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INTEGRABILITY AND REDUCTION OF
NORMALIZED PERTURBED
KEPLERIAN SYSTEMS

by

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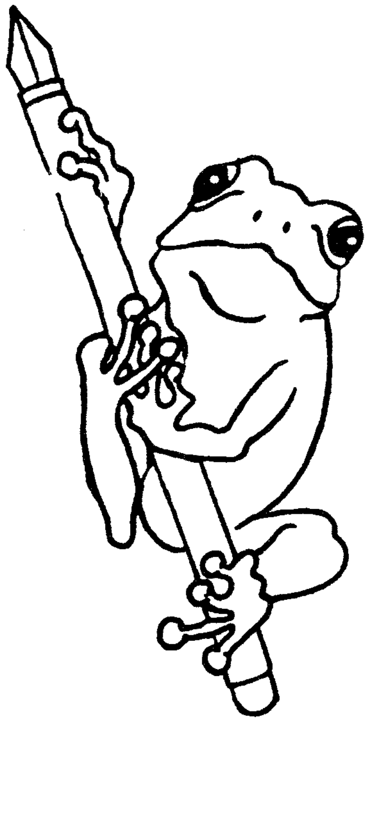

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ABSTRACT

In this paper it is shown that under certain conditions integrable formal normal forms can be obtained for perturbed 3-dimensional Keplerian systems. The two formal integrals allow us to reduce the obtained integrable approximation to a one degree of freedom system and analyze its qualitative behavior. As an example the lunar problem is considered.

Key words & phrases :

Integrability, constrained normal form, equivariant normalization, reduced phase space, perturbed Keplerian system, lunar problem.

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Table of Contents

§ 0. Introduction

In this paper we develop an algorithm for further normalization of normal forms for perturbed Keplerian systems. This process called equivariant normalization respects the symmetries obtained by earlier normalization. Our approach is similar to ideas in [3] where further normalization of Hamiltonian systems near equilibrium points is discussed. In this paper we consider formal power series perturbations of Keplerian systems. Under certain conditions on the lower order terms of the perturbation further normalization is possible. When considering 3-dimensional perturbed Keplerian systems an integrable normal form is obtained after normalizing twice.

A first step towards the normalization of perturbed Keplerian systems was made in [1] where it is shown that Hamiltonian systems on $\mathbb{R}^{2n}$ with formal power series Hamiltonian $H = H_0 + \varepsilon H_1 + \varepsilon^2 H_2 + \cdots$, $H_k \in C^\infty(\mathbb{R}^{2n})$, can be normalized if the Hamiltonian vector field X_{H_0} corresponding to the zeroeth order term H_0 has periodic flow. The normalization comes down to averaging over the periodic solutions of X_{H_0}. The resulting normal form up to order m is $\tilde{H} = H_0 + \varepsilon \tilde{H}_1 + \cdots$, $L_{H_0} \tilde{H}_k = \{\tilde{H}_k, H_0\} = 0$, $0 \le k = m$, where $\{\cdot, \cdot\}$ is the standard Poisson bracket in $\mathbb{R}^{2n}$. We speak of a normal form with respect to H_0.

A second step was made in [4] where the algorithm of [1] is adjusted in order to be able to normalize Hamiltonian systems which are constrained to some symplectic submanifold of $\mathbb{R}^{2n}$. In [4] it is also shown that regularized perturbed Keplerian systems can be considered within the framework of constrained systems to which this constrained normalization algorithm applies.

The third and final step is further normalization of the obtained constrained normal form. This comes, mutatis mutandis, down to just applying the constrained normalization algorithm again to the obtained normal form. Let $\tilde{H}$ be in constrained normal form up to order m with respect to H_0. Under certain conditions on $\tilde{H}_1$, $\tilde{H}$ can now be normalized with respect to $\tilde{H}_1$. The formal power series Hamiltonian $\hat{H}$ obtained after double normalization up to order m, now commutes, up to order m, with $\tilde{H}_1$ as well as with H_0 because the second normalization does not interfere with the earlier obtained symmetry with respect to H_0. Consequently, when H_0 corresponds to a perturbed 3-dimensional Keplerian system, we have after truncation at order $m \ge 2$ an integrable system.

The integrable approximation obtained after truncation can be analyzed by reduction to a one degree of freedom system. Hereto we apply the reduction process twice, first with respect to the X_{H_0}-flow, and second with respect to the $X_{\tilde{H}_1}$-flow. All the possible two dimensional phase spaces which one can obtain this way are described.

The contents of the paper is as follows. After some preliminaries in Section 1 equivariant normalization is considered in Sections 2, 3 and 4. In Section 2 we consider equivariant normalization from the point of view of normalization on orbit spaces. In Section 3 from the point of view of averaging over tori. In Section 4 we give a detailed treatment of further normalization of formal power series Hamiltonians with polynomial coefficients. In Section 5 it is shown how in the 3-dimensional case integrable normal forms can be reduced to one degree of freedom systems. In Sections 6 and 7 the example of the lunar problem is considered.

A first example of further normalization is found in [2], where on an ad hoc basis a second order integrable approximation for the lunar problem was found. The general approach presented in this paper was developed when considering the orbiting dust model [5], which is another example to which the method of this paper applies.

§ 1. Preliminaries - a first normalization and reduction

In this section we review some known results about constrained normalization and reduction of perturbed Keplerian systems. The main references are [1] and [4].

Let $M \subset \mathbb{R}^{2n}$ be a submanifold given as the zero set of $2m$ functions $f_1 = \cdots = f_{2m} = 0$, $m < n$. Let ω be the standard symplectic form on $\mathbb{R}^{2n}$, and suppose that M is a symplectic manifold with symplectic form $\omega_M = \omega\,|_M$, the restriction of ω to M. The Hamiltonian system on $(\mathbb{R}^{2n}, \omega)$ with Hamiltonian function $H : \mathbb{R}^{2n} \to \mathbb{R}$ is denoted by $(\mathbb{R}^{2n}, \omega, H)$, and $(\mathbb{R}^{2n}, \omega, H)$ constrained to M is $(M, \omega_M, H\,|_M)$. Now let H be a formal power series, that is, $H = \sum_{k=0}^{\infty} \varepsilon^k H_k$.

When

(c1) X_{H_0} has periodic flow

We may normalize H on $\mathbb{R}^{2n}$ by using the algorithm of [1]. When in addition

(c2) M is invariant under the flow of X_{H_0}

$H\,|_M$ can be normalized on M, which we call constrained normalization (see[4]). The constrained normalization procedure comes down to the following. We start with normalizing H on $\mathbb{R}^{2n}$ using the algorithm of [1]. At each step we adjust the normalizing transformation such that (1) it leaves M invariant and (2) it only changes the normal form by adding terms which vanish on M. As a consequence $H\,|_M$ is normalized on M by restricting the transformations to M.

The technique of constrained normalization can be applied to perturbed Keplerian systems of arbitrary dimension. We will restrict to the 3-dimensional case for convenience.

