Hamiltonian Paths in Neighbor-swap Graphs of Permutations

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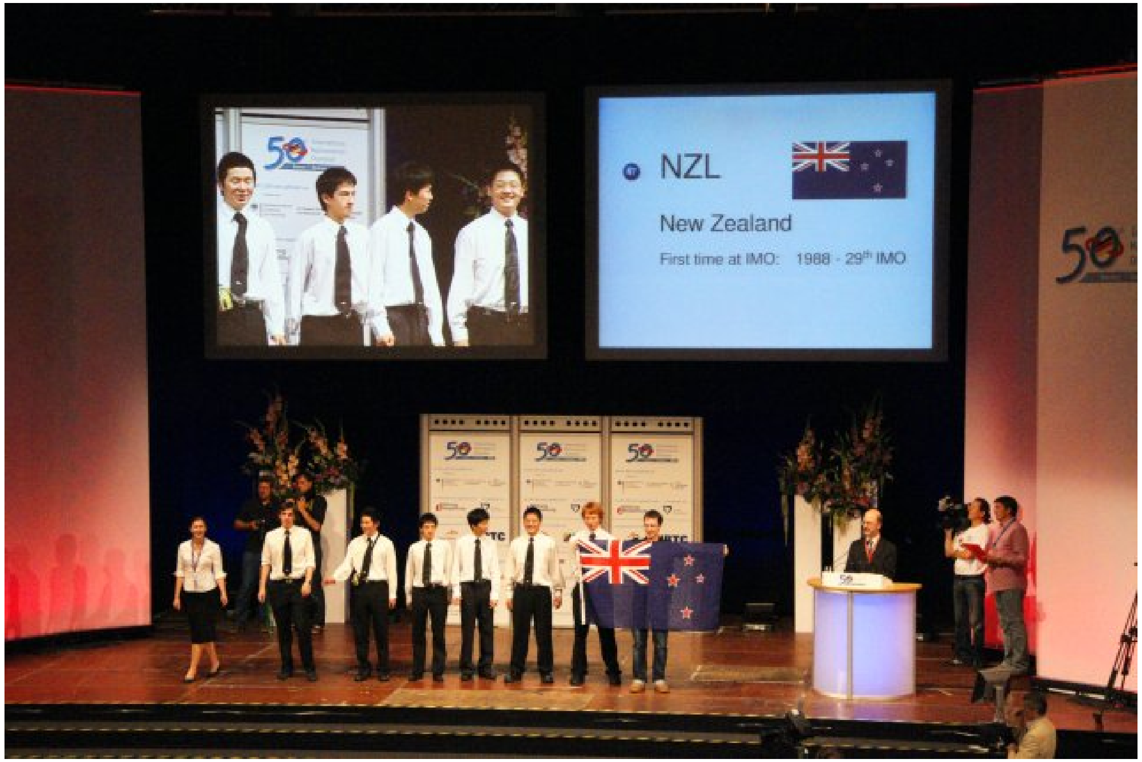
In honor of Andries Brouwer's 65th Birthday

Historic Introduction

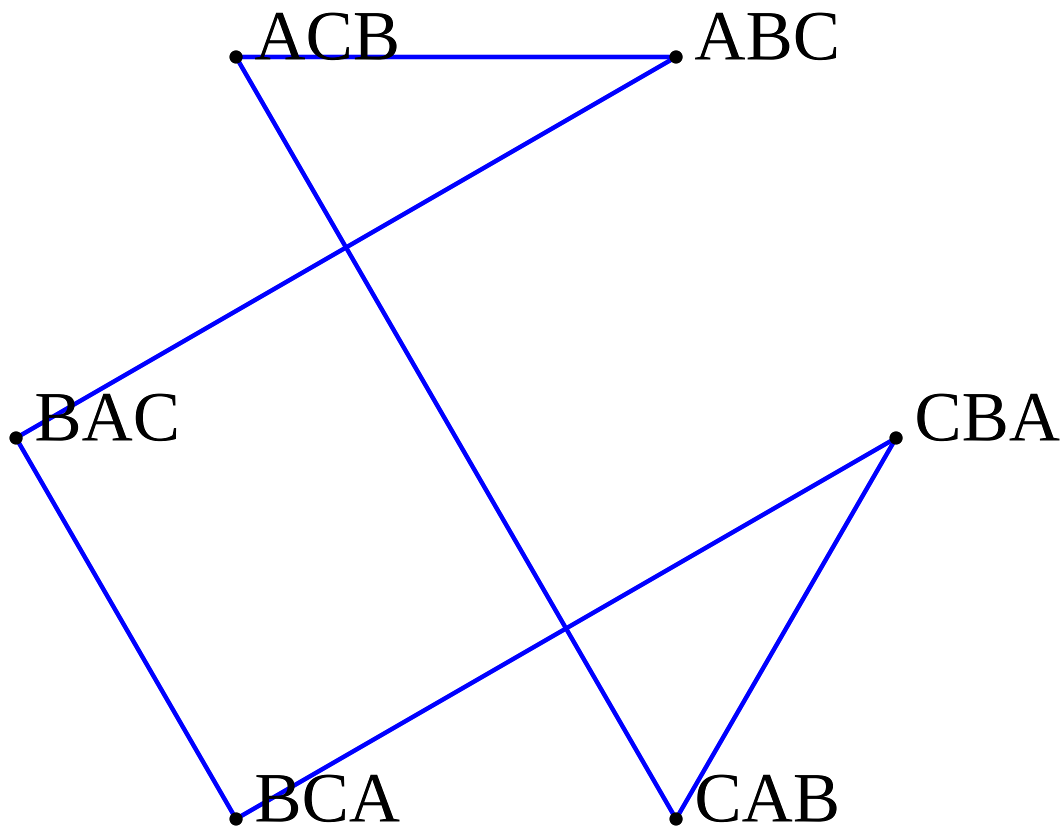
- International Mathematical Olympiad (IMO) 2009, Germany

