A Note on Local Morse Theory in Scale Space and Gaussian Deformations

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Abstract. In this note we study the local behavior of singularities occurring in scale space under Gaussian blurring. Based on ideas from singularity theory for vector fields this is done by considering deformations or unfoldings. To deal with the special nature of the problem the concept of Gaussian deformation is introduced. Using singularity theory the stability of these deformations is considered. New concepts of one-sided stability and one-sided equivalence are introduced. This way a classification of stable singularities is obtained which agrees with those known in literature.

1 Introduction

In computer vision (see [6], [7]) the main problem is to identify and manipulate objects in a computer screen image. In general the image is given by a "pixel intensity" function. By embedding this function in a one-parameter family a scale-space representation is obtained, with the parameter representing the scale. The scale on which the image is considered is changed by applying "blurring". In most literature Gaussian blurring is considered. Starting with an intensity function $u_0(x)$ on $\mathbb{R}^n$ Gaussian blurring yields a family of intensity functions $u(x;\tau)$, where τ can be considered the scale parameter. When considering Gaussian blurring these functions have to satisfy the diffusion equation

$$\frac{\partial u}{\partial \tau} = \Delta u \ , \tag{1}$$

with $u(x,0) = u_0(x)$, $u_0 : \mathbb{R}^n \to \mathbb{R} \in C^0(\mathbb{R}^n,\mathbb{R})$. In [5] it is shown that the axioms of scale space allow also blurring with respect to the operators $-(-\Delta)^\alpha$, $0 < \alpha < 1$. Through blurring a less detailed image is obtained by diffusion of the intensity function which is equivalent to considering a larger scale. This way the image can be blurred in such a way that it only contains the relevant information. At some stages one may however want to reconstruct parts of the image in more detail.

In the following we want to understand the behavior of the intensity functions, especially the qualitative changes the intensity functions will undergo when changing the scale parameter. We will at first restrict to the case of two dimensional images, where local qualitative changes will be studied. Local qualitative changes occur when a critical point of the intensity function changes its nature. If the intensity functions are required to be sufficiently differentiable one can interpret the level lines of the intensity function as integral curves of an Hamiltonian system or equipotential lines of a gradient vector field. The scale parameter can now be considered as a deformation parameter of some planar Hamiltonian system or gradient vector field. Consequently, the questions concerning the local behavior of critical points of smooth functions is similar to studying bifurcations of vector fields. The intensity functions are however parameter dependent in a particular way, i.e. the parameter is introduced by Gaussian blurring. To deal with this we introduce the new concept of semigroup deformations and show that this is the proper framework for studying bifurcations under Gaussian blurring. We will illustrate the theory by applying it to Gaussian deformations, that is, work with the semigroup generated by the Laplace operator. For semigroups generated by the operators $-(-\Delta)^{\alpha}$, $0 < \alpha < 1$, the theory should apply as well. Because in this case the computations will be somewhat more complicated this question will not be addressed in this paper. When considering the local behavior of critical points of smooth function on $\mathbb{R}^2$ we may assume the critical point to be at the origin. This can be obtained by allowing shifts or working modulo addition of constants. Also the gradient vanishes at the critical point and consequently the nature of the critical point is determined by the eigenvalues of the hessian. By Morse theory the generic critical points are centers and saddles. The idea taken from the study of singularities of vector fields is now to start from a (possibly non-generic) critical point, unfold the function and derive results concerning its stability. This way this paper provides an alternative way to obtain results concerning stable singularities in Gaussian scale space which can be found in [3] where a complete singularity theoretic treatment is given of Gaussian blurring. In a much more general context results are obtained in [4] for many other generalizations, including working wit other semigroups. Compared to the singularity theoretic approach in [3] we circumvent the problem of defining the right group of transformations by applying geometric arguments. These geometric arguments allow us to obtain the results concerning this special case in a straightforward way from the results known in singularity theory. The results obtained are in a slightly different form compared to Damon's. The origin of these differences lies in the fact that the underlying group of transformations is different from the ones chosen in [3] as we exploit full $\mathcal{A}$-equivalence.

2 Preliminaries on Stability, Deformations and Unfoldings

We will start with recalling some facts from singularity theory and/or bifurcation theory ([1,2,10]) in order to reveal the precise meaning of the terminology used.

