Answers to the questions in the examination (theoretical part) of March 20th, 2003,

for the course

Proving with Computer Assistance

(2R844/2IF44)

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Question 1 a

On the one hand:

$SKKx \equiv (\lambda xyz\ .\ xz(yz))KKx$
$\to_\beta (\lambda yz\ .\ Kz(yz))Kx$
$\to_\beta (\lambda z\ .\ Kz(Kz))x$
$\to_\beta Kx(Kx) \equiv (\lambda xy\ .\ x)x(Kx)$
$\to_\beta (\lambda y\ .\ x)(Kx)$
$\to_\beta x$,

whereas on the other hand:

$Kxy \equiv (\lambda xy\ .\ x)xy$
$\to_\beta (\lambda y\ .\ x)y$
$\to_\beta x\ .$

Hence, $SKKx =_\beta Kxy$.

Question 1 b

Assume such a term H exists.

Consider the term $(\lambda u\ .\ Huv)v$. Note that variable u is different from variable v by Barendregt's convention.

This term contains two redexes:
(1) the full term,
(2) Huv.

Reducing the *first* redex leads to:
$(\lambda u\ .\ Huv)v \to_\beta Hvv \twoheadrightarrow_\beta K$, because $v =_\beta v$ (see the first property of H).

Reducing the *second* redex leads to (see now the second property of H and note that $u \neq_\beta v$):
$(\lambda u\ .\ Huv)v \twoheadrightarrow_\beta (\lambda u\ .\ S)v \to_\beta S\ .$

It follows that $K =_\beta S$, and since both are normal forms, this leads to a contradiction by the Church-Rosser theorem.

(To be precise: by Church-Rosser there exists T such that $K \twoheadrightarrow T$ and $S \twoheadrightarrow T$, but K and S are both normal forms (no redexes occur in either $\lambda xy\ .\ x$ or $\lambda xyz\ .\ xz(yz)$), so $K \equiv T$ and $S \equiv T$, hence $K \equiv S$, which is clearly not so. Contradiction.)

So no such H exists.

Question 2

(1) $\boxed{x : (\sigma \to \tau) \to \sigma}$

(2) $\boxed{y : \sigma \to \tau}$

(3) $\boxed{z : \varphi}$

{ (axiom) on (1): }

(4) $x : (\sigma \to \tau) \to \sigma$

{ (axiom) on (2): }

(5) $y : \sigma \to \tau$

{ $\to$–elim on (4) and (5): }

(6) $xy : \sigma$

{ $\to$–elim on (5) and (6): }

(7) $y(xy) : \tau$

{ $\to$–intro on (7): }

(8) $\lambda z : \varphi \,.\, y(xy) \;:\; \varphi \to \tau$

{ $\to$–intro on (8): }

(9) $\lambda y : \sigma \to \tau \,.\, \lambda z : \varphi \,.\, y(xy) \;:\; (\sigma \to \tau) \to (\varphi \to \tau)$

{ $\to$–intro on (9): }

(10) $\lambda x : ((\sigma \to \tau) \to \sigma) \,.\, \lambda y : \sigma \to \tau \,.\, \lambda z : \varphi \,.\, y(xy) \;:$

$((\sigma \to \tau) \to \sigma) \to ((\sigma \to \tau) \to (\varphi \to \tau))$

Hence,

$$\emptyset \;\vdash\; \lambda x : (\sigma \to \tau) \to \sigma \,.\, \lambda y : \sigma \to \tau \,.\, \lambda z : \varphi \,.\, y(xy) \;: \;((\sigma \to \tau) \to \sigma) \to ((\sigma \to \tau) \to (\varphi \to \tau))\,,$$

so $\lambda x : (\sigma \to \tau) \to \sigma \,.\, \lambda y : \sigma \to \tau \,.\, \lambda z : \varphi \,.\, y(xy)$ is legal in the empty context.

