Perfectness of bipartite graphs

Consider a bipartite graph $G = (V, E)$. We make the following observations:

1. each induced subgraph of $G$ is also a bipartite graph;

2. the complement of each induced subgraph of $G$ is also the complement of a bipartite graph;

3. a bipartite graph with at least one edge has: maximum clique size two, color number two (without any edge, clique size and color number are equal to one), hence a bipartite graph $G$ is perfect;

4. a direct, ‘constructive’ proof of perfectness of the complement of $G$ given below

Proof of complementary perfectness.
Let $\bar{G} = (U \cup W, \bar{F})$, be the complement of $G$ with where $U$ and $W$ are the color classes of $G$. Then $U$ and $W$ each induce a complete subgraph of $\bar{G}$. Let $\omega$ denote the maximum clique size in $\bar{G}$, then we need to show that we can color the vertices of $\bar{G}$ with $\omega$ colors.

1. Partition $U = U_1 \cup U_2$ and $W = W_1 \cup W_2$, where $U_2 \cup W_2$ forms the largest clique, of size $\omega$. We need $\omega$ distinct colors to color $U_2 \cup W_2$, and now claim that the coloring of $U_2 \cup W_2$ can be extended to a complete coloring without any extra colors.

2. Consider any subset $U'$ of $U_1$, and let $W' := \{v \in W_2 | \exists u \in U', uv \notin F\}$, that is $W'$ is the set of neighbors of $U'$, in $G$ restricted to $U_1 \cup W_2$. Then each vertex in $W_2 \setminus W'$ must be adjacent to each vertex in $U'$. As the clique $U_2 \cup W_2$ is of maximum size, $|U'| \leq |W'|$, otherwise exchanging these two would increase the clique size.

3. From matching theory we now know that there is a matching in $G$ of size $|U_1|$, from $U_1$ to $W_2$. In other words, for each $u \in U_1$ there is a distinct mate $m(u) \in W_2$ such that $\{u, m(u)\} \in E$, or rather, $\{u, m(u)\} \notin F$. Moreover $m(u) \neq m(u')$, for $u \neq u'$. Give $u \in U_1$ the same color as $m(u)$.

4. Similarly, we can color the vertices in $W_1$, using the colors of $U_2$. 

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