Let $\mathcal{G}$ be a group of transformations acting on the space of functions. (or a local group at 0 acting on the space of germs of functions). For instance right-left action of origin preserving diffeomorphisms $\mathcal{G} = \mathcal{A} = \mathit{Diff}_n \times \mathit{Diff}_1$, with Diff_n the group of C^∞ diffeomorphisms from $\mathbb{R}^n$ to itself, acting by $g \cdot f(x) = \psi \circ f \circ \phi^{-1}(x)$, $g \in \mathcal{G}$, $\phi \in \mathit{Diff}_n$ and $\psi \in \mathit{Diff}_1$.

Also other groups can be chosen. To illustrate this we will give the groups used by Damon [3]. Damon [3] introduces $\mathcal{G} = \mathcal{H}$, with $\mathcal{H}$ the group of pairs (ϕ, c), $\phi \in \mathit{Diff}_{n+1}$ of the form $\phi(x,t) = (\phi_1(x,t), \phi_2(t))$ with $\phi_2'(0) > 0$ and $c \in \mathit{Diff}_1$, and acting by $g \cdot f(x,t) = f \circ \phi(x,t) + c(t)$. In addition $\mathcal{G} = \mathcal{IS}$ is introduced in [3], with $\mathcal{IS}$ the group of pairs (ϕ, ψ), $\phi \in \mathit{Diff}_{n+1}$ of the form $\phi(x,t) = (\phi_1(x,t), \phi_2(t))$ with $\phi_2'(0) > 0$ and $\psi \in \mathit{Diff}_2$ of the form $\psi(y,t) = (\psi_1(y,t), t)$ with $(\partial\psi_1/\partial y)(0,0) > 0$ and $\psi_1(0,t) = 0$, and acting by $g \cdot f(x,t) = \psi_1(f \circ \phi(x,t), t) + c$. The groups $\mathcal{H}$ and $\mathcal{IS}$ should be considered as local groups of germs of diffeomorphisms at the origin acting on germs of functions. Damon describes $\mathcal{IS}$-equivalence as $\mathcal{A}$-equivalence of unfoldings preserving the target.

Note that Damon considers groups of diffeomorphisms on $\mathbb{R}^{n+1}$, that is, the parameter is included. In the sequel we will start with considering group actions of groups that do not depend on the scale parameter. The parameter will be introduced by unfolding and considering equivalence of unfoldings.

Taking the scale parameter t in $\mathcal{IS}$ equal to a constant this group reduce to to $\mathcal{A}$ with adding constants, i.e we obtain the group $\mathcal{IS}_c$ which is the group of triples (ϕ, ψ, c), $\phi \in \mathit{Diff}_n$, $\psi \in \mathit{Diff}_1$ with $(\partial\psi/\partial y)(0) > 0$ and $\psi(0) = 0$, c a constant, and acting by $g \cdot f(x) = \psi(f \circ \phi(x)) + c$.

Definition 1. *Two functions f , h from $\mathbb{R}^n$ to $\mathbb{R}$ are $\mathcal{G}$-* ***equivalent*** *if*

$$f = g \cdot h$$

for some $g \in \mathcal{G}$.

Note that the notion of equivalence depends on the choice of the group of transformations chosen.

Definition 2. *Let $f_0 : \mathbb{R}^n \to \mathbb{R}$. A* ***s-parameter unfolding*** *of f_0 is a map $f : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R} \times \mathbb{R}^s$ such that*

- ***i.*** *$f(x,u) = (\tilde{f}(x,u), u)$, where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^s$, $\tilde{f}(x,u) \in \mathbb{R}$,i.e. $\pi \circ f = \pi$, where $\pi : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^s$ and $\pi : \mathbb{R} \times \mathbb{R}^s \to \mathbb{R}^s$ are the canonical projections.*
- ***ii.*** *$f_0(x) = \tilde{f}(x,0)$.*

In practice one often calls $\tilde{f}(x,u)$ an unfolding of f_0, although, if one wants to be precise $\tilde{f} : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}$ is actually a **deformation** of f_0. The **constant unfolding** is the unfolding f of f_0 with $\tilde{f}(x,u) = f_0(x)$.

