2IF25 Formal Methods

Written examination January 19, 2010: workout of assignment 2

$\vdash (\exists x P(x)) \vee (\exists x Q(x)) \Rightarrow \exists x(P(x) \vee Q(x))$
$\mathit{TERM} \quad y_0 : \mathit{nil}$

$(\Rightarrow R)$
$\exists x P(x) \vee \exists x Q(x) \vdash \exists x(P(x) \vee Q(x))$
$\mathit{TERM} \quad y_0 : \mathit{nil}$

$(\vee L), (\exists R), (\exists R)$
$\exists x P(x) \vdash P(y_0) \vee Q(y_0), \exists x(P(x) \vee Q(x))$
$\exists x Q(x) \vdash P(y_0) \vee Q(y_0), \exists x(P(x) \vee Q(x))$
$\mathit{TERM} \quad y_0 : \exists x(P(x) \vee Q(x))$

$(\exists L), (\vee R); (\exists L), (\vee R)$
$P(y_1) \vdash P(y_0), Q(y_0), \exists x(P(x) \vee Q(x))$
$Q(y_2) \vdash P(y_0), Q(y_0), \exists x(P(x) \vee Q(x))$
$\mathit{TERM} \quad y_0 : \exists x(P(x) \vee Q(x));\ y_1 : \mathit{nil};\ y_2 : \mathit{nil}$

$(\exists R); (\exists R)$
$P(y_1) \vdash P(y_0), Q(y_0), P(y_1) \vee Q(y_1), P(y_2) \vee Q(y_2), \exists x(P(x) \vee Q(x))$
$Q(y_2) \vdash P(y_0), Q(y_0), P(y_1) \vee Q(y_1), P(y_2) \vee Q(y_2), \exists x(P(x) \vee Q(x))$
$\mathit{TERM} \quad y_0 : \exists x(P(x) \vee Q(x));\ y_1 : \exists x(P(x) \vee Q(x));\ y_2 : \exists x(P(x) \vee Q(x))$

$(\vee R), (\vee R); (\vee R), (\vee R)$
$P(y_1) \vdash P(y_0), Q(y_0), P(y_1), Q(y_1), P(y_2), Q(y_2), \exists x(P(x) \vee Q(x))$ axiom
$Q(y_2) \vdash P(y_0), Q(y_0), P(y_1), Q(y_1), P(y_2), Q(y_2), \exists x(P(x) \vee Q(x))$ axiom
$\mathit{TERM} \quad y_0 : \exists x(P(x) \vee Q(x));\ y_1 : \exists x(P(x) \vee Q(x));\ y_2 : \exists x(P(x) \vee Q(x))$

Omitting superfluous substitutions from the above yields the following proof tree.

$$
\dfrac{\dfrac{
\dfrac{\dfrac{\dfrac{P(y_1) \vdash P(y_1), Q(y_1), \exists x(P(x) \vee Q(x))}{P(y_1) \vdash P(y_1) \vee Q(y_1), \exists x(P(x) \vee Q(x))}\,(\vee R)}{P(y_1) \vdash \exists x(P(x) \vee Q(x))}\,(\exists R)}{\exists x P(x) \vdash \exists x(P(x) \vee Q(x))}\,(\exists L)
\qquad
\dfrac{\dfrac{\dfrac{Q(y_2) \vdash P(y_1), Q(y_1), \exists x(P(x) \vee Q(x))}{Q(y_2) \vdash P(y_1) \vee Q(y_1), \exists x(P(x) \vee Q(x))}\,(\vee R)}{Q(y_2) \vdash \exists x(P(x) \vee Q(x))}\,(\exists R)}{\exists x P(x) \vdash \exists x(P(x) \vee Q(x))}\,(\exists L)
}{(\exists x P(x)) \vee (\exists x Q(x)) \vdash \exists x(P(x) \vee Q(x))}\,(\vee L)}{\vdash (\exists x P(x)) \vee (\exists x Q(x)) \Rightarrow \exists x(P(x) \vee Q(x))}\,(\Rightarrow R)
$$

part b) Consider the structure $\mathcal{M}$ with domain $\{y_0, y_1\}$ and interpretation function I given by

$$I(P)(y_0) = \textit{false}, \quad I(Q)(y_0) = \textit{false}, \quad I(P)(y_1) = \textit{true}, \quad I(Q)(y_1) = \textit{true}$$

Then we have $\mathcal{M} \vDash \exists x P(x)$, $\mathcal{M} \vDash \exists x Q(x)$ but $\mathcal{M} \nvDash \exists x (P(x) \vee Q(x))$.