Question 3

(1) $\boxed{nat : *}$

(2) $\boxed{\alpha : *}$

(3) $\boxed{\beta : *}$

(4) $\boxed{x : nat \to \alpha}$

(5) $\boxed{y : \alpha \to (nat \to \beta)}$

(6) $\boxed{z : nat}$

{ $\to$–elim on (4) and (6): }

(7) $xz : \alpha$

{ $\to$–elim on (5) and (7): }

(8) $y(xz) \ : \ nat \to \beta$

{ $\to$–elim on (8) and (6): }

(9) $y(xz)z \ : \ \beta$

{ $\to$–intro on (9): }

(10) $\lambda z : nat \,.\, y(xz)z \quad : \quad nat \to \beta$

{ $\to$–intro on (10): }

(11) $\lambda y : (\alpha \to (nat \to \beta)) \,.\, \lambda z : nat \,.\, y(xz)z \quad :$
$(\alpha \to (nat \to \beta)) \to (nat \to \beta)$

{ $\to$–intro on (11): }

(12) $\lambda x : nat \to \alpha \,.\, \lambda y : (\alpha \to (nat \to \beta)) \,.\, \lambda z : nat \,.\, y(xz)z \quad :$
$(nat \to \alpha) \to (\alpha \to (nat \to \beta)) \to (nat \to \beta)$

{ Π–intro on (12): }

(13) $\lambda \beta : * \,.\, \lambda x : nat \to \alpha \,.\, \lambda y : (\alpha \to (nat \to \beta)) \,.\, \lambda z : nat \,.\, y(xz)z \quad :$
$\Pi \beta : * \,.\, (nat \to \alpha) \to (\alpha \to (nat \to \beta)) \to (nat \to \beta)$

{ Π–intro on (13): }

(14) $\lambda \alpha : * \,.\, \lambda \beta : * \,.\, \lambda x : nat \to \alpha \,.\, \lambda y : (\alpha \to (nat \to \beta)) \,.\, \lambda z : nat \,.\, y(xz)z \quad :$
$\Pi \alpha : * \,.\, \Pi \beta : * \,.\, (nat \to \alpha) \to (\alpha \to (nat \to \beta)) \to (nat \to \beta)$

Hence, the inhabitant asked for is:

$\lambda \alpha : * \,.\, \lambda \beta : * \,.\, \lambda x : nat \to \alpha \,.\, \lambda y : (\alpha \to (nat \to \beta)) \,.\, \lambda z : nat \,.\, y(xz)z$

Question 4

The 'universal validity' (that is, for *all* propositions p) of $(\neg p \Rightarrow p) \Rightarrow p$ in classical logic, can be shown in λC by giving an inhabitant of:

$\Pi p : *_{(p)} \,.\, ((p \to \bot) \to p) \to p$,

in context $\quad c \;:\; \Pi\gamma : *_{(p)} \,.\, ((\gamma \to \bot) \to \bot) \to \gamma$.

In the derivation below, we omit the arguments $\{\ldots\}$.

$c \;:\; \Pi\gamma : *_{(p)} \,.\, ((\gamma \to \bot) \to \bot) \to \gamma$

$p : *_{(p)}$

$x : (p \to \bot) \to p$

$y : p \to \bot)$

$xy \;:\; p$

$y(xy) \;:\; \bot$

$\lambda y : p \to \bot \,.\, y(xy) \;:\; (p \to \bot) \to \bot$

$cp \;:\; ((p \to \bot) \to \bot) \to p$

$cp(\lambda y : p \to \bot \,.\, y(xy)) \;:\; p$

$\lambda x : ((p \to \bot) \to p) \,.\, cp(\lambda y : p \to \bot \,.\, y(xy)) \;:$
$\qquad ((p \to \bot) \to p) \to p$

$\lambda p : * \,.\, \lambda x : ((p \to \bot) \to p) \,.\, cp(\lambda y : p \to \bot \,.\, y(xy)) \;:$
$\qquad \Pi p : * \,.\, ((p \to \bot) \to p) \to p$

Hence, the inhabitant asked for is:

$\lambda p : * \,.\, \lambda x : ((p \to \bot) \to p) \,.\, cp(\lambda y : p \to \bot \,.\, y(xy))$