When considering unfoldings of functions $f(x,u)$ we may also consider a group of transformations $\tilde{\mathcal{G}}$ consisting of unfoldings of $\mathcal{G}$ acting on these unfolded functions. A transformation $g(x,u) \in \tilde{\mathcal{G}}$ is an **unfolding of the identity** if $g(x,0) = id$. We denote the unfoldings of the identity by $\mathcal{G}_{un}$.

Let f be a s-parameter unfolding of f_0. Consider a map χ given by $\chi : v \to u = \chi(v)$, i.e. a transformation acting on the parameter space is considered. The **pull-back** of f by χ is the t-parameter unfolding

$$\chi^* f : \mathbb{R}^n \times \mathbb{R}^t \to \mathbb{R} \times \mathbb{R}^t; (x,v) \to (\tilde{f}(x,\chi(v)), v) \ .$$

Definition 3. *Two s-parameter unfoldings f and h of f_0 are* ***equivalent*** *if there exists a $g \in \mathcal{G}_{un}$ such that*

$$h \ = \ g \cdot f \ .$$

An unfolding is ***trivial*** *if it is equivalent to the constant unfolding. An unfolding f of f_0 is* ***universal*** *if every unfolding of f_0 is equivalent to $\chi^* f$ for some mapping χ.*

For $\mathcal{A}_{un}$ equivalence this means that

$$h = \psi \circ f \circ \varphi^{-1} \tag{2}$$

with

$$\varphi : \mathbb{R}^{n+s} \to \mathbb{R}^{n+s}; (x,t) \to (\tilde{\varphi}(x,t), t) \ ,$$

and

$$\psi : \mathbb{R}^{1+s} \to \mathbb{R}^{1+s}; (y,t) \to (\tilde{\psi}(y,t), t) \ ,$$

where both $\tilde{\varphi}$ and $\tilde{\psi}$ are unfoldings of the identity, i.e. $\tilde{\varphi}(x,0) = x$ and $\tilde{\psi}(x,0) = x$.

Two arbitrary unfoldings f and h of f_0 are equivalent if h is $\mathcal{G}_{un}$ equivalent to $\chi^* f$ for some C^∞ map χ.

With abuse of language we wil say in the remainder of this paper that unfoldings f and h are $\mathcal{G}$-equivalent, where the action of the maps is as given above.

Like before a trivial or universal unfolding gives rise to a trivial or universal deformation.

Moreover

Definition 4. *A function f_0 is* ***stable*** *if any unfolding of f_0 is trivial.*

The notion of stability can be rephrased as follows. A function f_0 is stable if all functions that are close to f_0 (in an appropriate topology) are equivalent to f_0, i.e. are in the $\mathcal{G}$-group orbit through f_0. Stability is usually considered through the equivalent notion of infinitesimal stability, i.e. formulated in terms of the tangent space to the orbit. Let f_0 be in $\mathcal{E}_0$, the space of smooth germs of functions at zero. Let $T_{\mathcal{G}}(f_0)$ denote the tangent space at f_0 to the $\mathcal{G}$-orbit through f_0.

Proposition 1. *$f_0 \in \mathcal{E}_0$ is* ***stable*** *if an only if $T_{\mathcal{G}}(f_0) = \mathcal{E}_0$.*

Thus f_0 is stable if and only if the complement to the tangent space to the $\mathcal{G}$-orbit at f_0 is empty. If f_0 is not stable then define

Definition 5. *The* ***$\mathcal{G}$-codimension*** *of $f \in \mathcal{E}_0$ is*

$$d(f, \mathcal{G}) := dim(\mathcal{E}_0 / T\mathcal{G}(f)\ ) \ .$$

If the co-dimension is non-zero the nontrivial deformations of f_0 can be found by unfolding f_0 in the directions which are in the complement to the tangent space. The tangent space can be given the form of a module of vector fields. Also describing the complement by a basis of vector fields X_i, we may consider the one-parameter groups e^{tX_i} acting on the space of functions. Then $f(x,t) = e^{tX_i} f_0(x)$ gives a deformation of f_0 in the direction of X_i with initial speed $X_i f_0$, where $X_i f_0$ is the derivative of f_0 along X_i. A deformation is a universal deformation if the unfolding directions span the complement to the tangent space to the orbit at f_0. Therefore a universal deformation is stable.