Consider $\mathbb{R}^8$ with coordinates (q, p) and standard symplectic form ω. Let

$$H_0(q,p) = (|\,p\,|^2\,|\,q\,|^2 - \langle q, p\rangle^2)^{1/2}\,, \tag{1.1}$$

$$C_8 = \{(q,p) \in \mathbb{R}^8 \mid H_0(q,p) = 0\}\,. \tag{1.2}$$

On $\mathbb{R}^8 \backslash C_8$ consider a Hamiltonian system with formal power series Hamiltonian $H = \sum_{k=0}^{\infty} \varepsilon^k H_k$, H_0 as in (1.1) and $H_k \in C^\infty(\mathbb{R}^8 \backslash C_8)$, $k \geq 1$. Let $\tilde{\omega}$ denote the restriction of ω to $\mathbb{R}^8 \backslash C_8$, let

$$T^+ S^3 = \{(q,p) \in \mathbb{R}^8 \mid |\,q\,|^2 = 1\,,\ \langle q, p\rangle = 0\,, p \neq 0\} \tag{1.3}$$

and let $\hat{\omega}$ be the restriction of ω to $T^+ S^3$. Then $(T^+ S^3, \hat{\omega})$ is a symplectic manifold. Because $H_0\,|_{T^+S^3} = |\,p\,|$, the system $(T^+ S^3, \hat{\omega}, H_0\,|_{T^+S^3})$ is precisely the regularized Kepler system for negative energy (see [6]).

Proposition 1 [4] Each formal power series perturbation of a Keplerian system with negative energy can be written as a constrained system $(T^+ S^3, \hat{\omega}, H\,|_{T^+S^3})$, where H is a formal power series on $I\!R^8 \setminus C_8$ with H_0 as in (1.1), provided the perturbation is regularized together with the Kepler system. (If not we have to exclude the collision set of the Kepler system from $I\!R^8 \setminus C_8$ and $T^+ S^3$),

Proposition 2 [4] The flow of X_{H_0}, H_0 as in (1.1), on $I\!R^8 \setminus C_8$ is periodic and leaves $T^+ S^3$ invariant.

From Propositions 1 and 2 it is clear that we may apply constrained normalization to $H\,|_{T^+S^3}$ on $T^+ S^3$. The following proposition allows us to determine what such a normal form looks like on $I\!R^8 \setminus C_8$. Write $F \approx G$ if $F\,|_{T^+S^3} = G\,|_{T^+S^3}$, and let $\{\{\ ,\ \}\}$ denote the Poisson bracket on $(T^+ S^3, \hat{\omega})$.

Proposition 3 [1] $\{\{ H_0\,|_{T^+S^3}\ ,\ F\,|_{T^+S^3}\ \}\} = 0$ if and only if $F \approx \hat{F}$, with $\hat{F}$ a formal power series of which the coefficients $\hat{F}_k$ are smooth functions in the homogeneous quadratic polynomials

$$S_{ij}\,(q, p) = q_i\, p_j - q_j\, p_i\ ,\quad 1 \leq i \overset{<}{\leq} j \leq 4 \tag{1.4}$$

Corollary 4 Let $\tilde{H}$ be a constrained normal form for H, up to order l, on $I\!R^8 \setminus C_8$, then there exists an $\hat{H} \approx \tilde{H}$ such that $\hat{H}_k$, $0 \leq k \leq l$, are smooth functions in the polynomials S_{ij}, $1 \leq i < j \leq 4$.

The S_{ij} together with the Poisson bracket on $I\!R^8$ span a Lie algebra isomorphic to $so(4)$. The action of the corresponding group $SO(4)$ leaves H_0 and $T^+ S^3$ invariant. This action corresponds to the well known symmetry group of the Kepler system generated by the momentum and Laplace vector.

Next we will consider reduction of Hamiltonian systems which on $T^+ S^3$ commute with H_0, that is, which on $I\!R^8 \setminus C_8$ have a formal power series Hamiltonian with coefficients smooth in the S_{ij}.

Consider the map

$$\rho : I\!R^8 \setminus C_8 \ \rightarrow\ I\!R^6 \setminus \{0\}\ ;\ (q, p)\ \mapsto\ (S_{12}, S_{13}, S_{23}, S_{34}, -S_{24}, S_{14}) \tag{1.5}$$

By Propositions 2 and 3 the restriction of ρ to $T^+ S^3$ is an orbit map for the flow of X_{H_0} on $T^+ S^3$. Consequently $M_l = \rho\,(T^+ S^3 \cap \{H_0 = l\})$ are the reduced phase spaces for the X_{H_0}-action (cf. [1]). The image of ρ is determined by the relation

$$S_{12}\, S_{34} - S_{13}\, S_{24} + S_{14}\, S_{23} = 0\,. \tag{1.6}$$

Furthermore we have

$$H_0(q, p)^2 = \sum_{1 \leq i < j \leq 4} S_{ij}^2 = l^2\,. \tag{1.7}$$

The equations (1.6) and (1.7) completely determine the reduced phase space M_l as a 4-dimensional variety in $I\!R^6$. The coordinate change on $I\!R^6$ given by

$$X_1 = S_{12} + S_{34}\ ,\ \ X_2 = S_{13} - S_{24}\ ,\ \ X_3 = S_{23} + S_{14}\ ,$$
$$Y_1 = S_{12} - S_{34}\ ,\ \ Y_2 = S_{13} + S_{24}\ ,\ \ Y_3 = S_{23} - S_{14}\ , \tag{1.8}$$

changes (1.6) and (1.7) to

$$X_1^2 + X_2^2 + X_3^2 = l^2\ ,\ \ Y_1^2 + Y_2^2 + Y_3^2 = l^2 \tag{1.9}$$

Consequently M_l is diffeomorphic to $S^2 \times S^2$. Identifying $\mathbb{R}^6$ with $so(4)^*$ ($*$ denoting dual) the linear coordinate change (1.8) is precisely the Lie algebra isomorphism between $so(4)^*$ and $(so(3)+so(3))^*$. The reduced phases spaces can be considered as co-adjoint orbits of $SO(3) \times SO(3)$ on the dual of its Lie algebra. The symplectic form on the reduced phase space corresponds to the Lie Poisson structure.

Now let $H = \sum_{k=0}^{\infty} \varepsilon^k H_k$, with H_0 as in (1.1) and H_k, $k \geq 1$, smooth in the S_{ij}. Then $H = \tilde{h} \circ \rho$, with $\tilde{h} = h\,|_{\rho(T^*S^3)}$, $h : \mathbb{R}^6 \rightarrow \mathbb{R}$. On M_l the reduced Hamiltonian is $\tilde{h}\,|_{M_l}$.

§ 2. Equivariant normalization - normalization on the orbit space

Consider instead of (1.5) the orbit map

$$\rho : T^+ S^3 \to I\!R^6 ; \ (q, p) \mapsto (X, Y), \tag{2.1}$$

with X and Y as defined in (1.8). The X_{H_0}-orbit space $\rho(T^+S^3)$ is given by the equation

$$X_1^2 + X_2^2 + X_3^2 = Y_1^2 + Y_2^2 + Y_3^2 , \ (X, Y) \neq 0 , \tag{2.2}$$

which is equivalent to (1.6). Let $A^\infty(I\!R^8)$ be the Lie subalgebra of $C^\infty(I\!R^8)$ which consists of the smooth functions in the quadratics X_i and Y_i, $i = 1, 2, 3$. For two functions F and G in $A^\infty(I\!R^8)$ we have

$$\{F(q, p), G(q, p)\} = \sum_{\substack{i,j=1 \\ i \neq j}}^{3} \{X_i , X_j\} \frac{\partial F}{\partial X_i} \frac{\partial G}{\partial X_j} + \{Y_i , Y_j\} \frac{\partial F}{\partial Y_i} \frac{\partial G}{\partial Y_j} . \tag{2.3}$$

