IMO 2007, Problem 1

Tom Verhoeff
4 August 2007

Introduction

I will present a calculational solution to Problem 1 of the 48th International Mathematical Olympiad (IMO) held in July 2007 in Hanoi, Vietnam [3]. This is the first of six problems at IMO 2007. On each of the two competition days, the contestants were given three problems to be solved in four and a half hours.

Problem Statement

The original problem statement reads:

**Problem 1.** Real numbers $a_1, a_2, \ldots, a_n$ are given. For each $i$ $(1 \leq i \leq n)$ define

$$d_i = \max\{a_j : 1 \leq j \leq i\} - \min\{a_j : i \leq j \leq n\}$$

and let

$$d = \max\{d_i : 1 \leq i \leq n\}.$$ 

(a) Prove that, for any real numbers $x_1 \leq x_2 \leq \cdots \leq x_n$,

$$\max\{|x_i - a_i| : 1 \leq i \leq n\} \geq \frac{d}{2}. \quad (\ast)$$

(b) Show that there are real numbers $x_1 \leq x_2 \leq \cdots \leq x_n$ such that equality holds in $(\ast)$. 


Solution

It surprises me a bit that the problem statement mentions no restrictions on \( n \), for example that it is a nonnegative integer. The case \( n = 0 \) is uninteresting (or ill-defined, if you prefer). So, let us assume \( 1 \leq n \).

I will not constantly repeat the range restriction on indices. Indices range over the interval \( \{1, 2, \ldots, n\} \). For maximum and minimum I use the notation \( \uparrow \) and \( \downarrow \).

The notation that I use is based on that of Edsger W. Dijkstra [1], who wrote a large collection of (mostly technical) essays [2], many of them with inspiring solutions to nice problems.

The problem statement can now be disentangled as follows:

Given is a sequence \( a \) of \( n \) real numbers. For each \( i \) define

\[
\begin{align*}
m.i &= (\downarrow j : i \leq j : a.j) \\
M.i &= (\uparrow j : j \leq i : a.j) \\
d.i &= M.i - m.i \\
d &= (\uparrow i :: d.i).
\end{align*}
\]

(a) Prove that, for any ascending sequence \( x \) of \( n \) real numbers,

\[
(\uparrow i :: |x.i - a.i|) \geq \frac{d}{2}.
\] \hspace{1cm} (1) \hspace{1cm}

(b) Show that there is an ascending sequence \( x \) of \( n \) real numbers with

\[
(\uparrow i :: |x.i - a.i|) = \frac{d}{2}.
\] \hspace{1cm} (2) \hspace{1cm}

Note the different ranges in the definitions of \( m.i \) and \( M.i \). For increasing \( i \), the range of \( m.i \) decreases (in terms of set containment), and the range of \( M.i \) increases. Hence, both \( m.i \) and \( M.i \) are ascending in \( i \):

\[
\begin{align*}
m.i &\leq m.j \\
M.i &\leq M.j
\end{align*}
\]

for \( i \leq j \). It is also worthwhile to observe that for all \( i \)

\[
m.i \leq a.i \leq M.i.
\] \hspace{1cm} (3) \hspace{1cm}

This yields \( d.i \geq 0 \) and, hence, also \( d \geq 0 \).
The goal of part (a) is to prove (1), which can be restated as

\[(\exists i :: |x.i - a.i| \geq d/2)\]  \hspace{1cm} (4)

The definition of \(d\) involves several quantifications. Let’s analyze it:

\[
d = (\uparrow i :: d.i)
\]

\[
\Rightarrow \{ \text{property of } \uparrow, \text{ guided by } \exists\text{-shape of goal } (4) \}
\]

\[
(\exists i :: d = d.i)
\]

\[
\equiv \{ \text{definition of } d.i \}
\]

\[
(\exists i :: d = M.i - m.i)
\]

\[
\equiv \{ \text{definitions of } M.i \text{ and } m.i, \text{ using a fresh dummy for } m.i \}
\]

\[
(\exists i :: d = (\uparrow j : j \leq i : a.j) - (\downarrow k : i \leq k : a.k))
\]

\[
\Rightarrow \{ \text{property of } \uparrow \text{ and } \downarrow, \text{ guided by } \exists\text{-shape of goal } (4) \}
\]

\[
(\exists i, j, k : j \leq i \leq k : d = a.j - a.k))
\]