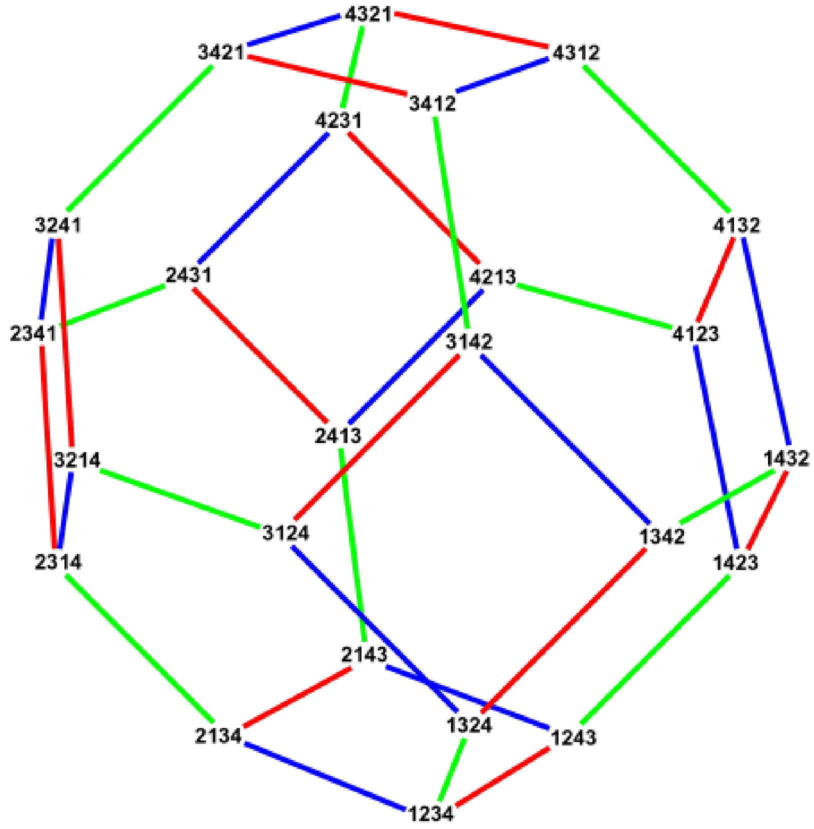
  Opening Ceremony: All (100^{+}) teams get stage time



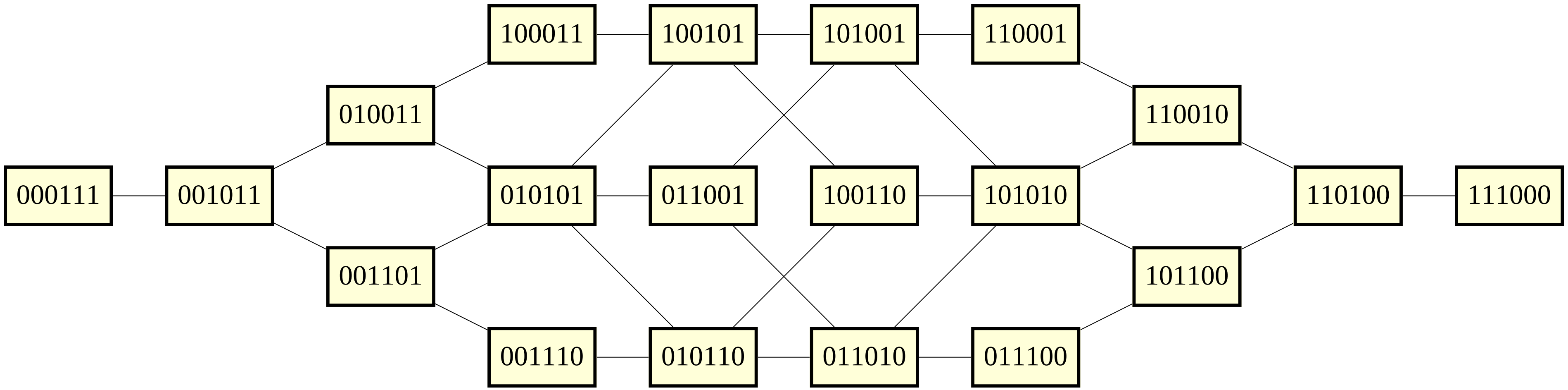
Present All Permutations of a String by Neighbor Swaps

All distinct symbols: possible (change ringing, in 17th C)

