Exercises for Chapter 1

Exercise 1
Consider the graph below. Compute a minimum spanning tree of this graph.

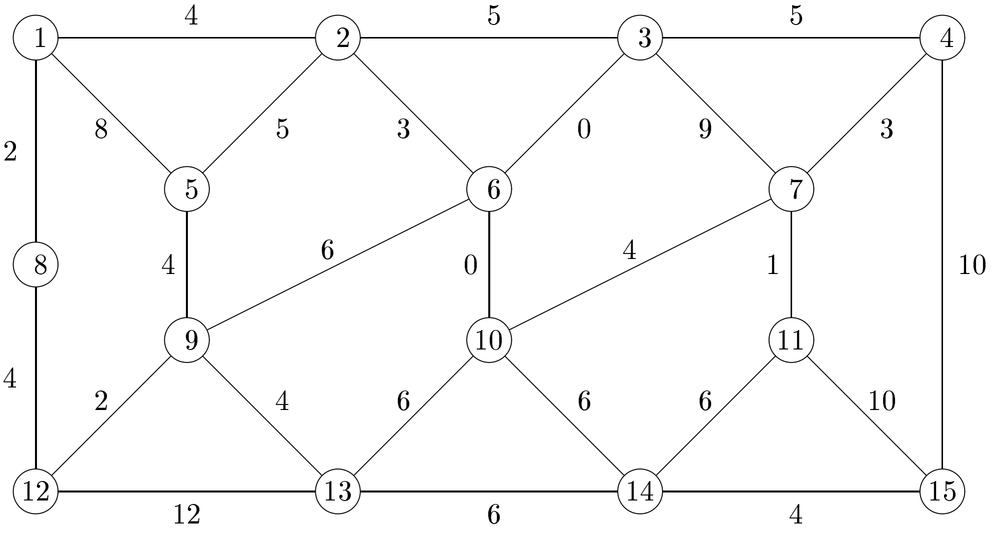


Figure 1: Graaf1

Exercise 2
Consider the graph for exercise 1. Compute the shortest path from point 1 to all other nodes.

Exercise 3
We are given a graph $G = (V, E)$ with weight function w; Let $T = (V, E')$ denote some minimum weight spanning tree.

(a) How can you determine how much you may decrease the weight of an edge $e \notin E'$ so that T remains the minimum weight spanning tree?

(b) With the use of Kruskal's algorithm some minimum weight spanning tree T has been found. Suppose that afterwards you realize that some edges have not been taken into account. Is it now possible to find a minimum spanning tree starting from T, or do you have to start all over?

Exercise 4
(a) Let $G = (V, A)$ be an arbitrary **directed** graph, on point set V and arc set A. Show that, if G is a cyclic, there exists a point $v \in V$, from which there no arc in A leaving v, that is, there is no point w, with $(v, w) \in A$.

(b) Suppose that there exists a point v without any incoming arcs. Does this imply that G is acyclic?

Exercise 5 (Bellman's algorithm)
Consider the shortest path problem defined on a graph $G = (V, E)$, with edge costs $c(e)$, for $e \in E$, and where one needs to find all shortest paths from source node s. Define $l(v, k)$ to be the length of the shortest path from s to v using at most k edges. Bellman's algorithm is

a dynamic programming algorithm that determines the shortest length paths from s to v. It is defines as follows:

Bellman's algorithm
STEP 1. Initialisation: $k \leftarrow 0$, define

$$l(v,k) = \begin{cases} 0 & \text{if } v = s \text{ and } k = 0 \\ \infty & \text{otherwise} \end{cases}$$

STEP 2. Set $l(v,k+1) = \min\{l(v,k), \min_{w \in V}\{l(w,k) + c(v,w)\}\}$ for all $v \in V$; $k \leftarrow k+1$.

STEP 3. If $k \leq K$, go to 2.

(a) How large should one set K such that in case there are no negative edge lengths $c(e)$, the length $l(v,K)$ denotes exactly the length of the shortest $s-v$-path, for every v?.