Now, take \(j\) and \(k\) with \(j \leq k\) such that

\[a.j - a.k = d\]. \hspace{1cm} (5)

Note that \(x.j \leq x.k\) and thus

\[x.k - x.j \geq 0\]. \hspace{1cm} (6)

Adding (5) and (6) yields

\[a.j - x.j + x.k - a.k \geq d\]. \hspace{1cm} (7)

Consequently (generalized pigeon-hole principle, or converse of monotonicity of addition),

\[a.j - x.j \geq d/2 \lor x.k - a.k \geq d/2\].

Using \(d \geq 0\), this establishes (4) and thereby settles part (a) of the problem.

**Part (b)**

Once we have part (a), the goal of part (b) can be weakened to

Show that there is an ascending sequence \(x\) of \(n\) real numbers with

\[(\uparrow i :: |x.i - a.i|) \leq \frac{d}{2}\]. \hspace{1cm} (8)
Equation (8) can be rephrased as

\[(\forall i :: |x.i - a.i| \leq d/2)\]  (9)

By definition of \(d\) we have

\[(\forall i :: M.i - m.i \leq d)\]  (10)

In view of (3) — \(m.i \leq a.i \leq M.i\) — it seems sweetly reasonable to take

\[x.i = \frac{M.i + m.i}{2},\]

that is, take \(x.i\) as midpoint of \(m.i\) and \(M.i\). It remains to ascertain that this sequence \(x\) is ascending and that it satisfies (9).

Ascendingness follows immediately from the ascendingness of \(m\) and \(M\). And (9) follows from (3) and (10). This is intuitively obvious, but here is a calculation:

\[
|a.i - x.i| \\
= \{ \text{definition of } x.i \} \\
|a.i - (M.i + m.i)/2| \\
= \{ \text{rearrange terms} \} \\
|((a.i - M.i)/2 + (a.i - m.i)/2| \\
\leq \{ \text{triangle inequality} \} \\
|(a.i - M.i)/2| + |(a.i - m.i)/2| \\
= \{ (3): m.i \leq a.i \leq M.i \} \\
(M.i - a.i)/2 + (a.i - m.i)/2 \\
= \{ \text{algebra} \} \\
(M.i - m.i)/2 \\
\leq \{ (10) \} \\
d/2
\]

Q.E.D.

**Conclusion**

I must confess that initially I drew some diagrams, only to find out that they did not really help me in giving a *rigorous* and *elegant* proof. In hindsight, studying the case \(n = 2\) actually suffices for this problem.
The definition of $d$ could have been simplified as follows, but it is not necessary to discover this explicitly:

\[
d = \begin{cases} 
\text{definition of } d \\
(\uparrow i :: d.i)
\end{cases}
\]

\[
= \begin{cases} 
\text{definition of } d.i \\
(\uparrow i :: M.i - m.i)
\end{cases}
\]

\[
= \begin{cases} 
\text{definition of } M.i \text{ and } m.i, \text{ using a fresh dummy for } m.i \\
(\uparrow i :: (\uparrow j : j \leq i : a.j) - (\downarrow k : i \leq k : a.k))
\end{cases}
\]

\[
= \{ -(\downarrow k :: E.k) = (\uparrow k :: -E.k) \}
\]

\[
= \{ \text{distribute + over } \uparrow (\text{nonempty range, twice}) \}
\]

\[
= \{ \text{change order of } \uparrow \text{ quantifications} \}
\]

\[
= \{ a.j - a.k \text{ does not depend on } i, \text{ the range for } i \text{ is nonempty} \}
\]

\[
= (\uparrow j, k : j \leq k : a.j - a.k))
\]

There are various alternative definitions for sequence $x$ in part (b), such as

\[
x.i = M.i - d/2
\]

\[
x.i = m.i + d/2
\]

In summary, the two “key” properties used in my solution are:

\[
b + c \geq d \Rightarrow b \geq \frac{d}{2} \lor c \geq \frac{d}{2}
\]

\[
b \leq a \leq c \Rightarrow \left| a - \frac{b + c}{2} \right| \leq \frac{c - b}{2}
\]

References