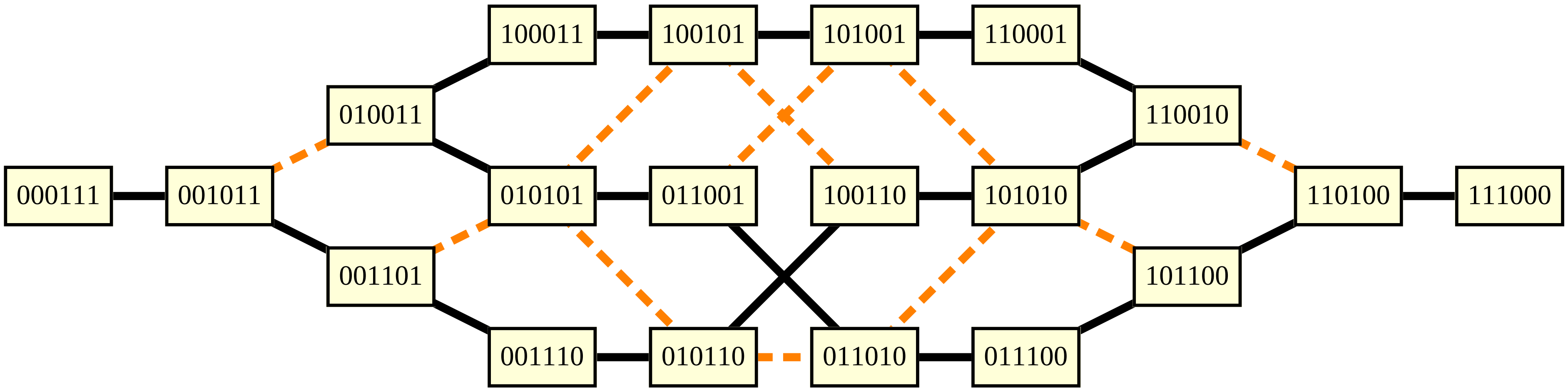




Present All Permutations of 0 0 0 1 1 1 by Neighbor Swaps

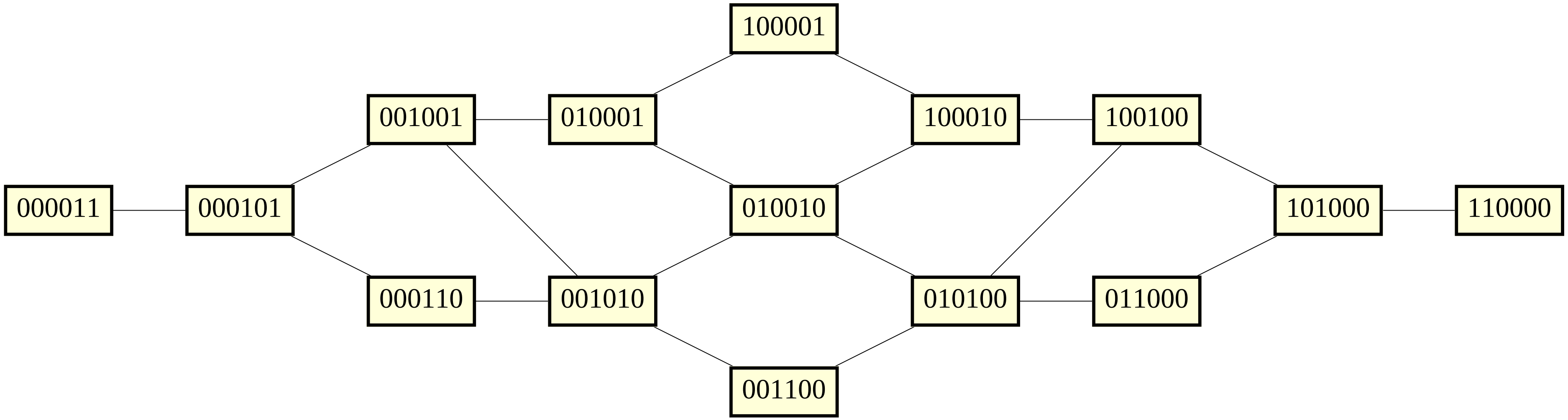


Can Present All Permutations of 0 0 0 1 1 1 by Neighbor Swaps

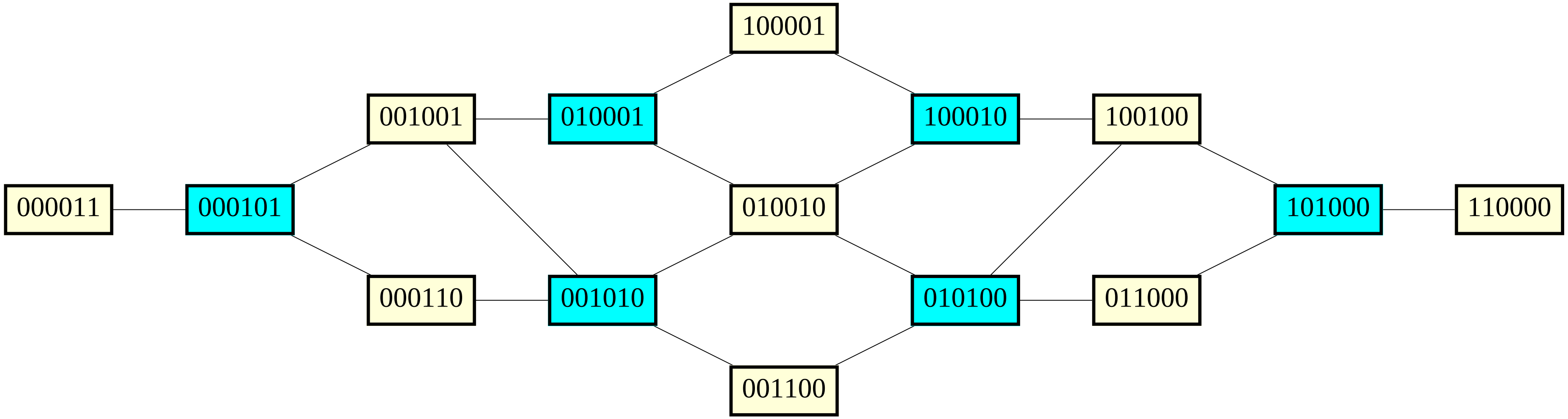


Cycle is impossible

Present All Permutations of 0 0 0 0 1 1 by Neighbor Swaps



Cannot Present Permutations of 0 0 0 0 1 1 by Neighbor Swaps



No edge between same colors: On any path, the two colors alternate

One color exceeds the other color by more than one

Conclusion: A Hamiltonian path does not exist

Present All Permutations of $0^{k_0} 1^{k_1}$ by Neighbor Swaps

Theorem (Eades, Hickey, Read, 1984): Path exists if and only if

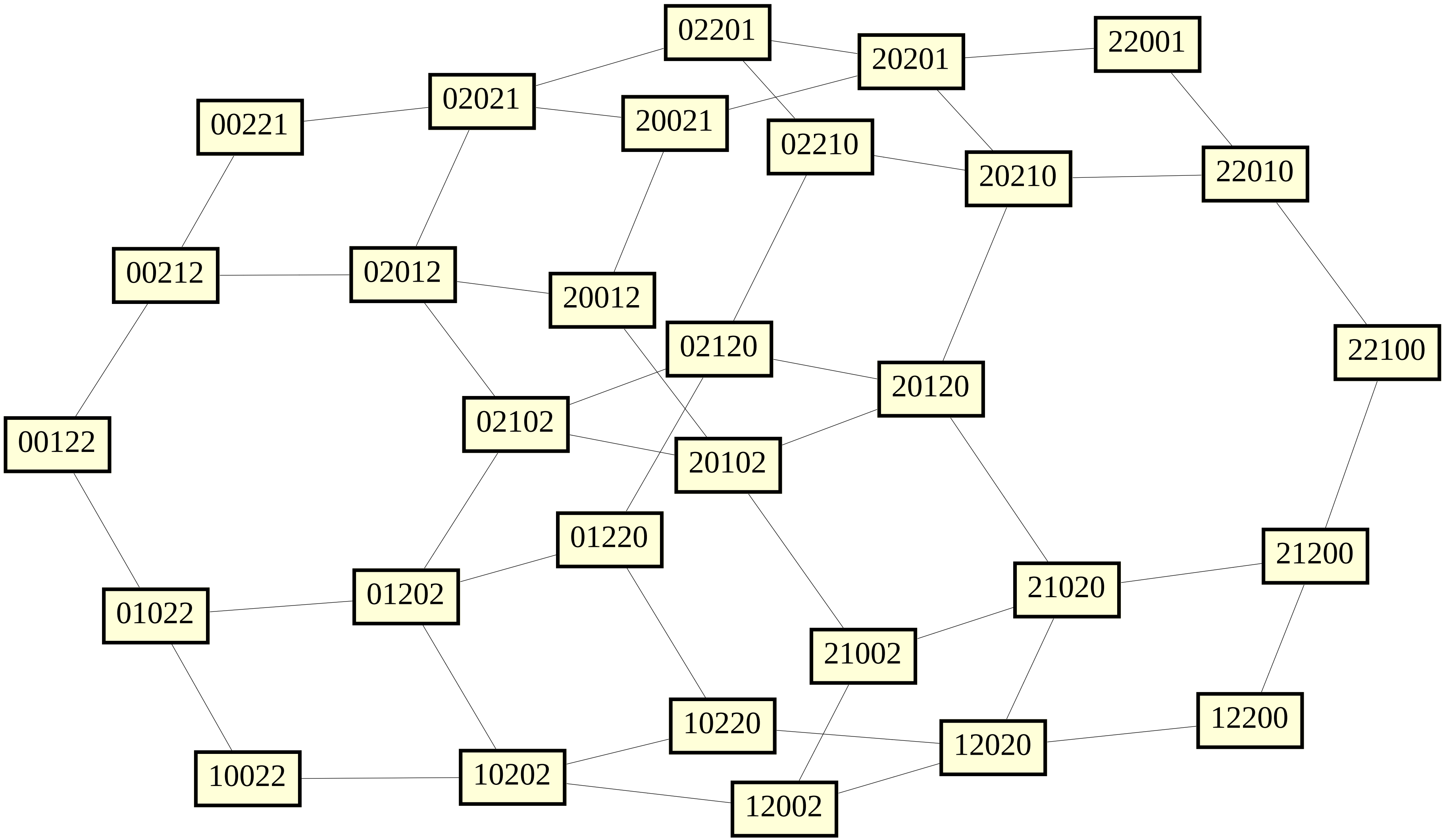