(b) Under what condition can you use the same algorithm, with the same K, even if some weights $c(e)$ have negative values?

Exercises for Chapter 2

Exercise 1
Consider the instance of the Chinese postman problem, given by the weighted graph in figure 1. Solve the Chinese postman problem; give all steps in your procedure and describe them briefly.

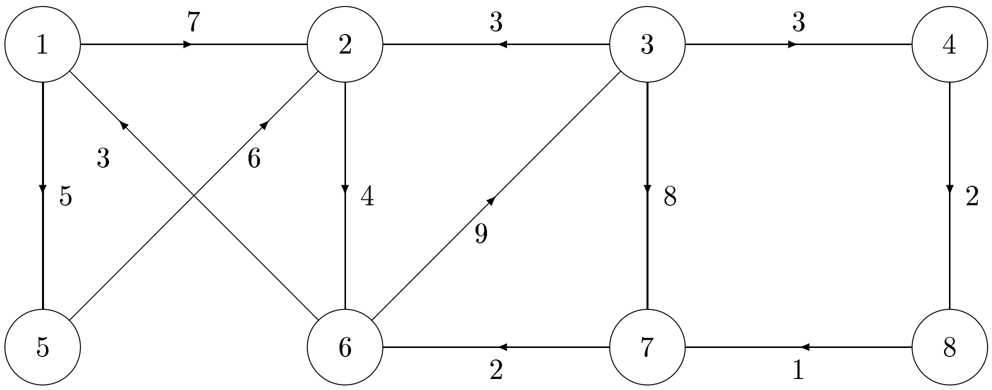
Exercise 2*(Exam 24 August 1994)*
Consider the Chinese postman problem on a **directed** graph: given a directed graph, find a minimum length tour that uses each arc at least once. (Note an arc can only be used in the right direction!) A cycle passing through all arcs exactly once is called a directed Euler cycle.

(a) Give conditions (necessary and sufficient) for a directed graph to have an Euler cycle.

(b) Solve the directed Chinese postman problem for the graph below on 8 points; the numbers besides the arcs denotes the arc lengths. Describe each step in your procedure.

Exercise 3
The **Direct Delivery** problem is defined as follows: a transport company residing in city 0 has to carry out n transport orders V_i, $(i = 1, \ldots, n)$. Transport order V_i amounts to picking up cargo at place a_i and delivering this at place b_i. It is not possible to combine orders. After picking up order i it has to be delivered in b_i immediately.

The time needed for carrying out task V_i is equal to $d(i)$ which accounts for travel time as well as time for loading and unloading etcetera. The time for moving a truck from place j to place k is given by c_{jk}. The transporter (who has only one transport vehicle) leaves place 0



at time 0. The goal is to find a solution in which all orders are carried out and the transport returns back to place 0 as early as possible.

For the instance given below, compute an optimal schedule. Is your strategy a general one in that it will solve every instance of the **Direct Delivery**-problem optimally?

V_j	Pick-up	Delivery
1	1	2
2	1	5
3	2	3
4	2	4
5	3	6
6	4	1
7	5	4
8	6	2
9	6	7

The following symmetric table gives the intercity distances.

	1	2	3	4	5	6	7
1	-	7	5	7	1	5	11
2	7	-	2	6	4	12	8
3	5	2	-	2	3	1	6
4	7	6	2	-	4	8	3
5	1	4	3	4	-	5	6
6	5	12	1	8	5	-	22
7	11	8	6	3	6	2	-

Exercise 4
Consider the road network of a city. One wants to position policemen at several street corners so as to be able to monitor all streets. From a street corner a policeman can observe each street starting from this crossing. The problem can be modeled by a graph where the set of nodes V denote the crossings, and edge ($e \in E$) between two nodes denote a road connecting the two crossings. Now the problem is to find a subset $V' \subseteq V$, such that each edge has at least one end point in V'.

