Number Guessing with Lies

Tom Verhoeff

Department of Mathematics & Computer Science
Software Engineering & Technology

www.win.tue.nl/~wstomv/edu/hci
The Game

1. Alice picks a number $N$ in the range 0 through 15.
2. Bob asks a series of Yes/No questions.
3. Alice answers each question, and may lie once.
4. Bob then tells the number $N$ and which answer was a lie (if any).

How can Bob do this?
Is your number one of these?

1  3  4  6  8  10  13  15
Question $Q_2$

Is your number one of these?

1 2 5 6 8 11 12 15
Question $Q_3$

Is your number one of these?

8 9 10 11 12 13 14 15
Is your number one of these?

1 2 4 7 9 10 12 15
Is your number one of these?
Is your number one of these?

- 2
- 3
- 6
- 7
- 10
- 11
- 14
- 15
Is your number one of these?

1    3    5    7    9    11    13    15
Let the answers be $a_i$ ($0 = \text{No}; 1 = \text{Yes}$) for $i = 1, \ldots, 7$

Compute

\[
\begin{align*}
p_1 &= a_1 + a_3 + a_5 + a_7 \pmod{2} \\
p_2 &= a_2 + a_3 + a_6 + a_7 \pmod{2} \\
p_3 &= a_4 + a_5 + a_6 + a_7 \pmod{2}
\end{align*}
\]

Compute $q = p_1 + 2p_2 + 4p_3$ (each $p_i$ is 0 or 1)

If $q = 0$, then there was no lie

If $q \neq 0$, then answer $a_q$ was a lie: flip $a_q$ (replace it by $1 - a_q$)

Alice’ secret number was $N = 8a_3 + 4a_5 + 2a_6 + a_7$
## How It Works

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Questions $Q_3$, $Q_5$, $Q_6$, and $Q_7$ do a *Binary Search*; works *without* lie.

The three other questions help detect a single lie:

\[
\begin{align*}
Q_1 & \quad Q_3 & \quad Q_5 & \quad Q_7 \\
Q_2 & \quad Q_3 & \quad Q_6 & \quad Q_7 \\
Q_4 & \quad Q_5 & \quad Q_6 & \quad Q_7
\end{align*}
\]

There are $8 = 2^3$ possibilities: no lie, or 7 possible lies.
Error-Correcting Hamming(7,4) Code

Less efficient solution repeats questions \(Q_3, Q_5, Q_6, Q_7\) three times.

We used a Hamming(7,4) code.

It has 4 data bits, 3 parity/check bits, and can correct one bit error.

The 4 data bits encode a value from 0 through 15.

Each question corresponds to the transmission of a bit.

A lie corresponds to a bit error.

Can be generalized to \(2^k - k - 1\) data bits and \(k\) parity/check bits.

Variation (for kids): lie every time except once.