Vertex-coloring

Consider a connected graph $G = (V, E)$. Let $\Delta(G) := \max_v \deg(v)$ denote the maximum degree, and let $\chi(G)$ denote the vertex coloring number of G. We make the following observations:

1. $\chi(G) \leq \Delta(G)+1$. We can color the vertices of the graph with 'colors' $1, \ldots, \Delta+1$ as follows: take any labeling $v_1, \ldots, v_n$, for $i = 1, \ldots, n$ assign v_i the lowest number not used for its neighbors among $v_1, \ldots, v_{i-1}$. As there are at most Δ such neighbors, a color from $1, \ldots, \Delta + 1$ is always available.

2. If G contains a vertex v of degree $\deg(v) < \Delta(G)$, then $\chi(G) \leq \Delta(G)$. To find the coloring with Δ colors, make a breadth first search of the graph, starting at node v. Label the vertices $v_1, \ldots, v_n$, where v_i is the ith vertex encountered. Hence $v_1 = v$. Now color the nodes in backward order. For each vertex v_i at most $\Delta - 1$ neighbors are among $v_{i+1}, \ldots, v_n$ that receive a color before v_i, hence there is a color available for v_i. Note that for v there are $\Delta - \deg(v)$ colors available.

3. (Brooks' theorem) If G is not K_m or C_{2m+1} for some m, $\chi(G) \leq \Delta(G)$. A constructive proof is given below.

Proof of Brooks' theorem.

1. If $\Delta = 0$, $G = K_1$; if $\Delta = 1$, $G = K_2$; if $\Delta = 2$, then G is a path, or an even cycle, and $\chi = 2$, or G is an odd cycle with $\chi = 3$. We may assume for the remainder that $\Delta \geq 3$.

2. If G has a one-node cut-set $\{v\}$, we consider the sub-graphs G_1 and G_2. Both can be colored with $\Delta(G)$ colors, as v has degree at most $\Delta(G) - 1$ in both G_1 and G_2.

3. If G has a two-node cut-set $\{u, v\}$, with $\{u, v\} \notin E$, then we can again color both subgraphs G_1 and G_2 with $\Delta(G)$ colors as both vertices have degree at most $\Delta(G)-1$, both in G_1 and in G_2. If in both graphs G_1 and G_2, at last one vertex u or v has degree at most $\Delta(G) - 2$, then both graphs can be colored with u and v in different colors, and the colorings can be combined to a coloring of G with $\Delta(G)$ colors. If in G_1, say, both u and v have degree $\Delta(G) - 1$, then in G_2 they have degree 1; say $uu' \in E$, $vv' \in E$, with $u' \neq v'$, for some $u', v' \in V(G_2)$. In this case replace the vertex cut-set by $\{u, v'\}$. That one does the job.

4. If the above does not apply, then take any node $w \in V$ of maximum degree. It must have two neighbors u, v say, such that $uv \notin E$, otherwise $G = K_n$. The graph $G - u - v$ is connected, and so its vertices can be labeled $v_1, \ldots, v_{n-2}$ by breadth first search starting from w setting $w = v_1$. Now set $u = v_{n-1}$, and $v = v_n$, and color the vertices in order $v_n, v_{n-1}, \ldots, v_1$. For each vertex a color from $1, \ldots, \Delta$ is available.