(a) Show that the minimum number of points in V' is at least the size of the largest matching in the graph.

(b) Give an example for which the minimum size of V' is strictly larger than that of a maximum matching.

Exercise 5
We are given an undirected bipartite graph $G = (V, E)$. A matching is a subset K of the edges such that each node in the graph (V, K) has degree at most one. Let, for each point $v \in V$ a positive integer b_v be given. A b-matching is a subset K of the edges such that each point $v \in V$ has degree at most b_v.

(a) Describe an algorithm to find a maximum cardinality b-matching of a graph. One may use (as a black-box) a subroutine that finds an alternating path.

(b) Formulate the problem of finding a maximum cardinality b-matching as a flow-problem in a network.

Exercise 6
Given is a directed graph $G = (V, A)$, wher V denotes the node set and A denotes the set of directed arcs. The problem is now to find a maximum number of mutually arc disjoint paths from source node $s \in V$ to sink node $t \in V$, $t \neq s$. Two directed paths are called arc-disjoint if they have no arc in common. Formulate this problem as a maximum flow problem.

Exercise 7*(Exam 28 August 1991)*
Given is an undirected graph $G = (V, E)$. A *coloring* of the edges is a partition of the edge set into subsets $E_1, \ldots, E_k$, such that each pair of edges in some subset E_i $(i = 1, \ldots, k)$, has no vertex in common. The cardinality of this coloring is k. When k is minimum, it is the coloring number of the graph.

(a) Show that the coloring number of the graph is greater than or equal to the highest degree of any vertex in the graph.

(b) Given is now a **bipartite** graph, with the property that each vertex has degree d. Show that the coloring number of this graph is exactly d. You may use the following theorem: inn a bipartite graph in which each vertex has degree $d > 0$, there exists a perfect matching.

Exercise 8
A group of 12 boys and 12 girls decides to follow ball room dancing classes. For each boy-girl pair, it has been determined whether or not they could be a dancing couple; see table below.

	A	B	C	D	E	F	G	H	I	J	K	L
1	-	+	-	+	-	-	-	-	-	-	-	-
2	+	+	-	+	-	-	-	-	-	-	-	-
3	+	+	-	-	-	-	-	-	-	-	-	-
4	-	+	-	+	+	-	-	-	-	-	-	-
5	-	-	+	+	+	+	-	+	-	-	-	-
6	-	-	-	-	+	-	+	-	-	-	-	-
7	-	-	+	-	-	+	+	+	+	-	-	-
8	-	-	-	-	-	-	+	-	-	+	-	-
9	-	-	-	-	-	+	-	+	+	-	+	-
10	-	-	-	-	-	-	-	-	+	-	+	+
11	-	-	-	-	-	-	-	-	-	+	-	+
12	-	-	-	-	-	-	-	-	-	+	-	+

(a) How many couples can be on the dance floor at the same time?

(b) Assuming that your answer is less than 12, can you give a convincing argument that no perfect partition into 12 couples exists?

Exercise 9
Consider the network below. The numbers along the arcs give the value of the current flow, and the capacity of the arc, respectively. Compute the maximum flow through this network, and prove maximality by using a minimum capacity cut. Describe each step of your procedure.