3 Deformations by Semi-Groups

If the unfolding directions are prescribed to follow the orbit of a semigroup, for instance because the deformation is governed by some partial differential equation, we speak of a semigroup deformation. Let S_τ, $\tau > 0$ be a semigroup, then the semigroup deformation of f_0 is a function $f(x,\tau) = S_\tau f_0$, with $f : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$. By extending the domain of the parameter τ to $\mathbb{R}$ we obtain $F(x,\tau)$ with $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$. We obtain the following theorem (cf [3] Lemma 4.5).

Theorem 1. *$F(x,\tau)$ is $\mathcal{G}$-stable if and only if either f_0 is stable or $d(f_0,\mathcal{G})$=1 and $\frac{\partial S_\tau f_0}{\partial \tau}|_{\tau=0}$ generates the complement of $T_\mathcal{G}(f_0)$ in $\mathcal{E}_0$.*

Proof. If f_0 is stable then any unfolding is trivial and hence stable. If f_0 is of co-dimension 1 then the unfolding $F(x,\tau)$ is stable if and only if it is universal. $\square$

In the above the notion of stability is used with respect to the full group action. A semigroup deformation, by definition, only exists for $\tau > 0$. That is, we have a **one-sided deformation**. As a consequence the initial speed $\frac{\partial S_\tau f_0}{\partial \tau}|_{\tau=0}$ of the deformation has to be taken with its direction. Therefore in the $\mathcal{G}$-codimension-one case we can at most obtain half of $\mathcal{E}_0$ because the diffeomorphisms acting on the deformation must respect the sign of τ . Consequently the deformation does not cover a full neighborhood of f_0 but only a halfspace. To cover the other half we need an other deformation, which, with respect to the full group, is equivalent to the previous one. The two are not equivalent if we restrict our group action to the proper halfspace and consider stability with respect to the restricted group action. To make this precise, if we have that $\mathbb{R}\{\frac{\partial S_\tau f_0}{\partial \tau}|_{\tau=0}\} + T_\mathcal{G}(f_0) = \mathcal{E}_0$ considered as modules, then $F(x,\tau)$ is a universal deformation. With $\mathbb{R}_+ = \{\tau \in \mathbb{R} | \tau > 0\}$ we obtain for a semigroup deformation $f(x,\tau) = S_\tau f_0$ the halfspace $\mathbb{R}_+\{\frac{\partial S_\tau f_0}{\partial \tau}|_{\tau=0}\} + T_\mathcal{G}(f_0)$ which is half of the tangent space if $F(x,\tau)$ is a nontrivial universal deformation.

Definition 6. *Two non-trivial one-parameter deformations $h(x,\tau)$ and $g(x,\tau)$ of $f(x)$ are one-sided $\mathcal{G}$-equivalent if they are $\mathcal{G}$-equivalent and*

$$\mathbb{R}_+\{ \frac{\partial h}{\partial \tau}\Big|_{\tau=0} \} + T_\mathcal{G}(f) = \mathbb{R}_+\{ \frac{\partial g}{\partial \tau}\Big|_{\tau=0} \} + T_\mathcal{G}(f) \; .$$

Now if f_0 and g_0 are $\mathcal{G}$-equivalent then also universal deformations f and g are equivalent. This need not be true for semigroup deformations because $\mathbb{R}_+\{\frac{\partial S_\tau f_0}{\partial \tau}|_{\tau=0}\}+ T_{\mathcal{G}}(f_0)$ and $\mathbb{R}_+\{\frac{\partial S_\tau g_0}{\partial \tau}|_{\tau=0}\}+T_{\mathcal{G}}(g_0)$ need not be equivalent, i.e. diffeomorphic by a τ dependent map which respects the sign of τ.

Definition 7. *Two non-trivial semigroup deformations $S_\tau f_1$ and $S_\tau f_2$ are one-sided $\mathcal{G}$-equivalent if f_1 and f_2 are $\mathcal{G}$-equivalent, i.e. if there exists a $g \in \mathcal{G}$ such that $f_1 = g \cdot f_2$, and $S_\tau f_1$ and $g \cdot S_\tau f_2$ are on-sided $\mathcal{G}$-equivalent as one parameter deformations of f_1*

Here in $g \cdot S_\tau f_2$ the action of $g \in \mathcal{G}$ is on the x variable only. Consequently $g \cdot S_\tau f_2$ is an unfolding of $g.f_2$.