- $k_0 \leq 1$ or $k_1 \leq 1$ (trivial because graph is a chain), or
- both k_0 and k_1 are odd

Proof

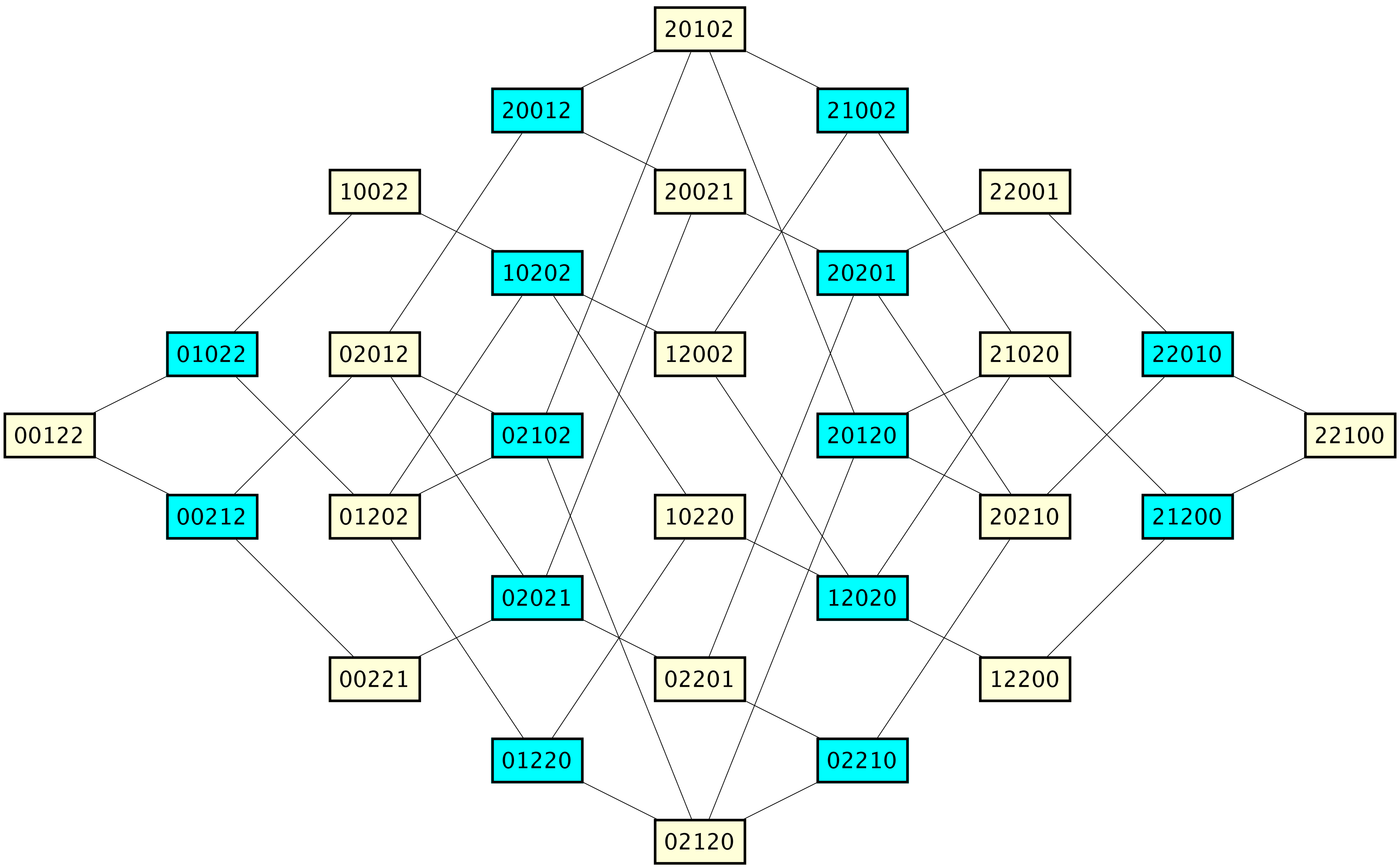
- "Only if": by (involved) coloring argument
- "If": by efficient (not so simple) recursive algorithm

  Alternative algorithms: Cor Hurkens, Ivo van Heck

Neighbor-swap Graph for Permutations of 0 0 1 2 2



Colored Neighbor-swap Graph for Permutations of 0 0 1 2 2



Present All Permutations of $0^{k_0} 1^{k_1} 2^{k_2} \cdots$ by Neighbor Swaps

Three or more kinds of objects

Theorem (Stachowiak, 1992): Path exists if and only if

- at least two of the k_i are odd

Proof

- "Only if": by extended coloring argument
- "If": by rather complex algorithm

  Based on linear extensions of posets

In fact, always a *cycle*, except for (even, 1, 1)

