Infinity in Mathematics & Computer Science

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Motivations for Working on Scientific Problems

- Direct application in real life
- Foundation for working on other problems
- Aid to acquisition of knowledge and development of skills
- Fun and enjoyment
Termination of Computations: A Recent Success

- Start with finite sequence over \( \{a, b, c\} \)
  
  \[
  bb\text{a}a
  \]

- Repeatedly replace subsequences:
  
  \[
  \begin{align*}
  aa & \rightarrow bc \\
  bb & \rightarrow ac \\
  cc & \rightarrow ab
  \end{align*}
  \]

Example:

\[
bb\text{a}a \rightarrow bbbc \rightarrow bacc \rightarrow baab \rightarrow bbcb \rightarrow accb \rightarrow aabb \rightarrow aaac \rightarrow abcc \rightarrow abab
\]

Does this terminate for every start sequence?
Termination of Computations: Famous Open Problems

- If $N$ even $\rightarrow \frac{N}{2}$, if $N$ odd $\rightarrow 3N + 1$ (Collatz)

  $3 \rightarrow 10 \rightarrow 5 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1 \rightarrow 4 \rightarrow \cdots$

- While $N$ is not a palindrome, add it to its reverse

  $152 \rightarrow 152 + 251 = 403 \rightarrow 403 + 304 = 707 \rightarrow 196 \rightarrow ?$
Marble Game 1

Each time take a marble:

- Remove it

Does this terminate? After how many steps?
Repeatedly take a marble:

- If blue, then remove

- If white, then replace by one blue marble
Repeatedly take a marble:

- If blue, then remove it
- If white, then replace by \textit{arbitrary} number of blue marbles
Marble Game Analysis

$W$ white marbles  $B$ blue marbles

**Game 1.** Terminates after $f_1(W, B) = W + B$ steps

**Game 2.** Terminates after $f_2(W, B) = 2 \cdot W + B$ steps

**Game 3.** Terminates after ten hoogste $f_3(W, B) = \omega \cdot W + B$ steps
Repeatedly take an $\mathbb{N}$-numbered lotto ball:

- Replace by arbitrary number of balls with smaller numbers
  
  i.e. $(n)$ replaced by $(<n) (<n) \cdots (<n)$
Operations on Natural Numbers

**Successor** (1 more): $a + 1$

**Addition** (repeated successor): $a + b = a + \cdots + a$

**Multiplication** (repeated addition): $a \times b = a + \cdots + a$

$a \times 0 = 0 \quad a \times 1 = a \quad a \times b = a \times b + a \quad a \times (b + c) = a \times b + a \times c$

**Exponentiation** (repeated multiplication): $a^b = a \times \cdots \times a$

$a^0 = 1 \quad a^1 = a \quad a^b = a^b \times a \quad a^{b+c} = a^b \times a^c$
Every natural number is *uniquely* expressible as

**sum of powers of 10 with coefficients < 10.**

Example:

\[
266 = 200 + 60 + 6 \\
= 2 \times 100 + 6 \times 10 + 6 \times 1 \\
= 2 \times 10^2 + 6 \times 10^1 + 6 \times 10^0
\]
Every natural number is *uniquely* expressible as a sum of powers of $B$ with coefficients $< B$.

Example with $B = 2$ (*binary*):

\[
266 = 256 + 8 + 2 = 2^8 + 2^3 + 2^1
\]

Example with $B = 3$ (*ternary*):

\[
266 = 243 + 18 + 3 + 2 = 3^5 + 2 \cdot 3^2 + 3^1 + 2 \cdot 3^0
\]
Super-Expansion in Base $B \geq 2$

1. Expand in base $B$.

2. **Repeatedly expand the exponents** in base $B$ as well.