Next consider $I\!R^6$ with coordinates $(x, y) = (x_1, x_2, x_3, y_1, y_2, y_3)$. Then $P = \rho(T^+ S^3)$ is defined by $P = \{(x, y) \in I\!R^6 \mid x_1^2 + x_2^2 + x_3^2 = y_1^2 + y_2^2 + y_3^2 , (x, y) \neq 0\}$. By a theorem of Schwarz [7] the pull back of ρ is surjective from $C^\infty(I\!R^6)$ onto $A^\infty(I\!R^8)$. In fact for $f \in C^\infty(I\!R^6)$ let $\tilde{f} = f \mid_P$, then by replacing (x, y) by (X, Y) $\tilde{f}$ pulls back to $f(X, Y) \in A^\infty(I\!R^8)$. It is now easy to see that the Poisson structure on $A^\infty(I\!R^8)$ given by the right hand side of (2.3) under ρ induces a Poisson structure on $C^\infty(I\!R^6)$ making ρ into a Poisson map. Because we have

$$\begin{aligned} &\{X_1 , X_2\} = 2X_3 , \ \{X_1 , X_3\} = -2X_2 , \ \{X_2 , X_3\} = 2X_1 , \\ &\{Y_1 , Y_2\} = 2Y_3 , \ \{Y_1 , Y_3\} = -2Y_2 , \ \{Y_2 , Y_3\} = 2Y_1 , \ \{X_i , Y_j\} = 0 , \end{aligned} \tag{2.4}$$

we obtain on $C^\infty(I\!R^6)$ the Poisson bracket

$$\begin{aligned} [f, g] = 2x_3 \frac{\partial f}{\partial x_1} \frac{\partial g}{\partial x_2} - 2x_2 \frac{\partial f}{\partial x_1} \frac{\partial g}{\partial x_3} + 2x_1 \frac{\partial f}{\partial x_2} \frac{\partial g}{\partial x_3} \\ -2x_3 \frac{\partial f}{\partial x_2} \frac{\partial g}{\partial x_1} + 2x_2 \frac{\partial f}{\partial x_3} \frac{\partial g}{\partial x_1} - 2x_1 \frac{\partial f}{\partial x_3} \frac{\partial g}{\partial x_2} \\ +2y_3 \frac{\partial f}{\partial y_1} \frac{\partial g}{\partial y_2} - 2y_1 \frac{\partial f}{\partial y_1} \frac{\partial g}{\partial y_3} + 2y_1 \frac{\partial f}{\partial y_2} \frac{\partial g}{\partial y_3} \\ -2y_3 \frac{\partial f}{\partial y_2} \frac{\partial g}{\partial y_1} + 2y_1 \frac{\partial f}{\partial y_3} \frac{\partial g}{\partial y_1} - 2y_1 \frac{\partial f}{\partial y_3} \frac{\partial g}{\partial y_2} , \end{aligned} \tag{2.5}$$

which has a natural restriction to P.

Note that $C^\infty(P)$ is Poisson isomorphic with $A^\infty(I\!R^8)/\mathbf{I}$, where $\mathbf{I}$ is the ideal (under multiplication) generated by the relation (2.2). In its turn $A^\infty(I\!R^8)/\mathbf{I}$ can be identified with $C^\infty(T^+ S^3)^{H_0}$, which is the space of C^∞ functions on $T^+ S^3$ invariant under the flow of X_{H_0}. Consequently $C^\infty(T^+ S^3)^{H_0}$ and $C^\infty(P)$ can be identified as Poisson algebras.

Given the Poisson structure (2.5) on $I\!R^6$ we define for $f \in C^\infty(I\!R^6)$ the Hamiltonian vector field X_f as the differential operator

$$X_f = [f, \cdot]$$

Let $h = h_0 + \varepsilon h_1 + \varepsilon^2 h_2 + \cdots$, $h_i \in C^\infty(\mathbb{R}^6)$, be a formal power series. If X_{h_0} has periodic flow then we may apply the normalization algorithm of [1] to normalize h.

Note that the fact that we are dealing with a Poisson structure instead of a symplectic structure has no influence on the algorithm. Furthermore note that the Poisson bracket of any function $f \in \mathbb{R}^6$ with $x_1^2 + x_2^2 + x_3^2 - y_1^2 - y_2^2 - y_3^2$ vanishes. As a consequence the normal form on P is found by just restricting the normalization on $\mathbb{R}^6$ to P.

Next consider on $\mathbb{R}^8$ a constrained normal form H corresponding to a perturbed Keplerian system, that is, $H = H_0 + \varepsilon H_1 + \varepsilon^2 H_2 + \cdots$, with H_0 given by (1.2), and $H_k \in A^\infty(\mathbb{R}^8)$. We can write $H(q, p) = \tilde{H}(X, Y)$. Under ρ H corresponds to a C^∞ function on P which is precisely $\tilde{H}(x, y)\,|_P$ where $\tilde{H}(x, y)$ is a smooth function on $\mathbb{R}^6 \setminus \{0\}$. Because $\tilde{H}_0$ commutes (with respect to $[\cdot, \cdot]$) with every $f \in C^\infty(\mathbb{R}^6 \setminus \{0\})$ its flow $\exp X_{\tilde{H}_0}$ acts as the indentity. Consequently $\tilde{h}(x, y) = \tilde{H}(x, y) - \tilde{H}_0(x, y) = \varepsilon \tilde{H}_1(x, y) + \varepsilon^2 \tilde{H}_2(x, y) + \cdots$ and $\tilde{H}(x, y)$ have equivalent flows on $\mathbb{R}^6 \setminus \{0\}$. Rescaling $\tilde{h}$ gives $h(x, y) = \tilde{H}_1 + \varepsilon \tilde{H}_2(x, y) + \cdots$, which can now be normalized provided $X_{\tilde{H}_1}$ has periodic flow on $\mathbb{R}^6 \setminus \{0\}$.

Recall that we can move forth and back between $A^\infty(\mathbb{R}^8)$ and $C^\infty(\mathbb{R}^6)$ by replacing (X, Y) by (x, y). Similarly a symplectic transformation $\exp L_{F(q,p)}$, with $L_F = \{\cdot, F\}$, $F \in A^\infty(\mathbb{R}^8)$ can be written as $\exp L_{\tilde{F}(X,Y)}$, $F(q, p) = \tilde{F}(X, Y)$, which corresponds to a Poisson diffeomorphism $\exp L_{\tilde{F}(x,y)}$ on $\mathbb{R}^6$. Consequently the normalization on the orbit space can be copied on $\mathbb{R}^8$. The normalizing transformations $\exp L_F$, $F \in A^\infty(\mathbb{R}^8)$, are equivariant with respect to the flow of X_{H_0}, and the resulting normal form commutes with H_0 as well as with H_1.

On $C^\infty(\mathbb{R}^6 \setminus \{0\})$ with bracket $[\cdot, \cdot]$ given by (2.5) we have that each $f \in C^\infty(\mathbb{R}^6 \setminus \{0\})$ commutes with $\tilde{H}_0(x, y)$ and $x_1^2 + x_2^2 + x_3^2 - y_1^2 - y_2^2 - y_3^2$. Consequently the reduced phase spaces M_l are invariant under the normalizing transformation. Therefore the normalization on P has a natural symplectic restriction to each reduced phase space M_l.