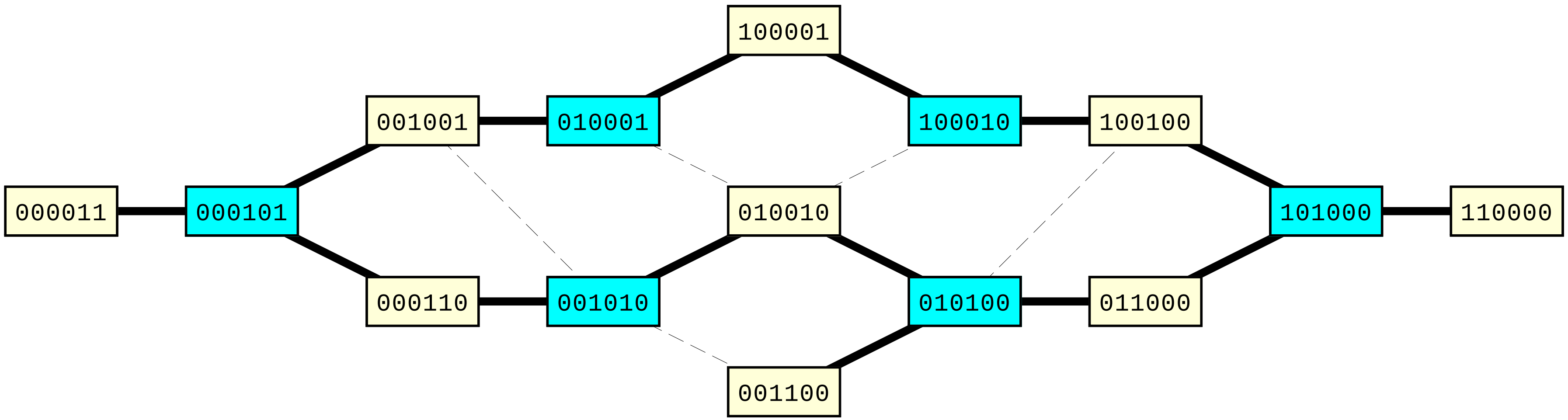
What can be salvaged if less than two of the k_i are odd?

Some requirement on the presentation method must be dropped.

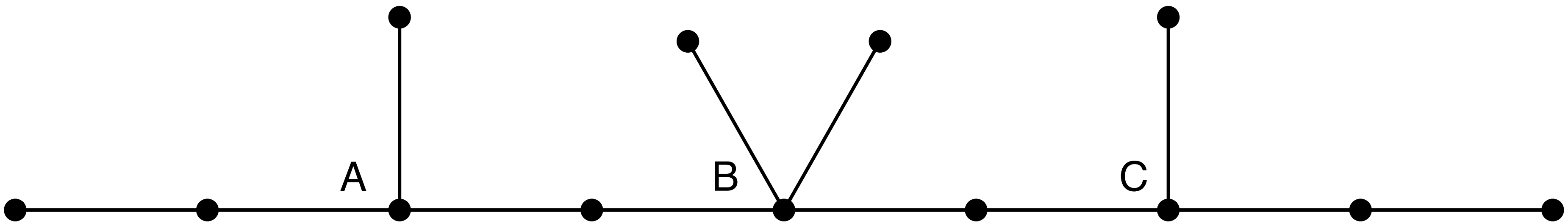
Conjecture (Lehmer, 1965): *Imperfect* Hamilton path always exists

Imperfect: Allow "spurs" (limited way of visiting a vertex twice)

Spur: path contains subsequence $v\,w\,v$



Spurs



Path with 4 spurs: *single* spurs at A and C; *double* spur at B

Terminology and Notation

- Signature: $(k_0, k_1, \ldots)$
- Arity: number of non-zero k_i in signature
- $n = k_0 + k_1 + \cdots$

Coloring and Counting

- *Inversion* in permutation: out-of-order symbol pair $\quad 2\,1\,1\,0 \to 5$
- *Partity* of permutation: parity of its number of inversions
- Neighbor swap changes number of inversions by 1, i.e., parity flips
- *Color* permutations by parity: *bipartite* graph
- $M(k_0, k_1, k_2, \ldots)$: number of permutations of $0^{k_0}\, 1^{k_1}\, 2^{k_2}\, \cdots$

$$M(k_0, k_1, k_2, \ldots) = \binom{n}{k_0\; k_1\; k_2\; \ldots} = \frac{n!}{k_0!\, k_1!\, k_2!\, \cdots}$$

- $D(k_0, k_1, \ldots)$: number of even minus number of odd permutations

$$D(k_0, k_1, \ldots) = \begin{cases} M(k_0 \div 2, k_1 \div 2, \ldots) & \text{if at most one } k_i \text{ is odd} \\ 0 & \text{if at least two } k_i \text{ are odd} \end{cases}$$

New Proof for $D(k_0, k_1, \ldots)$

- Stutter permutation: $e_1\, e_2 \mid e_3\, e_4 \mid \ldots \mid e_{2j-i}\, e_{2j} \mid \ldots$

  with $\forall i : 2i \leq n : e_{2i-1} = e_{2i}$ $\leftarrow$ *left-index-odd* (lio) pairs

- Stutter permutations are *even*

- A stutter permutation has at most two odd k_i

  Number of stutter permutations equals $M(k_0 \div 2, k_1 \div 2, \ldots)$

- Non-stutter permutations can be paired even-to-odd

  Reverse left-most lio pair whose elements *differ*

- $D(k_0, k_1, \ldots) =$ number of stutter permutations

Lemmata for Stutter Permutations

1. The number of odd permutations never exceeds the number of even permutations.
2. The number of non-stutter permutations is even.
3. There are no stutter permutations when the signature has two or more odd k_i.
4. There is exactly one stutter permutation when the signature is unary, or when it is binary and one $k_i = 1$ and the other is even, that is, when the graph is linear.
5. A stutter permutation of arity two or more is at distance 1 (in the neighbor-swap graph) from a non-stutter permutation.
6. The distance (in the neighbor-swap graph) between two distinct stutter permutations is a multiple of 4.

