1. Answer A

There are no strings $w \in \Sigma^* \setminus \Sigma^+$ such that $S \Rightarrow^*_G w$.
So $L(G) = L(G') = \emptyset$.

2. Statements A, C, and E are equivalent to the statement "Grammar G is ambiguous." Statement B implies that G is ambiguous, but is not equivalent; statement D is not equivalent.

3. Answer B

In grammar B, the string aa has two different parse trees:

![Parse trees for the string aa]

4. a. If $L$ is a regular language, then

\[
\exists m \left( m > 0 : \forall w \in L \wedge |w| \geq m : \exists x, y, z \left( |w| = xy^z \wedge |xy| \leq m \wedge |y| \geq 1 : \forall i \left( i \geq 0 : xy^i z \in L \right) \right) \right)\]

5. Let $m > 0$.

Choose $w = a^m b^{2m}$, then $w \in L$, $|w| = 3m \geq m$.
Let $x, y, z$ be strings such that $w = xy^z$, $|xy| \leq m$, $|y| > 1$.
From the form of $w$ and $|xy| \leq m$ it follows that $x$ and $y$ consist of $a$'s only: $x = a^p$, $y = a^q$, $p + q = m$, $q \geq 1$.
Choose $i = 2$.

\[
xy^2 = a^p a^{2q} a^{m-p-2q} b^{2m} = a^{m+q} b^{2m} \notin L
\]

Since $2(m+q) = 2m+2q > 2m$ due to $q \geq 1$.
So property $\otimes$ does not hold for $L$ and hence $L$ is not regular.
(4 steps)

**Invariant Table**

<table>
<thead>
<tr>
<th>State</th>
<th>Input</th>
<th>Stack</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>(q_0)</td>
<td>(a^{2k})</td>
<td>(k)</td>
<td>(k &gt; 0)</td>
</tr>
<tr>
<td>(q_1)</td>
<td>(a^{2k+1})</td>
<td>(k+1)</td>
<td>(k &gt; 0)</td>
</tr>
<tr>
<td>(q_2)</td>
<td>(a^{2k} \cdot m)</td>
<td>(k \cdot m)</td>
<td>(0 \leq m \leq k)</td>
</tr>
<tr>
<td>(q_2)</td>
<td>(a^{2k} \cdot l + t)</td>
<td>(\epsilon)</td>
<td>(k, l &gt; 0)</td>
</tr>
<tr>
<td>(q_3)</td>
<td>(a^{2k} \cdot l + t)</td>
<td>(\epsilon)</td>
<td>(k, l &gt; 0)</td>
</tr>
</tbody>
</table>

\((6)\) \(G = (\{S\}, \{a, b\}, R, S)\)

where \(R = \{ S \to aaSb, S \to aaS, S \to \epsilon \}\)

\((6)\) \(S \Rightarrow^P a^{2^p} S \Rightarrow^1 a^{2^p} a^{q} S \&^1 \Rightarrow a^{2^{(p+q)}} \&^1\)

- Applying \(S \Rightarrow^p a^{2^p} S\)
- Applying \(S \Rightarrow^1 a^{q} S \&^1\)
- Applying \(S \Rightarrow^1 \epsilon\)

**We prove by induction on \(n\) that for all \(n \geq 0\):**

- \(S \Rightarrow^n S \Rightarrow^n\)

**Proof by Induction on \(n\):**

- **Base:** \(n = 0\):
  - Let \(S \Rightarrow^0 S \Rightarrow^0\), then \(S = a^{2^n} S \Rightarrow^0\)

- **Step:** \(n = k+1\):
  - Assume the property holds for \(k\) (IH).
  - Let \(S \Rightarrow^k S \Rightarrow^k\). Pick \(y'\) such that \(S \Rightarrow^k y' \Rightarrow^k\).
  - This follows that \(y'\) contains at least one variable.
  - By the induction hypothesis (applied to \(S \Rightarrow^k y'\) we have that \(y' = a^{2^n} S \Rightarrow^m\) for some \(m: 0 \leq m \leq n\).
  - Case distinction on the production rule applied in the last step \(y' \Rightarrow^k y\):
    - \(S \Rightarrow^k a^{2^n} S \Rightarrow^1\) was applied:
      \(y = a^{2^n} a^{2^n} S \Rightarrow^m = a^{2(n+1)} S \Rightarrow^m\)
      where \(0 \leq m \leq k+1\) since \(0 \leq m \leq k+1\).
- $S \to aaS$ was applied:
  \[ y = a^{2k} aaS \#^m = a^{2k+1} S \#^m \]
  where $0 \leq m \leq k$ since $0 \leq m \leq k$

- $S \to \varepsilon$ was applied:
  \[ y = a^{2k} \varepsilon \#^m = a^{2k} \#^m = a^{2(k+1)-1} \#^m \]
  where $0 \leq m \leq k = (k+1)-1$

(end of proof by induction)

We now show that $L(G) \subseteq L$:

Let $w \in L(G)$, then $S \Rightarrow^* w$ and $w \in \Sigma^*$. Pick $k > 0$ such that $S \Rightarrow^k w$.

Using the above property and $w \in \Sigma^*$, it follows that $w = a^{2(k-1)} \#^m$ for some $0 \leq m \leq k$.

Hence $w \in L$.

**Alternative solution for $\varepsilon$**

\[
\begin{array}{c}
\text{State:} & a^{[2/1]} & b^{[2/1]} \\
\text{Input:} & a^{[2/1]} & b^{[2/1]} \\
\text{Stack:} & 2 \#^k \\
\text{Condition:} & k > 0
\end{array}
\begin{array}{c}
\text{State:} & b^{[2/1]} & a^{[2/1]} \\
\text{Input:} & b^{[2/1]} & a^{[2/1]} \\
\text{Stack:} & 2 \#^k \\
\text{Condition:} & k > 0
\end{array}
\]

**Invariant table**

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<td>$2 #^k$</td>
<td>$k &gt; 0$</td>
</tr>
<tr>
<td></td>
<td>$a^{2k+1}$</td>
<td>$1 #^k$</td>
<td>$k &gt; 0$</td>
</tr>
<tr>
<td>q_1</td>
<td>$a^{2k} #^m$</td>
<td>$2 #^{k-m}$</td>
<td>$0 \leq m \leq k$</td>
</tr>
<tr>
<td></td>
<td>$a^{2k} #^{k+1}$</td>
<td>$\varepsilon$</td>
<td>$k, l \geq 0$</td>
</tr>
<tr>
<td>q_2</td>
<td>$a^{2k} #^{k+1}$</td>
<td>$\varepsilon$</td>
<td>$k, l \geq 0$</td>
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