§ 3. Equivariant normalization - averaging over Tori

Although not really necessary we restricted ourselves in the foregoing section to 3-dimensional perturbed Keplerian systems. In this section we will start from a more general point of view.

Let M be a symplectic submanifold of $\mathbb{R}^{2n}$ as in Section 1, that is, M is given as the zero set of an even number of functions. By [4] a constrained normal form on M is the restriction to M of some formal power series

$$H = H_0 + \sum_{k=1}^{\infty} \varepsilon^k H_k, \quad H_k \in ker\ L_{H_0} \subset C^\infty(\mathbb{R}^{2n}) . \tag{3.1}$$

Consider $N^\infty(\mathbb{R}^{2n}) = \{F \in C^\infty(\mathbb{R}^{2n}) | \{F, H_0\} = 0, \quad \exp L_{\varepsilon F}(M) \subset M, \varepsilon \in \mathbb{R}\}$, that is, $N^\infty(\mathbb{R}^{2n})$ is the space of C^∞-functions in $ker\ L_{H_0}$, the flow of which leave M invariant.

Lemma 5 $N^\infty(\mathbb{R}^{2n})$ is a Lie subalgebra of $ker\ L_{H_0}$.

Proof. Follows by using the Jacobi identity and Lemma 1 of [4].

Theorem 6. Let $G : M \to \mathbb{R}$ be a Hamiltonian on M and let H be as in (3.1) such that $H|_M$ is a constrained normal form for G. Then $(\exp L_{\varepsilon^2 F} H)|_M$, with $F \in N^\infty(\mathbb{R}^{2n})$, is also a constrained normal form for G (with respect to H_0).

Thus we can use transformations $\exp L_{\varepsilon^2 F}$, $F \in N^\infty(\mathbb{R}^{2n})$, to further normalize an obtained normal form. We have

$$\exp L_{\varepsilon F} H = H_0 + \sum_{k=1}^{\infty} \varepsilon^k \tilde{H}_k ,$$

with

$$\tilde{H}_1 = \{F, H_0\} + H_1 = H_1$$

and

$$\tilde{H}_2 = \frac{1}{2} \{F, \{F, H_0\}\} + \{F, H_1\} + H_2 = \{F, H_1\} + H_2 \tag{3.2}$$

Let $H_1 \in N^\infty(\mathbb{R}^{2n})$. We may restrict L_{H_1} to $ker\ L_{H_0} \subset C^\infty(\mathbb{R}^{2n})$. If in addition there is a splitting $ker\ L_{H_0} = ker\ L_{H_1} \oplus im\ L_{H_1}$ then it is clear from (3.2) that we may normalize H_2 (constrained) with respect to H_1. Raising the power of ε by one this process can be repeated to normalize H_3 etc.

The following theorem shows that under certain conditions a splitting $ker\ L_{H_0} = ker\ L_{H_1} \oplus im\ L_{H_1}$ exists. It is a generalization of Proposition 1.1 in [1].

Recall that the flow of X_{H_0} is supposed to be periodic.

Theorem 7. Suppose H_0 and H_1 to be functionally independent. If M is fibered with 2-tori which are invariant under the flow of X_{H_0} and X_{H_1}, and on which the flow of X_{H_1} is periodic or quasi-periodic, then there exists a splitting $ker\ L_{H_0} = ker\ L_{H_1} \oplus im\ L_{H_1}$.

Proof. Clearly there is a linear combination $\alpha H_0 + \beta H_1$, $\beta \neq 0$, which has periodic flow. Choose α and $\beta \neq 0$ such that the period is minimal. The result of the theorem is now obtained by averaging over the periodic solutions of $\alpha H_0 + \beta H_1$ as in [1]. It is obvious that this can be done completely in $ker\ L_{H_0}$. $\square$

Note that H is supposed to be in constrained normal form with respect to H_0, that is H is obtained by averaging over the periodic orbits of X_{H_0}. Of course both averaging processes can be combined to one process of averaging over the invariant 2-tori.

In the case of 3-dimensional perturbed Keplerian systems $N^{\infty}(\mathbb{R}^{2n})$ and $ker\ L_{H_0}$ are replaced by $A^{\infty}(\mathbb{R}^8)$. The conditions on the flow of X_{H_1} come down to the condition that the reduced X_{H_1}-flow must be periodic. As a consequence normalization on the orbit space P is, on $\mathbb{R}^8$, precisely averaging over tori.

§ 4. Equivariant normalization - the case of polynomial coefficients

Consider a formal power series

$$H = H_0 + \sum_{k=1}^{\infty} \varepsilon^k H_k \tag{4.1}$$

with H_0 as in (1.1), and $H_k \in A^\infty(\mathbb{R}^8)$ polynomial. Let $V_{2n} \subset A^\infty(\mathbb{R}^8)$ be the set of polynomials of degree $2n$ in (q, p) (and thus of degree n in (X, Y)). Recall that V_2 is isomorphic to $so(4)$. $\sigma : F \to L_F$, with $L_F = \{\cdot\, , F\}$, is, upon a minus sign, the adjoint representation. Changing the representation space from V_2 to V_{2n} we obtain just another algebraic representation of the adjoint representation. Consequently L_F is semisimple on V_{2n} if and only if it is semisimple on V_2. If L_F acts as a semisimple linear operator on V_{2n} then we have $V_{2n} = \mathit{ker}\, L_F \oplus \mathit{im}\, L_F$. Thus we can normalize H with respect to H_1 if we can show that L_{H_1} acts semisimply on V_2. The following statement is obvious.

Theorem 7 L_{H_1} acts as a semisimple linear operator on V_2 if and only if $H_1(q, p) = I(q, p)\, F(q, p)$ with I in the center of $A^\infty(\mathbb{R}^8)$ and $F \in V_2$.

Corollary 8 If H_1 is of the form given in Theorem 7 then H can be equivariantly normalized with respect to H_1.

Suppose H_1 is as in Theorem 7. We can write the factor F as

$$F = \sum_{j=1}^{\infty} \alpha_j X_j + \gamma_j Y_j\,. \tag{4.2}$$

With $R \in A^\infty(\mathbb{R}^8)$ we have $\exp L_R\, H_1 = I \exp L_R\, F$. If $R \in V_2$ we may consider $\exp L_R$ as being an element of $SO(3) \times SO(3)$ if we identify V_2 with $so(3) + so(3)$. The action of $\exp L_R$ is then identified with the co-adjoint action. Consequently we may put H_1 by a linear coordinate change in the form

$$\tilde{H}_1 = I\,(\alpha X_1 + \gamma Y_1) = I \cdot \tilde{F}$$

We have $L_{\tilde{H}_1} = I\, L_{\tilde{F}}$. If we choose $\{X_1, X_2, X_3, Y_1, Y_2, Y_3\}$ as a basis for V_2 then the matrix of $L_{\tilde{F}}$ is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma \\ 0 & 0 & 0 & 0 & -\gamma & 0 \end{pmatrix} \tag{4.3}$$

which is semisimple. The matrix (4.3) is precisely the matrix of the vector field corresponding to $\tilde{F}$ on the orbit space $P = \rho(T^+ S^3)$, with respect to the Poisson structure (2.5). The corresponding flow on the orbit space is periodic if $\alpha / \gamma \in \mathbb{Q}$ (which corresponds to the cases considered in

Sections 2 and 3), and densely fills a 2-torus if α/γ is irrational. We will call these the resonant and the nonresonant case respectively. We may consider the reduced system corresponding to $\tilde{H}_1$ as two coupled harmonic oscillators.

