	10	8	11	9	4	5	1	3	3-4, 4-3
0	2	8	6	10	7	11	9	3	5	1	4	3-4, 4-3
0	6	2	7	8	10	11	5	9	4	3	1	3-4, 4-3
0	6	2	7	10	8	11	5	9	4	1	3	3-4, 4-3
0	6	2	8	7	10	11	5	9	3	4	1	3-4, 4-3
0	7	2	8	6	10	11	4	9	3	5	1	3-4, 4-3

Table 5: arrangement with most-perfect features

This yields the table 5, in which the remark section, as before, indicates how to use the conversion table 3 to get even more polynomials with the property of providing a most-perfect arrangement.

As an example, take the first row and 12 times the last row of Table 5 to get

$$x = (0, 2, 6, 7, 8, 10, 11, 9, 5, 4, 3, 1), \text{ and}$$

$$y = (0, 84, 24, 96, 72, 120, 132, 48, 108, 36, 60, 12).$$

The resulting square  $M_{12,p}$  has magic row and column sums, magic bent-diagonals, and has complementary entries in opposite quadrants, as seen from Figure 14.

## 8 Conclusions

The existence of Franklin Magic Squares of order  $n = 4k$ , with  $n \neq 4$  and  $n \neq 12$  has been shown. Multiples of 8 pose no problems. Orders  $20 + 8k$  are more difficult to realize, but not impossible. We have described four methods by which one can construct many Franklin Magic Squares. We are not aware of any Franklin Magic Square that does not fit into one of these four schemes.

The non-existence of a 12 by 12 Franklin Magic Square has been demonstrated by an exhaustive search that was only possible by maximal use of symmetry arguments as well as aggressive pruning.

I like to thank Andries Brouwer and Tonny Hurkens for fruitful discussions, and of course Arno van den Essen, and students Petra, Jesse and Willem, for the hype and interest they created.

$$M_{12,p} =$$

1	142	7	137	9	134	12	135	6	140	4	143
60	87	54	92	52	95	49	94	55	89	57	86
25	118	31	113	33	110	36	111	30	116	28	119
48	99	42	104	40	107	37	106	43	101	45	98
73	70	79	65	81	62	84	63	78	68	76	71
24	123	18	128	16	131	13	130	19	125	21	122
133	10	139	5	141	2	144	3	138	8	136	11
96	51	90	56	88	59	85	58	91	53	93	50
109	34	115	29	117	26	120	27	114	32	112	35
108	39	102	44	100	47	97	46	103	41	105	38
61	82	67	77	69	74	72	75	66	80	64	83
132	15	126	20	124	23	121	22	127	17	129	14

Figure 14: Most-perfect 12 by 12 Magic Square with bent-diagonals