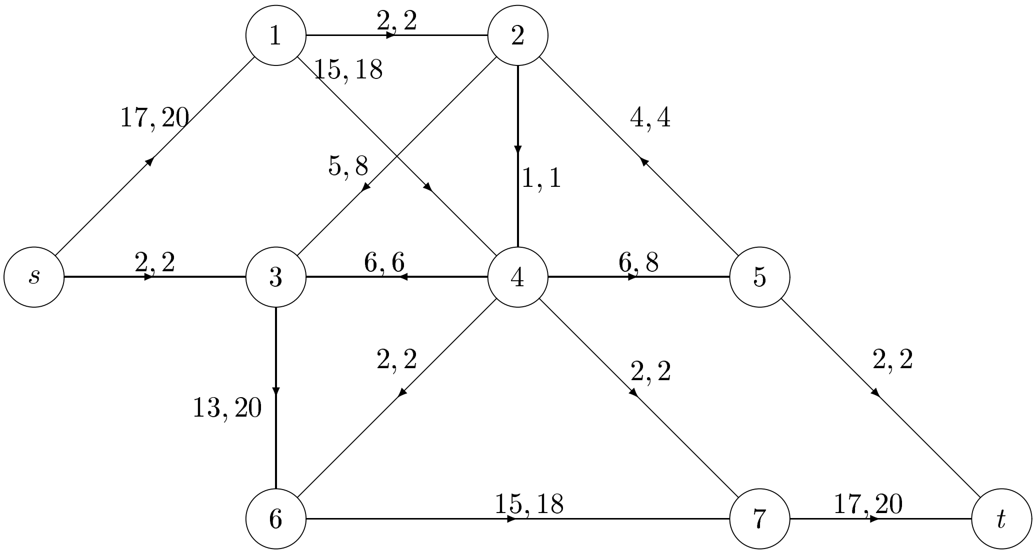


Figure 2: Network

Exercises for Chapter 3

Exercise 1
Simplify the following problem as much as possible.

$$
\begin{array}{lrcl}
\text{Minimize} & 2x_1 - x_2 + x_3 & & \\
\text{s.t.} & x_1 + 2x_2 - x_3 & \leq & 4 \\
& -x_1 - x_2 + 2x_3 & \geq & 2 \\
& x_2 + x_3 & = & 3 \\
& x_1, \quad x_2, \quad x_3 & \geq & 0
\end{array}
$$

Exercise 2 *(Exam 24 August 1994)*

(a) Solve the following instance of the linear programming problem and prove optimality of your solution.

$$\begin{array}{lrcrcrcrcl} \text{Maximize} & z & = & x_1 & + & 2x_2 & + & x_3 & & \\ \text{s.t.} & & & x_1 & + & x_2 & & & \leq & 6 \\ & & & x_1 & + & x_2 & + & x_3 & \leq & 12 \\ & & & & & x_2 & + & x_3 & \leq & 4 \\ & & & x_1 & , & x_2 & , & x_3 & \geq & 0 \end{array}$$

(b) Solve the following instance of the linear programming problem:

$$\begin{array}{lrcrcrcrcl} \text{Maximize} & z & = & x_1 & + & x_2 & + & x_3 & & \\ \text{s.t.} & & & x_1 & - & x_2 & & & \leq & 4 \\ & & & -4x_1 & + & x_2 & + & 3x_3 & \leq & 2 \\ & & & x_1 & & & - & x_3 & \leq & 2 \\ & & & x_1 & , & x_2 & , & x_3 & \geq & 0 \end{array}$$

(c) prove that the following linear programming problem instance does not have any feasible solution.

Note: do not try to do this by introducing an auxiliary variable, this will lead to cumbersome arithmetics.

$$\begin{array}{lrcrcrcrcl} \text{Maximize} & z & = & x_1 & & & & & & \\ \text{s.t.} & & & x_1 & + & 2x_2 & + & 3x_3 & \leq & 12 \\ & & & x_1 & - & 3x_2 & - & x_3 & \leq & -6 \\ & & & -x_1 & + & x_2 & - & x_3 & \leq & -4 \\ & & & x_1 & , & x_2 & , & x_3 & \geq & 0 \end{array}$$

Exercise 3
Formulate the dual of each problem defined in Exercise 2.

Exercise 4
Solve each of the following problems by use of an auxiliary variable.