We can thus determine $ker\ L_{\tilde{F}}$ in a straightforward way. In the nonresonant case $ker\ L_{\tilde{F}}$ is generated by

$$X_1\,,\, Y_1\,,\, X_2^2+X_3^2\,,\, Y_1^2+Y_2^2\,. \tag{4.4}$$

In the resonant case we may without loss of generality restrict ourselves to the case $\alpha \in \mathbb{N}\setminus\{0\}$, $\gamma \in \mathbb{Z}\setminus\{0\}$, g.c.d. $(\alpha,\gamma)=1$. In order to compute the kernel of $L_{\tilde{F}}$ we introduce complex conjugate variables $z_1 = X_2 + i X_3$, $z_2 = Y_2 + i Y_3$, $\zeta_1 = \bar{z}_1 = X_2 - i X_3$, and $\zeta_2 = \bar{z}_2 = Y_2 - i Y_3$. We obtain

$$L_{\tilde{F}} = -i\,\alpha\Big(z_1 \frac{\partial}{\partial z_1} - \zeta_1 \frac{\partial}{\partial \zeta_1}\Big) - i\,\gamma\Big(z_2 \frac{\partial}{\partial z_2} - \zeta_2 \frac{\partial}{\partial \zeta_2}\Big)$$

It is now easily found that $ker\ L_{\tilde{F}}$ is generated by

$$\pi_1 = X_1\,,\, \pi_2 = Y_1\,,\, \pi_3 = z_1\zeta_1 = X_1^2 + X_2^2\,,\, \pi_4 = z_2\zeta_2 = Y_2^2 + Y_3^2\,, \tag{4.5}$$

and

$$\begin{array}{lll} z_1^\gamma\, \zeta_2^\alpha\,, & \zeta_1^\gamma\, z_2^\alpha\,, & \text{if}\ \gamma > 0\,, \\ z_1^{|\gamma|}\, z_2^\alpha\,, & \zeta_1^{|\gamma|}\, \zeta_2^\alpha\,, & \text{if}\ \gamma < 0\,. \end{array} \tag{4.6}$$

(4.6) gives rise to the real generators

$$\pi_5 = \begin{cases} \frac{1}{2}\,(z_1^\gamma\, \zeta_2^\alpha + \zeta_1^\gamma\, z_2^\alpha)\,, & \text{if}\ \gamma > 0\,, \\ \frac{1}{2}\,(z_1^{|\gamma|}\, z_2^\alpha + \zeta_1^{|\gamma|}\, \zeta_2^\alpha)\,, & \text{if}\ \gamma < 0\,. \end{cases} \tag{4.7}$$

$$\pi_6 = \begin{cases} -\frac{1}{2}\, i\,(z_1^\gamma\, \zeta_2^\alpha + \zeta_1^\gamma\, z_2^\alpha)\,, & \text{if}\ \gamma > 0\,, \\ -\frac{1}{2}\, i\,(z_1^{|\gamma|}\, z_2^\alpha + \zeta_1^{|\gamma|}\, \zeta_2^\alpha)\,, & \text{if}\ \gamma < 0\,. \end{cases} \tag{4.8}$$

In both cases we have among the generators the relation

$$\pi_5^2 + \pi_6^2 = \pi_3^{|\gamma|}\, \pi_4^\alpha\,. \tag{4.9}$$

Provided H_1 fulfills the conditions of Theorem 7 this characterizes the terms that will appear in the normal form.

Note that by the results of Schwarz [7] $A^\infty(\mathbb{R}^8) \cap ker\ L_{\tilde{H}_1}$ consists of all smooth functions in the generators $\pi_1\,,\,\cdots\,,\,\pi_6$. Consequently in the resonant case, that is, X_{H_1} has periodic flow, this characterizes the normal forms even if the coefficients H_k, $k \geq 2$ are not polynomial.

§ 5. Reduction to one degree of freedom

Consider a Hamiltonian

$$H = H_0 + \varepsilon H_1 + \sum_{n=2}^{\infty} \varepsilon^n H_n \tag{5.1}$$

with H_0 give by (1.1),

$$H_1 = (m X_1 \pm k Y_1) I ,$$

where $m, k \in I\!N \setminus \{0\}$, g.c.d. $(m, k) = 1$ and I is in the center of $A^{\infty}(I\!R^8)$, and with $\{H_n, H_0\} = \{H_n, H_1\} = 0$, $n \geq 2$. Thus we suppose H to be in normal form with respect to H_0 as well as H_1. Consequently the coefficients H_n, $n \geq 2$, are smooth function in the generators $\pi_1, \cdots, \pi_6$ given in (4.5), (4.7) and (4.8).

In Section 1 we obtained a first reduced phase space by using the orbit map for the X_{H_0}-flow on $T^+ S^3$, which is given by

$$\rho : T^+ S^3 \to I\!R^6 ;\ (q, p) \mapsto (X, Y) . \tag{5.2}$$

The orbit space $P = \rho(T^+ S^3)$ is defined by the equation (2.2), i.e. $X_1^2 + X_2^2 + X_3^2 = Y_1^2 + Y_2^2 + Y_3^2$. The reduced phase spaces $M_l = \rho(T^+ S^3 \cap \{H_0 = l\})$ are given by equations (1.9), i.e. $X_1^2 + X_2^2 + X_3^2 = l^2$, and $Y_1^2 + Y_2^2 + Y_3^2 = l^2$. Writing $H = \tilde{H} \circ \rho$ we obtain in a trivial way the reduced Hamiltonian $\tilde{H}$ on M_l. On M_l we have the symplectic structure induced by the Poisson structure (2.5).

The flow of the reduced vector field $X_{\tilde{H}_1}$ on M is periodic and $[\tilde{H}, \tilde{H}_1] = 0$. Thus we may apply reduction with respect to the $X_{\tilde{H}_1}$ flow.

The orbit map for the $X_{\tilde{H}_1}$-flow on M_l is given by

$$\pi : M_l \to I\!R^6 ; (X, Y) \mapsto (\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6) . \tag{5.3}$$

The orbit space $P_l = (M_l)$ is determined by the following equations and inequalities obtained from (1.9), (4,5), and (4.9)

$$\begin{aligned} &\pi_1^2 + \pi_3 = l^2 ,\ \pi_2^2 + \pi_4 = l^2 , \\ &\pi_5^2 + \pi_6^2 = \pi_3^k \pi_4^m ,\ \pi_3 \geq 0 ,\ \pi_4 \geq 0 . \end{aligned} \tag{5.4}$$

The reduced phase spaces $P_{l,c}$ are given by (5.4) and

$$m\, \pi_1 \pm k\, \pi_2 = c . \tag{5.5}$$

Because I is constant on M_l (5.5) is equivalent to $\tilde{H}_1$ = constant. The reduced phase spaces $P_{l,c}$ are 2-dimensional semi-algebraic varieties. From (5.4) and (5.5) we obtain the following description of $P_{l,c}$ in (π_5, π_6, π_1)-space

$$\pi_5^2 + \pi_6^2 = (l^2 - \pi_1^2)^k \left[l^2 - \left[\frac{c - m\,\pi_1}{k} \right]^2 \right]^m ; \tag{5.6}$$

$$-l \leq \pi_1 \leq l \,, \quad \frac{-k\,l + c}{m} \leq \pi_1 \leq \frac{k\,l + c}{m} \,, \tag{5.7}$$

which holds for the plus as well as for the minus sign in (5.5).

