Edge-coloring

Consider a non-trivial graph $G = (V, E)$, possibly with parallel edges. Let $\Delta(G) := \max_v \deg(v)$ denote the maximum degree, and let $\chi'(G)$ denote the edge coloring number of G. We make the following observations:

1. $\Delta(G) \leq \chi'(G) = \chi(L(G)) \leq \Delta(L(G)) + 1 \leq 2\Delta(G) - 1$. Here we use the trivial upper bound for the vertex-coloring number of an arbitrary graph.

2. If G is bipartite, then $\chi'(G) = \Delta(G)$. Proof given in notes, trivial for $\Delta(G) \leq 2$.

3. (Vizings theorem) If G is simple, then $\chi'(G) \leq \Delta(G) + 1$. For a graph G on at most three nodes, the statement is trivial. Consider a smallest counter-example, on $|V(G)| > 3$ vertices. Select an arbitrary node v, then by definition, $G - v$ is $\Delta(G-v)+1$-edge-colorable, and hence it is certainly $\Delta(G)+1$-edge-colorable. Apply the following Lemma, with $k = \Delta(G)+1$. It follows that G is $\Delta(G)+1$-edge-colorable. The Lemma is applicable since every node has degree at most $(\Delta(G) + 1) - 1$.

4. (Lemma) If G is simple, $G - v$ is k-edge colorable, v and each of its neighbors have degree at most k, and at most one neighbor of v has degree equal to k, then G is k-edge-colorable as well. The proof, based on induction to k is given below.

Proof of Lemma.

1. For $k = 1$, the Lemma is evidently true. Node v has either degree 0, or degree 1.

2. For $k > 1$, we may assume that one neighbor of v has degree exactly k, while the other neighbors have degree exactly $k - 1$. For each edge we are missing at neighbor w we can introduce an extra vertex w' and an edge $\{w, w'\}$. Introducing these extra edges does not change the k-colorability of $G - v$.

3. Now consider any edge-coloring of $G - v$ with k colors. For such coloring, we define $X_i := \{u \in V | uv \in E, \text{no edge } uw \in G - v \text{ has color } i\}$. That is, X_i is the set of neighbors of v *missed* by color i. Now we have that $\sum_{i=1}^{k} |X_i| = 2\deg(v) - 1 \leq 2k - 1$. Hence the average $|X_i| < 2$, and there is at least one *odd* $|X_i|$. Choose that coloring for which $\sum_i |X_i|^2$ is minimum.

4. If $|X_i| = 1$, for some i, let $X_i = \{w\}$. Leave out, from G, edge $\{v, w\}$ and all edges colored i. We then obtain a graph G', with the property that v and its neighbors have degree one less than in G, and the property that $G' - v$ is $(k-1)$-edge-colorable. Hence, by induction, the Lemma applies, and we find that G' is $(k - 1)$-edge-colorable. Put back the removed edges and color them all k. This gives us the required k-coloring of G.

5. If the above does not apply, there must be a color i with $|X_i| = 0$, and a color j with $|X_j| \geq 2$, and $|X_j|$ odd. Consider the components in the sub-graph of $G - v$ obtained by deleting all edges not colored i or j. There is at least one such component that contains exactly one node from X_j. Interchanging colors i and j in this component maintains a k-edge-coloring of $G - v$, but increases $|X_i|^2$ from 0 to 1, whereas it decreases $|X_j|^2$ from $(2t + 1)^2$ to $(2t)^2$, for some $t \geq 1$. This decreases the sum of squares by $4t$, contradicting its minimality.