Tom's Conjecture

- There always exists an imperfect Hamilton path,

  with the stutter permutations as spurs.

- More specifically, there exists a Hamilton *cycle* on the *non-stutter* permutations, except when

  1. the arity is zero or one, or
  2. the arity is two, and at least one of the k_i is odd, or
  3. the signature is a permutation of $(2k, 1, 1)$

  In these cases, there exists a Hamilton *path*.

- Proven for arity at most 2

Proof for ‘except when’

A Hamilton cycle is impossible in the indicated cases, because

1. the graph is a singleton;
2. at least one of the permutations $0^{k_0}1^{k_1}$ and $1^{k_1}0^{k_0}$ is a non-stutter permutation, and it has only one neighbor;
3. the edges $0^i120^j \sim 0^i210^j$, for $0 \le i, j$ and $i + j = k_0$, form a *disconnecting set*, and there is an odd number of them.

   A Hamiltonian cycle would need to cross over an even number of times between the two components connected by these edges; in one component 1 precedes 2 in all its permutations, and in the other 2 precedes 1.

Proof for Binary Case

By induction on $n = k_0 + k_1$.

Distinguish three cases for (k_0, k_1):

1. odd-even

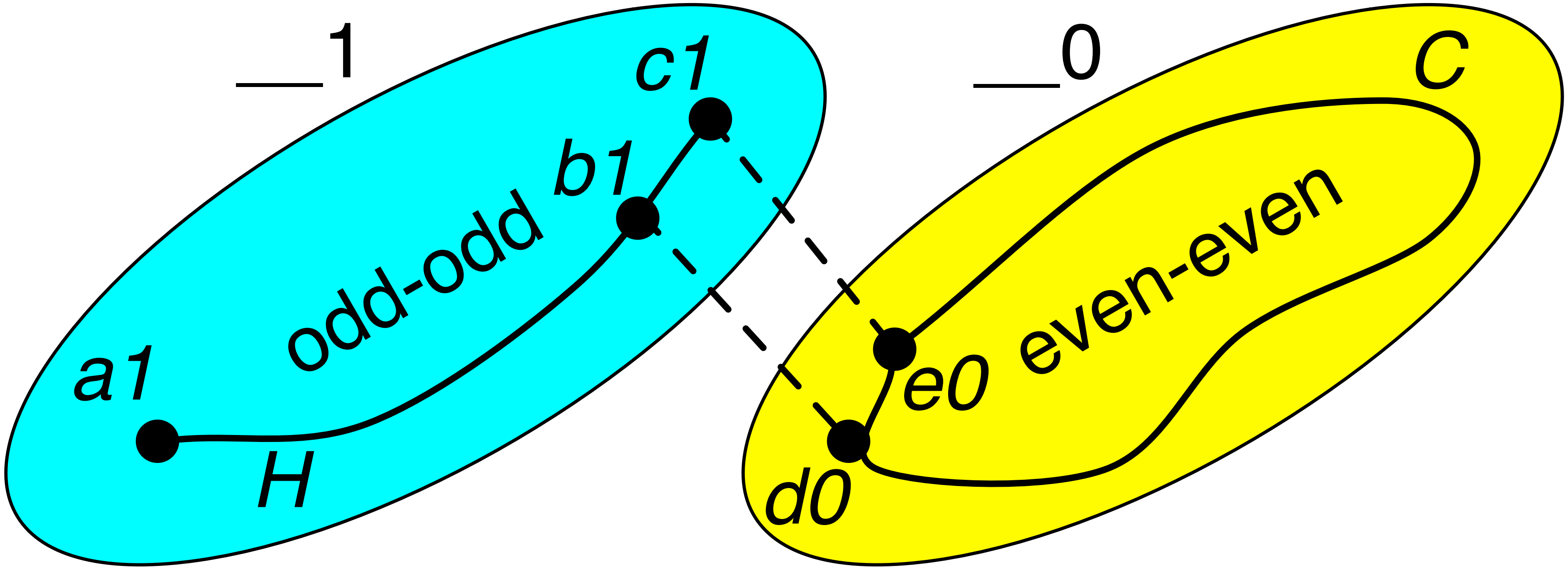
2. even-even

3. odd-odd

   We already knew this, but my construction is more elegant

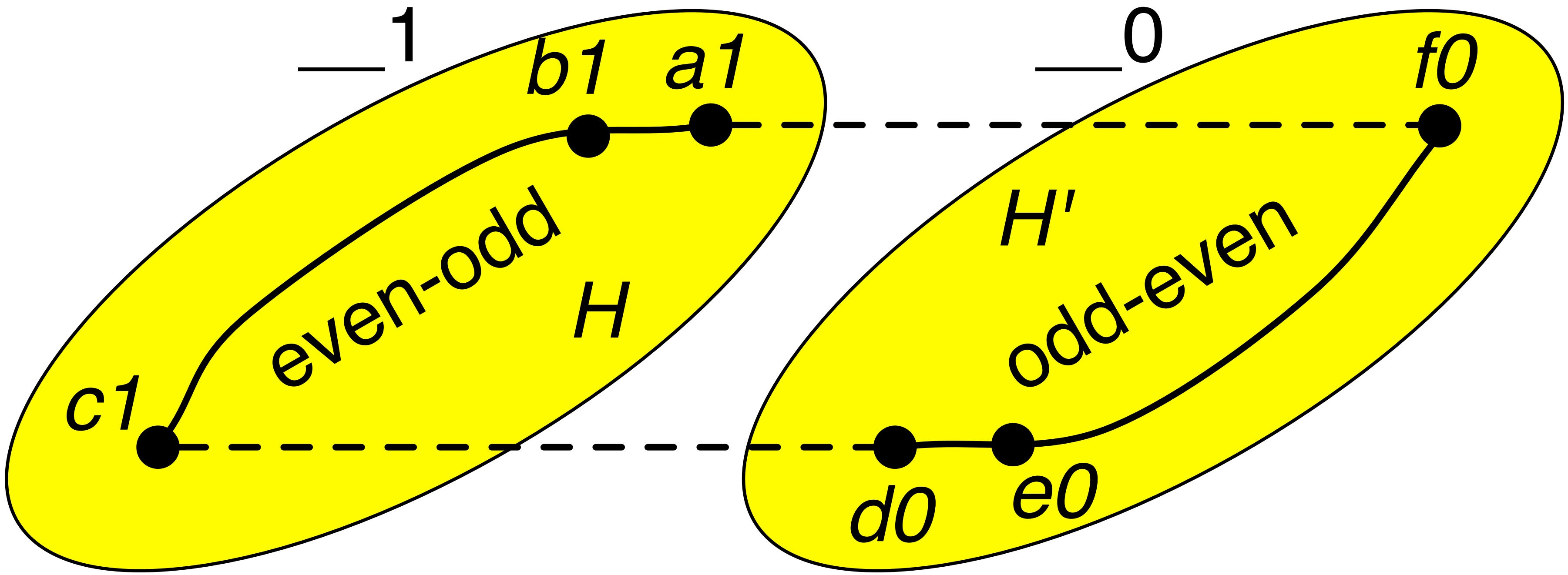
Theorem must be strengthened to help induction, by stating explicit edges that are on the path/cycle.