Equation (5.6) describes a surface of revolution. The inequalities (5.7) restrict to a part of this surface. The bounds for π_1 are zeroes of the right hand side of (5.6). As a consequence the different types of reduces phase spaces $P_{l,c}$ are distinguished by the position of the zeroes of the right hand side of (5.6) relative to each other, and their multiplicities. All the possibilities are listed in Tables I, II, III, together with the corresponding bounds for the parameter c. We have to distinguish between the cases $0 < m < k$ (Table I), $0 < k < m$ (Table II), and $m = k = 1$ (Table III).

We obtain that for $-(k+m)\,l < c < (k+m)\,l$ the reduced phase spaces are obtained by rotating around the π_1-axis the part of the graph of the right hand side of (5.6) which lies in between the two middle roots.

The reduced phase spaces turn out to be sphere like surfaces which we denote by $S(s, n)$ here s is the multiplicity of the smallest root (south pole), and n is the multiplicity of the largest root (north pole). In fact $S(s, n)$ is a topological 2-sphere which is smooth except for two cusp like singularities, one of contact order $(n-2)$ at the north pole and one of contact order $(s-2)$ at the south pole. $S(1, 1)$ is a smooth two sphere, while $n = 2$ or $s = 2$ gives a cone-like singularity. An important fact to notice is that the nature of the reduced phase space differs with the parameter c (= energy of $X_{\tilde{H}_1}$), and that in general the reduced phase space is not a differentiable manifold.

TABLE I $0 < m < k$

bounds for c	ordering of zeroes	respective multiplicities	reduced phase space
$c < -(m+k)l$	$\frac{-kl+c}{m} < \frac{kl+c}{m} < -l < l$	m, m, k, k	$\varnothing$
$c = -(m+k)l$	$\frac{-kl+c}{m} < \frac{kl+c}{m} = -l < l$	$m, m+k, k$	point
$-(m+k)l < c < (m-k)l$	$\frac{-kl+c}{m} < -l < \frac{kl+c}{m} < l$	m, k, m, k	$S(k, m)$
$c = (m-k)l$	$\frac{-kl+c}{m} < -l \; \frac{kl+c}{m} = l$	$m, k, m+k$	$S(k, m+k)$
$(m-k)l < c < (k-m)l$	$\frac{-kl+c}{m} < -l < l \; \frac{kl+c}{m}$	m, k, k, m	$S(k, k)$
$c = (k-m)l$	$\frac{-kl+c}{m} =< -l < l \; \frac{kl+c}{m}$	$m+k, k, m$	$S(m+k, k)$
$(k-m)l < c < (k+m)l$	$-l < \frac{-kl+c}{m} < l < l \; \frac{kl+c}{m}$	k, m, k, m	$S(m, k)$
$c = (k+m)l$	$-l < \frac{-kl+c}{m} = l < \frac{kl+c}{m}$	$k, m+k, m$	point
$c > (k+m)l$	$-l < l < \frac{-kl+c}{m} < \frac{kl+c}{m}$	k, k, m, m	$\varnothing$

TABLE II $0 < k < m$

bounds for c	ordering of zeroes	respective multiplicities	reduced phase space
$c < -(m+k)l$	$\frac{-kl+c}{m} < \frac{kl+c}{m} < -l < l$	m, m, k, k	$\varnothing$
$c = -(m+k)l$	$\frac{-kl+c}{m} < \frac{kl+c}{m} = -l < l$	$m, m+k, k$	point
$-(m+k)l < (k-m)l$	$\frac{-kl+c}{m} < -l < \frac{kl+c}{m} < l$	m, k, m, k	$S(k, m)$
$c = (k-m)l$	$\frac{-kl+c}{m} = -l < \frac{kl+c}{m} < l$	$m+k, m, k$	$S(m+k, m)$
$(k-m)l < c < (m-k)l$	$-l < \frac{-kl+c}{m} < \frac{kl+c}{m} < l$	k, m, m, k	$S(m, m)$
$c = (m-k)l$	$-l < \frac{-kl+c}{m} < \frac{kl+c}{m} = l$	$k, m, m+k$	$S(m, m+k)$
$(m-k)l < c < (m+k)l$	$-l < \frac{-kl+c}{m} < l < \frac{kl+c}{m}$	k, m, k, m	$S(m, k)$
$c = (m+k)l$	$-l < \frac{-kl+c}{m} = l < \frac{kl+c}{m}$	$k, m+k, m$	point
$c > (m+k)l$	$-l < l < \frac{-kl+c}{m} < \frac{kl+c}{m}$	k, k, m, m	$\varnothing$

TABLE III $k = m = 1$

bounds for c	ordering of zeroes	respective multiplicities	reduced phase space
$c < -2\,l$	$-l+c < l+c < -l < l$	1, 1, 1, 1	$\varnothing$
$c = -2\,l$	$-l+c < l+c = -l < l$	1, 2, 1	point
$-2\,l < c < 0$	$-l+c < -l < l+c < l$	1, 1, 1, 1	$S(1, 1)$
$c = 0$	$-l = -l < l = l$	2, 2	$S(2, 2)$
$0 < c < 2\,l$	$-l < -l+c < l < l+c$	1, 2, 1	$S(1, 1)$
$c = 2\,l$	$-l < -l+c = l < l+c$	1, 2, 1	point
$c > 2\,l$	$-l < l < -l+c < l+c$	1, 1, 1, 1	$\varnothing$

§ 6. The lunar problem - normal form

In this section we will show that the theory of the previous sections applies to the three dimensional lunar problem. We obtain a second order integrable normal form for the lunar problem which can be analyzed in a straightforward way. Our results partly cover earlier results of Kummer [2] who also analyzed a second order integrable normal form for the lunar problem. We will not compare the two normal forms because in general normal forms are not unique ([3]) and it can be quite hard to find the diffeomorhpism mapping two given normal forms to each other.

The Hamiltonian for the lunar problem is given by

$$K(x, y) = \tfrac{1}{2} \, | y |^2 - \frac{1}{| x |} - \lambda(x_1 y_2 - x_2 y_1) -$$

$$- \lambda^2 \frac{(1-\nu)}{2} (3x_1^2 - | x |^2) + O(\nu^{-1}\lambda^4) \tag{6.1}$$

which we consider on the energy surface $K = -\frac{1}{2} \, k_2$, $k > 0$. λ is the perturbation parameter and ν is the relative earth mass. After regularization and constrained normalization we obtain (see [4] corrected version), up to order two, in terms of X_i, Y_i, $i = 1, 2, 3$,

$$H(q, p) = H_0(q, p) + \lambda H_1(q, p) + \lambda^2 H_2(q, p)$$

with $H_0(q, p)$ given by (1.1), and

$$H_1 = -\frac{1}{2k} H_0 (X_1 + Y_1) \tag{6.2}$$

$$\begin{aligned} H_2 = & -\tfrac{3}{4} \, \frac{(1-\nu)}{k^3} H_0 (X_3 - Y_3)^2 + \tfrac{1}{2} \, \frac{(1-\nu)}{k^3} H_0^3 + \frac{1}{16 \, k^3} H_0 (X_1 + Y_1)^2 \\ & + \frac{3}{16} \, \frac{(1-\nu)}{k^3} H_0 (2H_0^2 - 2(X_3 Y_3 + X_2 Y_2 + X_1 Y_1)) - \\ & - \frac{3}{16} \, \frac{(1-\nu)}{k^3} H_0 (2H_0^2 + 2(X_1 Y_1 + X_2 Y_2 - X_3 Y_3)) \\ & - \frac{1}{128 \, k^2} \left(\tfrac{1}{2} + \frac{1}{H_0}\right) (X_1 + Y_1)^2 \, (2H_0^2 - 2(X_3 Y_3 + X_2 Y_2 + X_1 Y_1)) \end{aligned} \tag{6.3}$$

