Regular languages

\[ L \subseteq \Sigma^* \text{ is regular iff } L = \mathcal{L}(N) \text{ for some NFA } N \]

**THEOREM** the following statement are equivalent

- \( L \) is regular
- \( L = \mathcal{L}(D) \) for some DFA \( D \)
- \( L = \mathcal{L}(r) \) for some RE \( r \)

**PROOF** combine earlier results
**THEOREM**

- if $L_1$ and $L_2$ are regular, then $L_1 \cup L_2$ is regular
- if $L$ is regular, then $L^*$ is regular
- if $L_1$ and $L_2$ are regular, then $L_1 \cap L_2$ is regular

**PROOF** use a suitable representation for $L_1$, $L_2$ and $L$
The product automaton

two DFAs $D_i = (Q_i, \Sigma, \delta_i, q_0^i, F_i)$ for $i = 1, 2$

product automaton

$$D_1 \times D_2 = (Q_1 \times Q_2, \Sigma, \delta, \langle q_0^1, q_0^2 \rangle, F_1 \times F_2)$$

where

$$\delta(\langle q_1, q_2 \rangle, a) = \langle q'_1, q'_2 \rangle \iff \delta_1(q_1, a) = q'_1 \land \delta_2(q_2, a) = q'_2$$
An example product automaton
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product automaton

$$D_1 \times D_2 = (Q_1 \times Q_2, \Sigma, \delta, \langle q_0^1, q_0^2 \rangle, F_1 \times F_2)$$

where

$$\delta(\langle q_1, q_2 \rangle, a) = \langle q_1', q_2' \rangle \iff \delta_1(q_1, a) = q_1' \land \delta_2(q_2, a) = q_2'$$

LEMMA $\mathcal{L}(D_1 \times D_2) = \mathcal{L}(D_1) \cap \mathcal{L}(D_2)$
two DFAs $D_i = (Q_i, \Sigma, \delta_i, q^i_0, F_i)$ for $i = 1, 2$

product automaton

$$D_1 \times D_2 = (Q_1 \times Q_2, \Sigma, \delta, \langle q^1_0, q^2_0 \rangle, F_1 \times F_2)$$

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$$\delta(\langle q_1, q_2 \rangle, a) = \langle q'_1, q'_2 \rangle \iff \delta_1(q_1, a) = q'_1 \land \delta_2(q_2, a) = q'_2$$

**Lemma** \( \mathcal{L}(D_1 \times D_2) = \mathcal{L}(D_1) \cap \mathcal{L}(D_2) \)

**Proof** for all $n \geq 0$ it holds that

$$\forall q_1, q'_1 \in Q_1, q_2, q'_2 \in Q_2 \forall w, w' \in \Sigma^* :$$

$$(q_1, w) \vdash^n_1 (q'_1, w') \land (q_2, w) \vdash^n_2 (q'_2, w')$$

$$\iff (\langle q_1, q_2 \rangle, w) \vdash^n (\langle q'_1, q'_2 \rangle, w')$$

\[\square\]
(i) The language accepted by the left automaton is the empty language

(ii) The string 11010111 is accepted by the right automaton
(i) The language accepted by the left automaton is the empty language

(ii) The string 11010111 is accepted by the right automaton

A. Both (i) and (ii) are true
B. (i) is true and (ii) is false
C. (i) is false and (ii) is true
D. Both (i) and (ii) are false
E. Can't tell
Two decision theorems

THEOREM

It can be decided whether or not a regular language is empty

THEOREM

It can be decided whether or not a string is member of a regular language
Two decision theorems

**THEOREM**

It can be decided whether or not a regular language is empty.

**PROOF**  
let $R_n$ = the set of states reachable from the initial state in at most $n$ steps.  
then $R_0 = \{q_0\}$ and $R_{n+1} = R_n \cup \{\delta(q, a) \mid q \in R_n, a \in \Sigma\}$.  
$R_{|Q|-1} \cap F = \emptyset$ iff the language is empty.