Odd-Even Case, Split on Trailing Bit



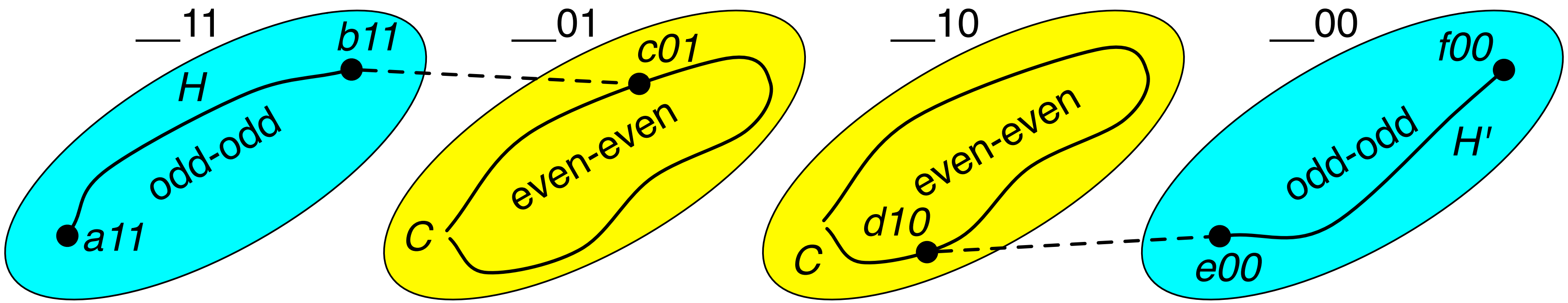
Special parallel edges: $b1 \sim c1$ and $d0 \sim e0$

Even-Even Case, Split on Trailing Bit



Special parallel edges: $b1 \sim a1$ and $d0 \sim e0$

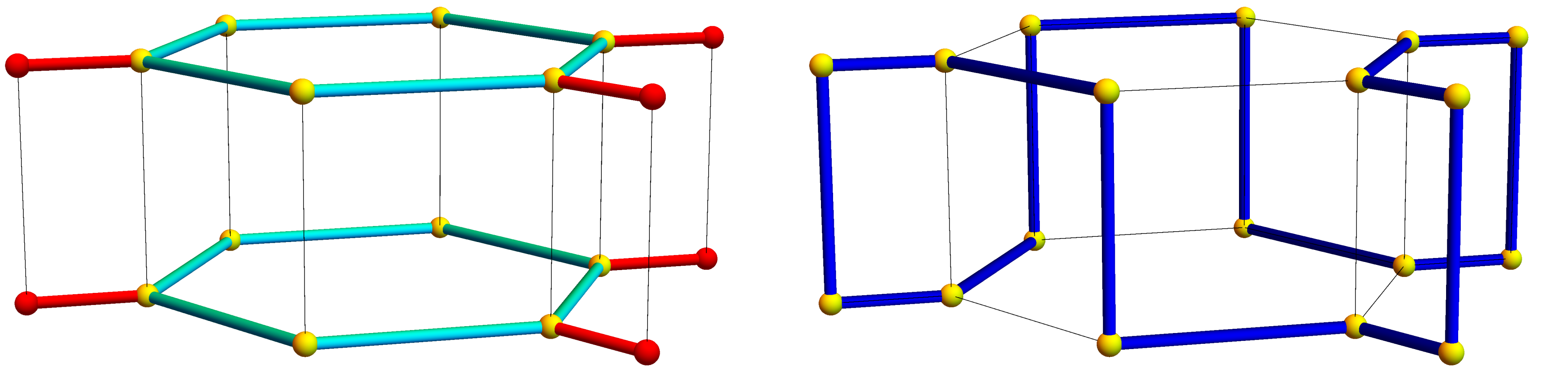
Odd-Odd Case, Split on 2 Trailing Bits



The two even-even parts are isomorphic and parallel

by swapping the trailing two bits

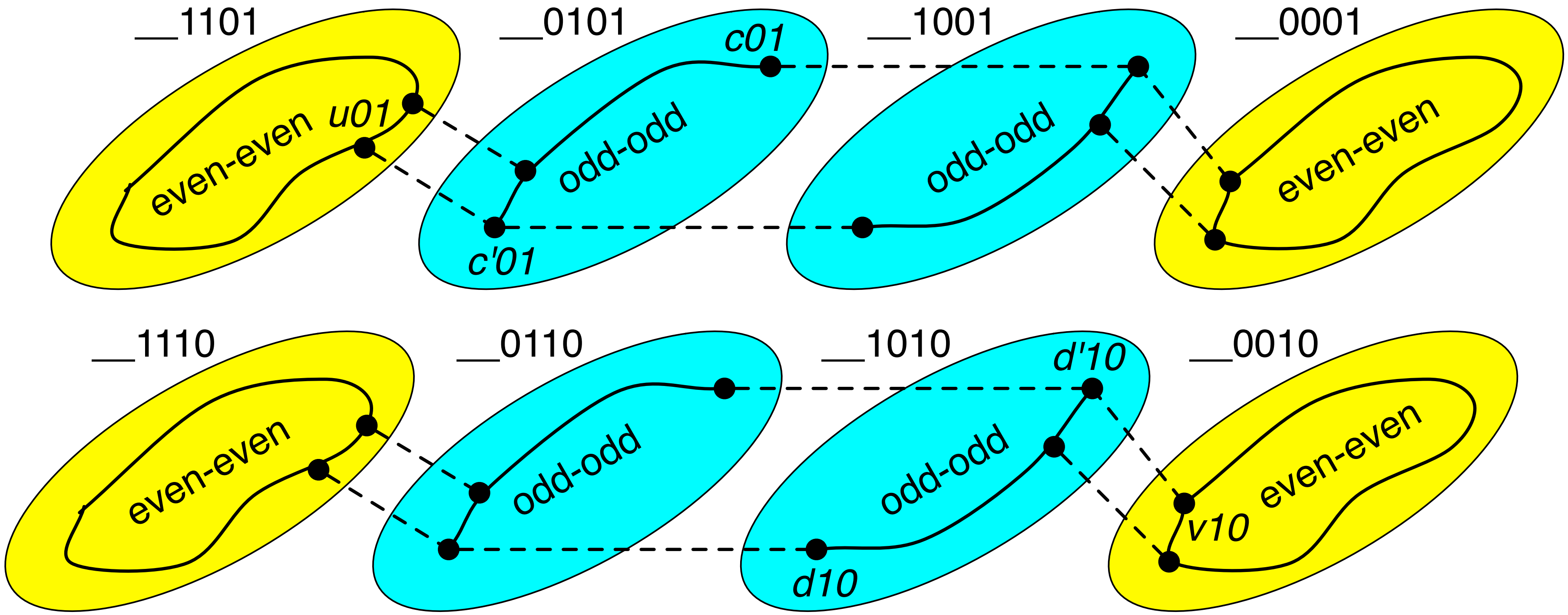
Combining Two Parallel Cycles with Spurs