where

$$2 \, H_0^2 = \sum_{i=1}^{3} X_i^2 + Y_i^2 \, .$$

For further normalization of H_2 we may apply the theory of Section 4. From (6.2) we see that we have a 1 : 1 resonance. Thus $ker \, L_{H_1} = ker \, L_{X_1 + Y_1}$ is generated by

$$\begin{aligned} &\pi_1 = X_1 \, , \; \pi_2 = Y_1 \, , \; \pi_3 = X_2^2 + X_3^2 \, , \; \pi_4 = Y_2^2 + Y_3^2 \, , \\ &\pi_5 = X_2 Y_2 + X_3 Y_3 \, , \; \pi_6 = X_2 Y_3 - X_3 Y_2 \, , \end{aligned} \tag{6.4}$$

with the relation

$$\pi_5^2 + \pi_6^2 = \pi_3 \pi_4 \tag{6.5}$$

To obtain the second order normal form with respect to H_0 and H_1 we have to split H_2 in a part in *ker* L_{H_1} and a part in *im* L_{H_1}, the part in *ker*L_{H_1} then is the desired normal form. We have

$$L_{X_1+Y_1}\,(X_3Y_2) = -2\,X_2\,Y_2 + 2\,X_3\,Y_3\,,$$

$$L_{X_1+Y_1}\,(X_2X_3) = 2\,X_2^2 - 2\,X_3^2\,,$$

$$L_{X_1+Y_2}\,(Y_2Y_3) = 2\,Y_3^2 - 2\,Y_2^2$$

Consequently we obtain the following splitting for X_3^2, Y_3^2, $X_3\,Y_3$,

$$\begin{aligned} X_3^2 &= \tfrac{1}{2}\,(X_2^2+X_3^2) - \tfrac{1}{2}\,(X_2^2-X_3^2) \\ Y_3^2 &= \tfrac{1}{2}\,(Y_2^2+Y_3^2) - \tfrac{1}{2}\,(Y_2^2-Y_3^2)\,, \\ X_3\,Y_3 &= \tfrac{1}{2}\,(X_2Y_2+X_3Y_3) - \tfrac{1}{2}\,(X_2Y_2-X_3Y_3)\,. \end{aligned}$$

Thus the normal form for H_2 is

$$\begin{aligned} \overline{H}_2 = &\tfrac{1}{2}\,\frac{(1-\nu)}{k^3}\,H_0^3 + \frac{1}{16\,k^2}\,H_0\,(X_1+Y_1)^2 - \frac{3}{8}\,\frac{(1-\nu)}{k^3}\,H_0\,(X_2^2+X_3^2) \\ &- \frac{3}{8}\,\frac{(1-\nu)}{k^3}\,H_0\,(Y_2^2+Y_3^2) + \frac{3}{8}\,\frac{(1-\nu)}{k^3}\,H_0\,(X_2Y_2+X_3Y_3) \\ &- \tfrac{3}{4}\,\frac{(1-\nu)}{k^3}\,H_0\,X_1\,Y_1 - \frac{1}{64\,k^2}\,(\tfrac{1}{2} + \frac{1}{H_0})\,H_0^2\,(X_1+Y_1)^2 \\ &+ \frac{1}{64\,k^2}\,(\tfrac{1}{2} + \frac{1}{H_0})\,(X_1+Y_1)^2\,(X_2Y_2+X_3Y_3) + \\ &+ \frac{1}{64\,k^2}\,(\tfrac{1}{2} + \frac{1}{H_0})\,(X_1+Y_1)^2\,X_1\,Y_1 \end{aligned} \tag{6.6}$$

The reduced phase spaces $P_{l,c}$ (see Section 5) are given by
$H_0 = l$, $X_1 + Y_1 = 2c$, $\pi_1^2 + \pi_3 = l^2$, $\pi_2^2 + \pi_4 = l^2$, $\pi_5^2 + \pi_6^2 = \pi_3\pi_4$.
We get that on $P_{l,c}$ our Hamiltonian is, modulo constants, equal to

$$\overline{H}_2 = \alpha\,(\pi_1 - c)^2 + \beta\,\pi_5\,, \tag{6.7}$$

with

$$\alpha = \frac{3}{2}\,\frac{(1-\nu)}{k^3}\,l - \frac{1}{16\,k^2}\,(\tfrac{1}{2} + \frac{1}{l})\,c^2\,,$$

$$\beta = \frac{3}{8}\,\frac{(1-\nu)}{k^3}\,l + \frac{1}{16\,k^2}\,(\tfrac{1}{2} + \frac{1}{l})\,c^2\,. \tag{6.8}$$

In fact (6.7) gives the reduced Hamiltonian on $P_{l,c}$ parametrized in (π_5, π_6, π_1)-space.

§ 7. The lunar problem - analysis of the integrable approximation

In the previous section we obtained a normal form for the lunar problem up to order two. Truncation gave us an integrable approximation of the lunar problem. Applying reduction as in Section 5 we obtain in (π_5, π_6, π_1)-space the reduced phase space $P_{l,c}$ given by (see (5.6) and (5.7)).

$$\pi_5^2 + \pi_6^2 = (l^2 - \pi_1^2)\,(l^2 - (2c - \pi_1)^2)\,, \tag{7.1}$$

$$-l \le \pi_1 \le l\,,\ 2c - l \le \pi \le l + 2c\,,\ |c| \le l\,,\ l > 0\,. \tag{7.2}$$

and the reduced Hamiltonian (see (6.7) and (6.8))

$$\overline{H}_2 = \alpha(\pi_1 - c)^2 + \beta\,\pi_{\cancel{6}5} \tag{7.3}$$

Substituting $\sigma_1 = \pi_1 - c$, $\sigma_2 = \pi_5$, $\sigma_3 = \pi_6$ we get

$$\sigma_2^2 + \sigma_3^2 = ((l - |c|)^2 - \sigma_1^2)\,((l + |c|)^2 - \sigma_1^2)\,, \tag{7.4}$$

$$|\sigma_1| \le l - |c|\ ,\ |c| \le l\,,\ l > 0\,, \tag{7.5}$$

$$\overline{H}_2 = \alpha\,\sigma_1^2 + \beta\,\sigma_2\,. \tag{7.6}$$

From Table III we see that the reduced phase space $P_{l,c}$ is $S(1,1)$, that is, a smooth S^2, for $0 < |c| < l$, and $S(2,2)$ for $c = 0$. The reduced system is a one degree of freedom system. The trajectories are precisely the intersections of the $\overline{H}_2$ level surfaces with the two dimensional reduced phase space $P_{l,c}$. We know the global phase portrait if we know the critical points of $\overline{H}_2$ on $P_{l,c}$. These critical points correspond to the stationary points of the reduced system. These critical points were determined for general α and β in [5]. We will not repeat the analysis but state the results using [5]. We have to take care of the fact that in the case of the lunar problem α and β depend on c, that is, the sign of $\alpha^2 - \beta^2$ might change with c. Furthermore $\beta \ne 0$, but α can be equal to zero.