3. Stop when all numbers $\leq B$.

Example:

<table>
<thead>
<tr>
<th>$B = 2$</th>
<th>$B = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>266</td>
<td>266</td>
</tr>
<tr>
<td>$= 2^8 + 2^3 + 2$</td>
<td>$= 3^5 + 2 \times 3^2 + 3^1 + 2$</td>
</tr>
<tr>
<td>$= 2^3 + 2^{2+1} + 2$</td>
<td>$= 3^{3+2} + 2 \times 3^2 + 3^1 + 2$</td>
</tr>
<tr>
<td>$= 2^{2+1} + 2^{2+1} + 2$</td>
<td></td>
</tr>
</tbody>
</table>
Goodstein Sequence of $N > 0$ and $B \geq 2$

1. **Super-expand** $N$ in base $B$.
   \[ 8 = 2^{2+1} \]

2. Replace each $B$ by $B + 1$.
   \[ 3^{3+1} = 81 \]

3. Decrease by 1; yields new $N$.
   \[ N' = 80 \]

4. Increase $B$ by 1; yields new $B$.
   \[ B' = 3 \]

5. Stop when $N = 0$, otherwise repeat from step 1.
**Goodstein Sequence for** \( N = 266 \) **and** \( B = 2 \)

<table>
<thead>
<tr>
<th>Step</th>
<th>( N )</th>
<th>( B )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( 2^{2^{2+1}} + 2^{2+1} + 2 )</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>( 3^{3^{3+1}} + 3^{3+1} + 3 - 1 )</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>( 443 \ldots 886 ) (39 digits)</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>( 3^{3^{3+1}} + 3^{3+1} + 2 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( 4^{4^{4+1}} + 4^{4+1} + 2 - 1 )</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>( 323 \ldots 681 ) (617 digits)</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>( 4^{4^{4+1}} + 4^{4+1} + 1 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( 5^{5^{5+1}} + 5^{5+1} + 1 - 1 )</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>( \ldots ) (( &gt; 10000 ) digits)</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>( 5^{5^{5+1}} + 5^{5+1} )</td>
<td></td>
</tr>
</tbody>
</table>
Goodstein’s Theorem (1944)

Every Goodstein sequence terminates with $N = 0$.

It can take a while:

- $N = 3, B = 2$ terminates after 5 steps
- $N = 4, B = 2$ terminates after $3 \times 2^{402653211} - 3 \approx 10^{10^8}$ steps
Theorem of Kirby and Paris (1982):

Goodstein’s Theorem cannot be proven from Peano’s Axioms.

‘Ordinary’ induction does not suffice: the sequence ‘grows too fast’.

Every proof of Goodstein’s Theorem involves (a form of) transfinite induction such as over the ordinal numbers.
Ordinal Numbers

Extend operations on numbers with a [limit operation ...]

0, 1, 2, 3, 1, ..., \(\omega\)

\(\omega + 1, \ \omega + 2, 1, ..., \omega + \omega = \omega \times 2\)

\(\omega \times 2 + 1, \ \omega \times 2 + 2, 1, ..., \omega \times 2 + \omega = \omega \times 3\)

\(1, \omega \times 4, 1, ..., \omega \times 5, 2, ..., \omega \times \omega = \omega^2\)

\(\omega^2 + 1, 1, ..., \omega^2 + \omega, 2, ..., \omega^2 + \omega^2 = \omega^2 \times 2\)

\(2, \omega^2 \times 3, 2, ..., \omega^2 \times 4, 3, ..., \omega^2 \times \omega = \omega^3\)

\(4, \omega^4, 5, ..., \omega^5, ..., \omega^\omega\)
Normal form of ordinal numbers $< \omega^\omega$

Generalize the base-$B$ expansion, taking $B = \omega$:

$$N = B^k \cdot c_k + B^{k-1} \cdot c_{k-1} + \cdots + B^2 \cdot c_2 + B \cdot c_1 + c_0$$

where $0 \leq c_i < B$, so now unbounded coefficients.

Normal form of $\alpha < \omega^\omega$:

$$\alpha = \omega^k \cdot c_k + \omega^{k-1} \cdot c_{k-1} + \cdots + \omega^2 \cdot c_2 + \omega \cdot c_1 + c_0$$

where $k$ and all $c_i$ are finite.

Solution to Lotto Ball Game: $c_i =$ number of balls with value $i$
Proof of Goodstein’s Theorem

Super-expand $N$ in base $B$.

Replace every $B$ by $\omega$.

The result $f_G(N, B)$ is an ordinal number $< \omega^{\omega^{\omega^{\cdots}}} = \epsilon_0$.

For example: $f_G(266, 2) = \omega^{\omega+1} + \omega^{\omega+1} + \omega$

Claim: If $N, B \rightarrow N', B'$ in the Goodstein sequence, then

$$f_G(N', B') < f_G(N, B)$$

Ordinal numbers are well ordered: every decreasing sequence ends.