**THEOREM**

It can be decided whether or not a string is member of a regular language.
THEOREM

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THEOREM

It can be decided whether or not a string is member of a regular language

PROOF compute \( q \) such that \((q_0, w) \vdash^* (q, \varepsilon)\)
Start pumping

The diagram shows a nondeterministic finite automaton (NFA) with states labeled $q_0$, $q_1$, $q_2$, $q_3$, $q_4$, and $q_5$. The transitions are labeled with symbols $a$ and $b$. The start state is $q_0$, and the accepting states are $q_3$, $q_4$, and $q_5$. The diagram illustrates the pumping lemma for regular languages.
Start pumping
THEOREM if $L \subseteq \Sigma^*$ is a regular language then

$$\exists m > 0: \forall w \in L, |w| \geq m: \exists x, y, z : w = xyz \land |xy| \leq m \land |y| > 0 : \forall i \geq 0 : xy^i z \in L.$$
The Pumping Lemma

**THEOREM** if \( L \subseteq \Sigma^* \) is a regular language then

\[
\exists m > 0 : \\
\forall w \in L, |w| \geq m : \\
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\]

**PROOF** suppose \( L = \mathcal{L}(D) \) for DFA \( D \)
take \( m \) the number of states of \( D \)
**LEMMA**  DFA $D$ over alphabet $\Sigma$

$$(q, w) \vdash^*_D (q', w') \iff (q, wv) \vdash^*_D (q', w'v)$$

for all states $q, q'$ of $D$ and all strings $w, w', v \in \Sigma^*$

**PROOF**  by induction on $n$:

$$(q, w) \vdash^n_D (q', w') \iff (q, wv) \vdash^n_D (q', w'v)$$
Example non-regular languages

Pumping Lemma: if $L \subseteq \Sigma^*$ is a regular language then

$$\exists m > 0 :$$

$$\forall w \in L, |w| \geq m :$$

$$\exists x, y, z : w = xyz \land |xy| \leq m \land |y| > 0 :$$

$$\forall i \geq 0 : x y^i z \in L$$

- the language $L_1 = \{ a^n b^n \mid n \geq 0 \}$ is not regular

- the language $L_2 = \{ uu \mid u \in \{a, b\}^* \}$ is not regular
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\]

- the language $L_1 = \{ a^n b^n \mid n \geq 0 \}$ is *not* regular
  for any $m > 0$, consider $a^m b^m$
- the language $L_2 = \{ uu \mid u \in \{a, b\}^* \}$ is *not* regular
Example non-regular languages

Pumping Lemma: if $L \subseteq \Sigma^*$ is a regular language then
\[\exists m > 0 : \forall w \in L, |w| \geq m : \exists x, y, z : w = xyz \land |xy| \leq m \land |y| > 0 : \forall i \geq 0 : xy^i z \in L\]

- the language $L_1 = \{a^n b^n \mid n \geq 0\}$ is \textit{not} regular
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- the language $L_2 = \{uu \mid u \in \{a, b\}^*\}$ is \textit{not} regular
  for any $m > 0$, consider $a^m ba^m b$
How many of the following languages are *not* regular?

- $L_1 = \{ uu \mid u \in \{a\}^* \}$
- $L_2 = \{ a^n b^m c^\ell \mid n \geq 0, m \geq 2, \ell \geq 4 \}$
- $L_3 = \{ a^n b^m c^\ell \mid n, m, \ell \geq 0, n = m \lor m = \ell \}$
- $L_4 = \{ a^n b^m c^\ell \mid n, m, \ell \geq 0, n \neq m \land m \neq \ell \}$
How many of the following languages are \textit{not} regular?

- \( L_1 = \{ uu \mid u \in \{a\}^* \} \)
- \( L_2 = \{ a^n b^m c^\ell \mid n \geq 0, \ m \geq 2, \ \ell \geq 4 \} \)
- \( L_3 = \{ a^n b^m c^\ell \mid n, m, \ell \geq 0, \ n = m \lor m = \ell \} \)
- \( L_4 = \{ a^n b^m c^\ell \mid n, m, \ell \geq 0, \ n \neq m \land m \neq \ell \} \)

A. 0  
B. 1  
C. 2  
D. 4  
E. Can’t tell
Pumping Lemma: if $L \subseteq \Sigma^*$ is a regular language then

$$\exists m > 0 :$$

$$\forall w \in L, |w| \geq m :$$

$$\exists x, y, z : w = xyz \land |xy| \leq m \land |y| > 0 :$$

$$\forall i \geq 0 : xy^i z \in L$$

- the language $L = \{ a^{n^2} | n \geq 0 \}$ is not regular
Pumping Lemma: if $L \subseteq \Sigma^*$ is a regular language then

$\exists m > 0:\,
\forall w \in L, |w| \geq m:\,
\exists x, y, z:\ w = xyz \wedge |xy| \leq m \wedge |y| > 0:\,
\forall i \geq 0:\ xy^i z \in L$

- the language $L = \{ a^{n^2} | n \geq 0 \}$ is not regular
  for any $m > 0$, consider $a^{m^2}$