$\overline{H}_2$ has a critical point on $P_{l,c}$ if the energy surface $\overline{H}_2 = L$ is tangent to $P_{l,c}$. Using Lagrange multipliers it is easily obtained that all critical points must be in the $\sigma_3 = 0$ plane, that is, on the topological circle $S^1_{l,c} = P_{l,c} \cap \{\sigma_3 = 0\}$. Putting $\sigma_3 = 0$ in (7.4) and eliminating σ_2 Using $h = \alpha\,\sigma_1^2 + \beta\,\sigma_2$, $\beta \ne 0$, we obtain

$$(\alpha^2 - \beta^2)\,\sigma_1^4 + 2(-\alpha h + \beta^2(l^2 + c^2))\,\sigma_1^2 + (h^2 - \beta^2(l^2 - c^2)^2) = 0 \tag{7.7}$$

$$|\sigma_1| \le l - |c|\,,\ \ |c| \le l\,,\ l > 0\,. \tag{7.8}$$

Now $(\sigma_1\,,\ \frac{1}{\beta}(h - \alpha\sigma_1^2),\ 0)$ is critical point of $\overline{H}_2$ on $P_{l,c}$ if and only if σ_1 is a double root of (7.7) which satisfies (7.8). Considering the discriminant locus of (7.7) taking the inequalities (7.8) into account gives that (7.7) has three branches of double roots given by (see [5])

$$h = \pm\left(\frac{3}{8}\frac{(1-\nu)}{k^3}l + \frac{1}{16k^2}\left(\frac{1}{2}+\frac{1}{l}\right)c^2\right)(l^2-c^2),\ |c| \le l,\ l > 0 \tag{7.9}$$

$$h = \left(\frac{3}{2}\frac{(1-\nu)}{k^3}l - \frac{1}{16k^2}\left(\frac{1}{2}+\frac{1}{l}\right)c^2\right)(l^2+c^2) -$$

$$-2\,l\,|c|\sqrt{\frac{135}{64}\frac{(1-\nu)}{k^6}l^2 - \frac{15}{64}\frac{(1-\nu)}{k^5}\left(\frac{l}{2}+1\right)c^2}\ ,$$

$$|c| \le l\sqrt{\frac{18(1-\nu)}{30(1-\nu)+k(l+2)}}\ ,\ l > 0 \tag{7.10}$$

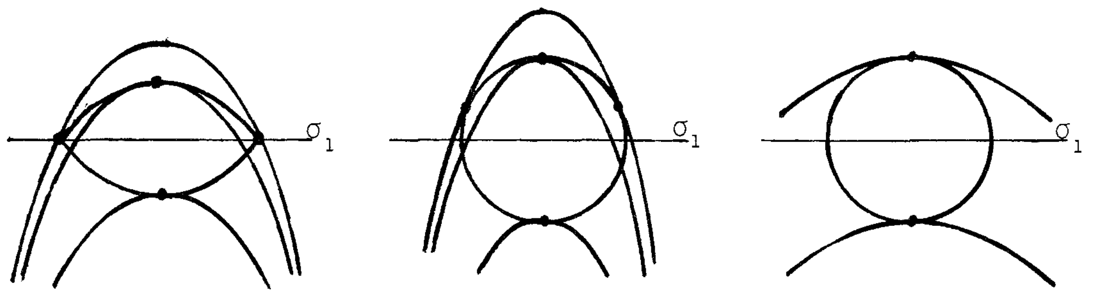
Note that for $c = c_0 = l\sqrt{\frac{18(1-\nu)}{30(1-\nu)+k(l+2)}}$; $c = -c_0$ the third branch attaches to the positive branch of (7.9). The third branch only exists for $\alpha^2 - \beta^2 \ge 0$, which is equivalent to $|c| < l\sqrt{\frac{18(1-\nu)}{k(l+2)}}$. Because $c_0 < l\sqrt{\frac{18(1-\nu)}{k(l+2)}}$ this last condition is always satisfied.

The tangency of $h = \alpha\,\sigma_1^2 + \beta\,\sigma_2$ and $S^1_{l,c}$ (in the (σ_1, σ_2)-plane) is sketched in Figure 1 for the different values of c. In Figure 2 the families of critical point are given in the parameter plane (c, h).

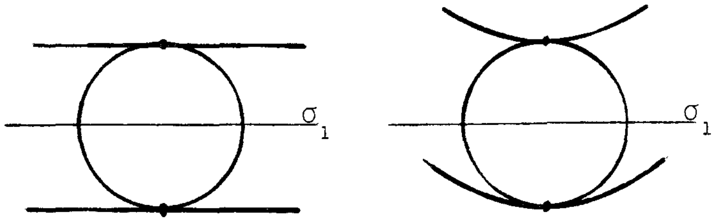
From Figure 1 it is immediately clear which critical points are elliptic and which are hyperbolic. This is indicated in Figure 2. Note that for the branch attached to h_0 (corresponds to (7.10)) each point corresponds to two critical points (see Fig. 1b.).

The results found are an extension of and in agreement with the results in [2]. Note that the points $(\pm l, 0)$ and $(0, h_0)$ (counted twice) corresponds to the four critical points of the only once reduced system on M_l.

For a further discussion of how the re· found here are related to the lunar problem in its original formulation we refer the reader to [7].

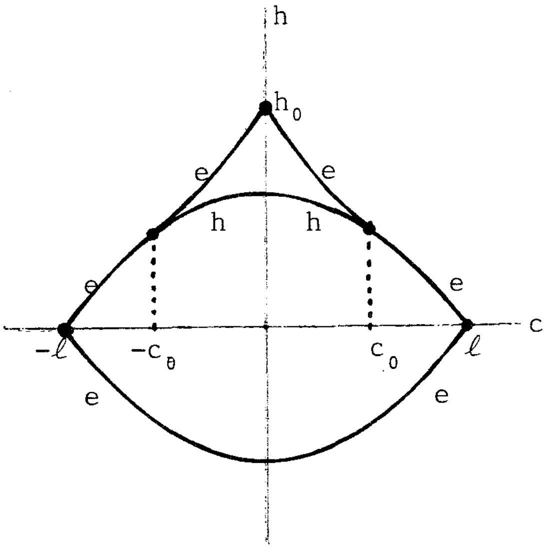


a: $c = 0$ b: $0 < c < c_0$ c: $c_0 \leq |c| < \ell\left(\frac{48(1-\nu)}{k(\ell+2)}\right)$



d: $|c| = \ell\left(\frac{48(1-\nu)}{k(\ell+2)}\right)^{\frac{1}{2}}$ e: $|c| > \ell\left(\frac{48(1-\nu)}{k(\ell+2)}\right)^{\frac{1}{2}}$

figure 1. Tangency of $\bar{H}_2$ and $S^1_{\ell,c}$.



$$c_0 = \ell\left(\frac{18(1-\nu)}{30(1-\nu)+k(\ell+2)}\right)^{\frac{1}{2}}$$

$$h_0 = \frac{3(1-\nu)}{2k^3}\ell^3$$

figure 2. Critical points of $\bar{H}_2$ on $P_{\ell,c}$ in the (c,h)-plane.

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