Definition 8. *A non-trivial semigroup deformation $S_\tau f_0$ is one-sided $\mathcal{G}$-stable if any semigroup deformation $S_\tau f_1$ such that $f_0 = g \cdot f_1$ for some $g \in \mathcal{G}$ and such that $\frac{\partial S_\tau f_0}{\partial \tau}(x,0)$ and $\frac{\partial g \cdot S_\tau f_1}{\partial \tau}(x,0)$ have the same sign as vectors in $\mathcal{E}_0$ is one-sided $\mathcal{G}$-equivalent to $S_\tau f_0$.*

Note that two one-sided stable semigroup deformations need not be one-sided equivalent. They might lie on different sides of the tangent space to the orbit through f_0.

For trivial deformations we have to adjust our definition.

Definition 9. *A trivial semigroup deformation $S_\tau f_0$ is two-sided $\mathcal{G}$-stable if f_0 is $\mathcal{G}$-stable.*

Note that in this case

$$\mathbb{R}_+\{\left.\frac{\partial S_\tau f_0}{\partial \tau}\right|_{\tau=0}\} + T_{\mathcal{G}}(f_0) = T_{\mathcal{G}}(f_0) = \mathcal{E}_0 \ .$$

Theorem 2.

(i) If for a semigroup deformation $S_\tau f_0$, $F(x,\tau)$ is a non-trivial $\mathcal{G}$-stable deformation then $S_\tau f_0$, is one-sided $\mathcal{G}$-stable.
(ii) If for a semigroup deformation $S_\tau f_0$, $F(x,\tau)$ is a trivial $\mathcal{G}$-stable deformation then $S_\tau f_0$ is two-sided $\mathcal{G}$-stable.

or phrased differently

Corollary 1. *If for a semigroup deformation $S_\tau f_0$, $F(x,\tau)$ is a non-trivial $\mathcal{G}$-universal deformation then $S_\tau f_0$, $\tau \in \mathbb{R}_+$, is one-sided $\mathcal{G}$-stable.*

Note that two one-sided stable semigroup deformations need not be one-sided equivalent. They might lie on different sides of the tangent space to the orbit through f_0.

4 Gaussian Deformations

In the case where one wants the deformation of f_0 to be a solution of the diffusion equation the deformation is completely prescribed by the diffusion equation giving $\frac{\partial u}{\partial \tau}$, that is, the direction in which one has to unfold is in fact given. The unfolding transformations are given by the semigroup $exp(\tau \Delta)$.

Definition 10. *Consider a function f_0. The one parameter deformations $f(x;\tau)$ of f_0 with the unfolding direction given by*

$$\frac{\partial u}{\partial \tau} = \Delta u \ ,$$

are

$$f(x,\tau) = exp(\tau \Delta) f_0 \ .$$

These deformations are called Gaussian deformations.

Note that $exp(\tau \Delta)$ is a holomorphic strongly continuous one-parameter semigroup, therefore its action on smooth functions is well defined. The Gaussian deformation can also be obtained by convolution with the Gaussian kernel.

The following theorem allows us to consider the notion of stability for such one-parameter semi-group deformations

Corollary 2. *$f(x,\tau) = exp(\tau \Delta) f_0$ is $\mathcal{G}$-stable if and only if either f_0 is stable or $d(f_0, \mathcal{G})$=1 and Δf_0 generates the complement of $T_{\mathcal{G}}(f_0)$ in $\mathcal{E}_0$.*

Thus the possible bifurcations f_0 can undergo as a consequence of Gaussian blurring are given by the singularities of co-dimension-1 for which $f(x,\tau) = exp(\tau \Delta) f_0$ is $\mathcal{G}$-stable.

Note that the constant and linear terms in f_0 do not influence the unfolding terms but linear terms and constant terms (i.e. depending only on t) can appear as unfolding terms. The linear terms in the deformation do influence the behavior of critical points. Therefore we will work modulo constant terms. Phrased differently we may include adding constant terms, which may be terms depending on t only, in the group action (compare the groups $\mathcal{H}$ and $\mathcal{IS}$ of Damon [3]). These constant terms change the intensity-level but not the qualitative behavior of the bifurcation.

