• Pumping Lemma for Regular Languages

If \( L \subseteq \Sigma^* \) is a regular language, then

\[
\exists m \in \mathbb{N} : m > 0 : \\
\forall w \in L : |w| \geq m : \\
\exists x, y, z \in \Sigma^* : w = xyz \land |xy| \leq m \land |y| > 0 : \\
\forall i \in \mathbb{N} : x y^i z \in L \] \]

Some languages can be proven to be not regular using the contraposition of the pumping lemma:

If

\[
\forall m \in \mathbb{N} : m > 0 : \\
\exists w \in L : |w| \geq m : \\
\forall x, y, z \in \Sigma^* : w = xyz \land |xy| \leq m \land |y| > 0 : \\
\exists i \in \mathbb{N} : x y^i z \notin L \] \]

then \( L \) is not a regular language.

• Proof that \( L = \{a^n b^n \mid n \geq 0\} \) is not regular:

Let \( m > 0 \).

Choose \( w = a^m b^m \), then \( w \in L \) and \( |w| = 2m \geq m \).

Let \( x, y, z \) be such that \( w = xyz \), \( |xy| \leq m \), \( |y| > 0 \).

It follows that \( x \) and \( y \) consist of \( a \)'s only.

Say \( x = a^k \), \( y = a^l \), and \( z = a^{m-k-l} b^m \).

for some \( k, l \) with \( k \geq 0 \), \( k + l = |xy| \leq m \), \( l > 0 \).

Choose \( i = 2 \). We now have

\[
xy^i z = x y^2 z = a^k a^{2l} a^{m-k-l} b^m = a^m + l b^m \notin L
\]

since \( m + l > m \).

due to \( l > 0 \).

\( L \) is not regular. It follows that \( L = \{a^n b^n \mid n \geq 0\} \) is not regular.