(a)

$$\begin{array}{lrcl} \text{Maximize} & 3x_1 + x_2 & & \\ \text{s.t.} & x_1 - x_2 & \leq & 0 \\ & -x_1 - x_2 & \leq & -3 \\ & 2x_1 + x_2 & \leq & 5 \\ & x_1, \ x_2 & \geq & 0 \end{array}$$

(b)

$$\begin{array}{lrcl} \text{Maximize} & 3x_1 + x_2 & & \\ \text{s.t.} & x_1 - x_2 & \leq & 0 \\ & -x_1 - x_2 & \leq & -3 \\ & 2x_1 + x_2 & \leq & 2 \\ & x_1, \ x_2 & \geq & 0 \end{array}$$

(c) Maximize $x_1 - 2x_2$

$$\begin{array}{lrcl} \text{s.t.} & 3x_1 + 4x_2 & \leq & 12 \\ & -2x_1 - \ x_2 & \leq & -2 \\ & x_1, \ \ x_2 & \geq & 0 \end{array}$$

Exercise 5

Formulate the following problem as a linear programming problem.

A company has a contract or delivering 20,000 radio sets in four weeks time. According to the contract each radio set delivered in the first week will pay NLG 20, a set delivered the second week will yield NLG 18, the third week NLG 16, and the last week week NLG 14,-. The material cost of a radio set is NLG 5,- per radio.

The radio sets are assembled by employees who can each make 50 sets a week. An employee costs NLG 200,- a week. However, an employee can, instead of making radios himself, teach three trainees per week to do that. A trainee costs only NLG 100 a week. Employees and trainees that are hired remain working at the company (for this period of four weeks).

The company desires to make a net profit as large as possible.

Use the following variables:
w_i: the number of employees in week i $(i = 1, \ldots, 4)$ making radio sets;
v_i: the number of employees in week i $(i = 1, \ldots, 4)$ teaching trainees;
l_i: the number of trainees in week i $(i = 1, \ldots, 4)$;
r_i: the number of radio sets assembled in week i $(i = 1, \ldots, 4)$.

Exercise 6

Given is a directed graph $G = (V, A)$, where V denotes the vertex set, and A the set of arcs. There are k nodes s_i, $(i = 1, \ldots, k)$, called *sources*, and l nodes t_j $(j = 1, \ldots, l)$ called *sinks*. A node cannot be both source and sink. In the network we want a flow from source node s_i of at most p_i units, $i = 1, \ldots, k$; and we want a flow to enter sink node t_j, of at least q_j units, for $j = 1, \ldots, l$. the capacity of an arc $a \in A$ is given by c_a $(a \in A)$, and the cost of using an arc a is w_a per unit of flow. The problem is to find a feasible flow in the network of minimum cost. Formulate this problem as a linear programming problem.

Exercise 7*(Exam 27 August 1992)*

Formulate the following problem as a linear programming problem.

A company hires employees from a pool of 80 people.Of those people 25 are considered skilled. A skilled employee can do twice as much as an ordinary employee. The company has n projects to put its employees on. For the next T days employees have to be assigned over the projects. The number of (ordinary) employees necessary for project i $(i = 1, ..., n)$ on day t $(t = 1, ..., T)$ is $w_{i,t}$ (Reminder: a skilled employee is as good as two ordinary ones); The cost of an ordinary employee are c_1 per day per project, the costs for a skilled employee are c_2 per day per project. For each project, every day at least u_i skilled employees are required for project i.

Use variables $x_{i,t}$ to denotes the number of skilled employees, and $y_{i,t}$ for the number of ordinary employees, assigned to project i on day t.

Exercise 8

A paint factory produces from three basic materials (I, II, and III) six different end products (A, B, C, D, E, and F). The sales price of the products depends in a complex way on the combination of basic materials used for its production and is given in the following table together with the relative contribution of the basic materials in the products.

product	sales price per m^3	basis I %	basis II %	basis III %
A	2	10	40	50
B	4	20	30	50
C	5	30	30	40
D	7	40	40	20
E	8	50	30	20
F	10	60	30	10

(Example: For $1.0m^3$ of product A $0.1m^3$ of basic component I, $0.4m^3$ of basics II and $0.5m^3$ of basic III are necessary.)