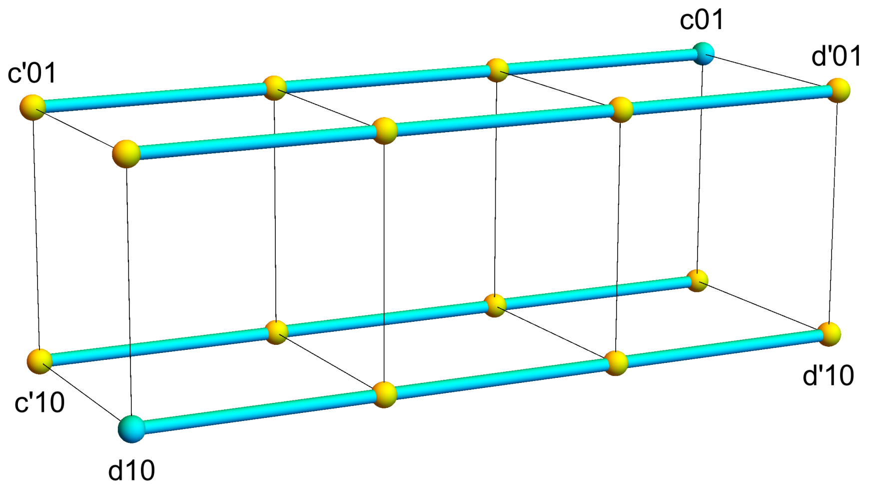
Parallel even-length cycles with single spurs (left)

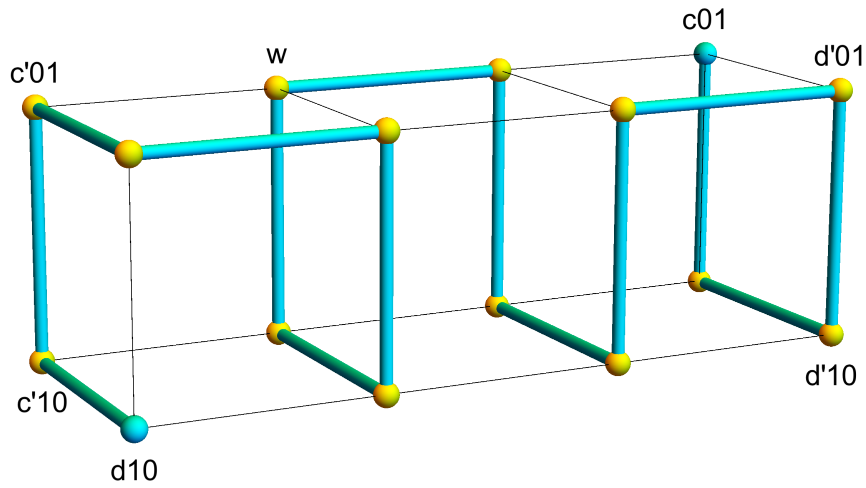
Combined into one cycle without spurs (right)

Break-down of the two parallel cycles for the even-even parts



Combining Four Doubly Parallel Paths





Four doubly parallel paths (left)

Combined into one modular path (right)

General Case, at most two odd

Vertex cycle cover for general case. Inductive structure:

- all even: split on trailing two numbers xy
  - $x = y$: all even, hence cycle
  - $x \neq y$: two odd; $\_xy$ and $\_yx$ are parallel, hence cycle
- all-but-one even, not (even, 2, 1): split on trailing number x
  - k_x odd: all even, hence cycle
  - k_x even: two odd, not (even, 1, 1), hence cycle
- (even, 2, 1), using paths for (even, 1, 1) and (odd, 2, 1)
- (even, 1, 1)

Conclusion

- Stutter permutations: candidates for spur tips
- Proven for binary case
- Evidence for general case: experimental and inductive structure

Why Is This Relevant?

- Exhaustive search across combinatorial objects
  - Cryptography: code breaking
- Hardware testing: traverse test patterns
- Statistics
- Genetic algorithms, to solve optimization problems

Present All Permutations of $0^{k_0}\,1^{k_1}$ by Prefix Rotations (2009)

1. Determine the shortest *prefix* that ends in $0\,1\,X$.

   Take the entire bit string, if such a prefix does not exist.

2. Rotate the prefix cyclicly by one position to the right:
   X (or last element) moves to the front.

$0\,0\,1\,1$

$0\,0\,1\,\underline{1} \to \underline{1}\,0\,0\,1$

$1\,0\,0\,\underline{1} \to \underline{1}\,1\,0\,0$

$1\,1\,0\,\underline{0} \to \underline{0}\,1\,1\,0$

$0\,1\,\underline{1}\,0 \to \underline{1}\,0\,1\,0$

$1\,0\,1\,\underline{0} \to \underline{0}\,1\,0\,1$

$0\,1\,\underline{0}\,1 \to \underline{0}\,0\,1\,1$