If we consider $\mathcal{A}$-equivalence than the stable functions are the Morse-functions $a_1 x^2 + a_2 y^2$. The Gaussian deformations of these Morse-functions are $a_1 x^2 + a_2 y^2 + 2t(a_1 + a_2)$. These are trivial $\mathcal{A}$-deformations. That is the initial speeds belong to the tangent space to the $\mathcal{A}$-orbit through the function. They are stable. Because there is no initial speed transversal to the orbit we have two-sided stability.

The standard form of an $\mathcal{A}$-co-dimension one function is $y^3 + x^2$, with universal $\mathcal{A}$-deformation $y^3 + ty + x^2$. For $t > 0$ there are no critical points while for $t < 0$ there are two critical points, a saddle and a node. We have creation or annihilation of critical points depending on the sign of t. In terms of vector fields this is known as the saddle-node bifurcation [8]. In catastrophe theory it is the fold catastrophe [11].

Now this is an $\mathcal{A}$-deformation while we have to consider Gaussian deformations. The Gaussian deformation of y^3+x^2 gives $y^3+6ty+x^2+2t$, $t>0$. Because it is equivalent to the universal $\mathcal{A}$-deformation it is one-sided $\mathcal{A}$-stable. The complement to the tangent space is spanned by the vector y. Now the saddle-node bifurcation is in this case a saddle and a center which exist for $t<0$ and join and disappear at $t=0$. Thus restricting to $t>0$ there is no structural change other then the critical point disappearing. Only by de-blurring, i.e. use $exp(-\tau\Delta)$, one sees that a saddle and a center are created. Thus blurring corresponds to annihilation.

In order to cover the codimension one case we need two one-sided $\mathcal{A}$-deformations obtained from semi-group deformations. Consider $y^3-6yx^2+x^2$ which has $\mathcal{A}$-codimension one and is actually $\mathcal{A}$-equivalent to y^3+x^2. Its universal $\mathcal{A}$-deformation is $y^3-6yx^2+ty+x^2$ and its Gaussian deformation is $y^3-6yx^2-6ty+x^2+2t$. The complement to the tangent space is spanned by the vector $-y$. Thus we obtain the complementary one-sided $\mathcal{A}$-stable deformation. In terms of catastrophe theory the latter is a fold embedded in the elliptic umbilic. Again there is a saddle-node bifurcation. This time the saddle and center are created at $t=0$.

More on catastrophe theory in the context of image analysis can be found in [9]

This classifies all the one-sided $\mathcal{A}$-stable Gaussian deformations. (cf [3] Theorem 3)

Theorem 3. *The $\mathcal{A}$-stable Gaussian deformations in $\mathbb{R}^2$ are listed in table 1.*

Table 1. $\mathcal{A}$-stable Gaussian deformations in $\mathbb{R}^2$.

Initial function	Gaussian deformation	Stability type
x^2+y^2	x^2+y^2+4t	two-sided $\mathcal{A}$-stable
x^2-y^2	x^2-y^2	two-sided $\mathcal{A}$-stable
y^3+x^2	$y^3+6ty+x^2+2t$	one-sided $\mathcal{A}$-stable
$y^3-6yx^2+x^2$	$y^3-6yx^2-6ty+x^2+2t$	one-sided $\mathcal{A}$-stable

A straightforward generalization to higher dimensions is obtained by adding quadratic Morse functions in the additional variables (see [2]) .

Theorem 4. *The $\mathcal{A}$-stable Gaussian deformations in $\mathbb{R}^n$ are listed in table 2. Here Q is a quadratic function as in (i) but with variables x_i, $i>1$, with $Q(t)$ its Gaussian deformation.*

If we considers functions $f(x,y)$ on $\mathbb{R}^2$ with a critical point at the origin and $f(0,0)=0$, then a normal form for the co-dimension 1 singularity is given by

Table 2. $\mathcal{A}$-stable Gaussian deformations in $\mathbb{R}^n$.