In the beginning of the week the paint factory has available 100 m^3 of each basic component. Now the problem is to find a production scheme (that is: amounts of end products to produce) that result in highest sales.

(a) Formulate this problem as a linear programming problem.

(b) prove that in the current situation it does not make sense to produce products A, C or E. (Do not solve the LP for your proof.)

(c) Now the company gets the opportunity to determine the amount of components I,II, and III to start with each week, limiting the total amount of components to a volume of 300 m^3. Formulate this new problem as a linear programming problem, given that the cost of each component is equal to one unit per m^3.

Exercise 9

The CUSTOMER DELIVERY PROBLEM is defined as follows. Given is a central organization of producers all making product Q. The organization receives customer orders from n customers. Customer j $(j = 1, \ldots, n)$ orders d_j quantities of product Q. Producer i can deliver at most K_i units of product Q. Moreover, the fixed cost for producer i amount to K_i, whereas the marginal costs are p_i per unit of product Q. This means, that, for producer i, delivery of x units of product Q costs 0, if $x = 0$, and $K_i + xp_i$ if $x > 0$. The transshipment cost for delivering the product from producer i to customer j are c_{ij} per unit of product Q delivered. Find a production and delivery plan minimizing total costs. Note: it is not strictly necessary that a customer gets its order delivered by one producer only.

(a) Show how you can solve the CUSTOMER DELIVERY PROBLEM by formulating it as an integer linear programming problem.

(b) Can you simplify the model for (a) if the fixed costs would all be zero?

Exercises for Chapter 4

Exercise 1 *(Exam 23 June 1992)*

Solve the following instance of the knapsack problem, both with branch-and-bound, and with dynamic programming. There are five items (5) and the knapsack capacity is 15.

j	1	2	3	4	5
c_j	8	12	7	15	12
a_j	4	8	3	6	5

What is the optimal solution in case the capacity $b = 13$?

Exercise 2 *(Exam 7 June 1994)*
Consider the following instance of the KNAPSACK problem, where b the capacity equals 10. Suppose that besides the 5 items mentioned, there are five identical copies of a sixth item. Each copy has a weight of 2. Taking one copy would yield a profit of 5 units, taking two or three will yield 9 units, and taking four or five copies will give a profit of 17. So is allowed to take three copies but taking two will give the same profit. How do you handle this when solving the knapsack by means of branch-and-bound.?

item i	1	2	3	4	5
c_i	3	4	7	11	12
a_i	2	3	4	5	6

Exercise 3 *(Exam 24 August 1994)*
Consider the following instance of the knapsack problem, in which n items are ordered so that $a_i \leq a_{i+1}$ and $c_i \geq c_{i+1}$ for $i = 1, \ldots, n-1$.

(a) Consider an arbitrary feasible solution to this problem, in which for certain indices $i < j$, item j is taken in the solution and item i is not. Prove that one can find a solution that is at least as good as the current one by replacing item j by item i.

(b) Prove that, irrespective of the capacity of the knapsack, the solution obtained by rounding down the optimal solution to the LP-relaxation is optimal.

(c) ten items have values and weights as given in the table below. Compute the optimal solution for $b = 40$.

item i	1	2	3	4	5	6	7	8	9	10
c_i	20	22	25	28	20	19	18	18	15	12
a_i	4	5	6	7	7	7	8	8	10	10

Exercise 4
On a construction site houses are being built in three types: type A is for families with children, type B for couples, and type C for single persons. Construction costs are 200,000, 140,000 and 80,000 NLG respectively. The total budget for the project is 48 million guilders.

In view of the shortage of living facilities for single persons, the number of type C apartments should be at least twice the number of those of type B. Similarly, the number of type B houses should be at least twice the number of those of type A.

How many houses of each type should be built, so that living space becomes available for as many people as possible. With this objective in mind, a family living in a type A house counts for four (type B for two, and type C for one).

Reformulate this problem as a knapsack problem and solve it.