$p = 1, \cdots, n$	Initial function	Gaussian deformation	Stability type
$(i)_p$	$\sum_{i=1}^{p} x_i^2 - \sum_{j=1}^{n-p} x_i^2$	$\sum_{i=1}^{p} x_i^2 - \sum_{j=1}^{n-p} x_i^2 + 2(2p-n)t$	two-sided $\mathcal{A}$-stable
$(ii)_p$	$x_1^3 + Q$	$x_1^3 + 6tx_1 + Q(t)$	one-sided $\mathcal{A}$-stable
$(iii)_p$	$x_1^3 - 6x_1x_i^2 + Q$	$x_1^3 - 6x_1x_i^2 - 6tx_1 + Q(t)$	one-sided $\mathcal{A}$-stable

y^3+x^2 which has universal deformation $y^3+\lambda y^2+x^2$. This universal deformation describes the transcritical bifurcation ([8]). If one considers Gaussian blurring the question is wether this deformation is equivalent to a Gaussian deformation. Therefore take $y^3 + 6ty + x^2 + 2t$ and apply the shift $y = z + \sqrt{-2t}$ (or $y = z - \sqrt{-2t}$), we get $z^3 + 3\sqrt{-2t}z^2 + x^2 + g(t)$. An additional shift on the target puts the Gaussian deformation in the required form $z^3+3\sqrt{-2t}z^2+x^2$. Thus for $t > 0$ we do not get any equivalence with (part of) the transcritical bifurcation. De-blurring gives $z^3 + 3\sqrt{2t}z^2 + x^2$. Thus we obtain the part of the transcritical bifurcation with $\lambda = 3\sqrt{2t} > 0$. We get equivalence of deformations but not for all values of the parameters. The mapping $\lambda^2 = 18t$ in parameter space reflects that the transcritical bifurcation is a folded fold. Note that using the shift $y = z-\sqrt{2t}$ gives the other half. However $z^3+3\sqrt{2t}z^2+x^2$ and $z^3-3\sqrt{2t}z^2-x^2$ are equivalent, so our equivalence class does not allow us to distinguish between the role of saddles and centers. Consequently both shifts $y = z + \sqrt{-2t}$ and $y = z - \sqrt{-2t}$ describe the same phenomena. In a similar fashion one can show that $y^3 - 6yx^2 - 6ty + x^2 + 2t$ is equivalent to half the transcritical bifurcation.

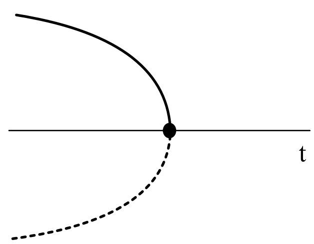

Fig. 1. Saddle-node bifurcation

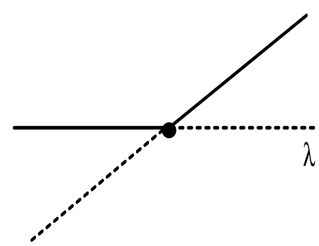

Fig. 2. Transcritical bifurcation

5 Conclusion

If we restrict to $\mathcal{A}$-equivalence and allow diffeomorphisms on the parameter space then combining all actions lead to

$$\tilde{\psi}_2(\psi_1(\tilde{f}(\tilde{\varphi}_3^{-1}(\varphi_1^{-1}(x), \varphi_2^{-1}(t))))\ ,$$

with $\varphi_1 \in Diff_n$, $\varphi_2 \in Diff_1$, $\psi_1 \in Diff_1$, $\tilde{\varphi}_3$ and $\tilde{\psi}_2$ as in (2), which comes down to $\mathcal{A}$-equivalence of map germs $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$. That is, unfoldings are also considered under right-left equivalence. The above approach was chosen to reveal the role of the directional preference invoked by the use of Gaussian deformations. It shows that one can actually take the singularity theoretic normal forms obtained for $\mathcal{A}$-equivalence and relate them to solutions of the diffusion equation. The above framework provides an alternative way to obtain the generic qualitative changes in Gaussian scale space in comparison to [3,4]. It indicates how to generalize to higher dimensions and how to deal with other operators than the Laplace-operator. Differences with the work of Damon occur because we exploit full $\mathcal{A}$-equivalence. This gives a larger group of transformations and therefore a less detailed classification. For instance there is no distinction between ellipses and circles. Furthermore working modulo constants makes it impossible to carefully keep track of the intensity levels.

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