An Invariance Theorem for Helices

Tom Verhoeff

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Definitions Helix $H(\psi, \phi)$ is defined by the helix turtle program (see [1, 2]) with unit step, roll angle ψ, and turn angle ϕ, that is, an indefinite repetition of $\mathit{Move}(1)\,;\mathit{Roll}(\psi)\,;\mathit{Turn}(\phi)$. This figure starts at the origin, and extends into infinity, winding around its axis. Figure 1 shows examples.

$\boxed{H(\psi, \phi)}$

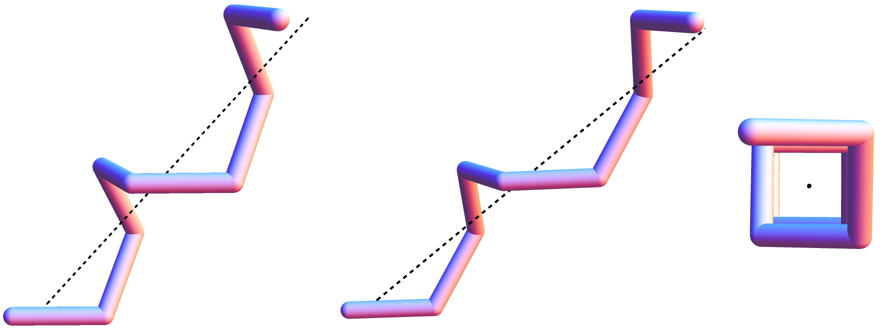

Figure 1: Helices $H(60°, \arccos(1/3))$ (left); $H(\arccos(1/3), 60°)$ (middle); $H(60°, \arccos(1/3))$ viewed along its axis ($\arccos(1/3) \approx 70.5°$)

Consider the parallel projection of $H(\psi, \phi)$ along its axis. This is a regular polygon (possibly infinite). The exterior angle of this polygon is denoted by $\theta(\psi, \phi)$. Figure 1, shows the square projection of $H(60°, \arccos(1/3))$.

$\boxed{\theta(\psi, \phi)}$

Invariance Theorem

$$\theta(\psi, \phi) \quad = \quad \theta(\phi, \psi) \tag{1}$$

That is, the exterior angle of the projection is invariant under swapping the roll angle and the turn angle of the helix.

Proof of Invariance Theorem Let P_i be the i-th vertex of helix $H(\psi, \phi)$, and let $\theta = \theta(\psi, \phi)$. Also see Figure 2. Observe that

- the exterior angle $P_i P_{i+1} P_{i+2}$ (the supplement of the angle at P_{i+1} in $\triangle P_i P_{i+1} P_{i+2}$) equals ϕ;
- the angle between the planes $P_i P_{i+1} P_{i+2}$ and $P_{i+1} P_{i+2} P_{i+3}$ equals ψ;

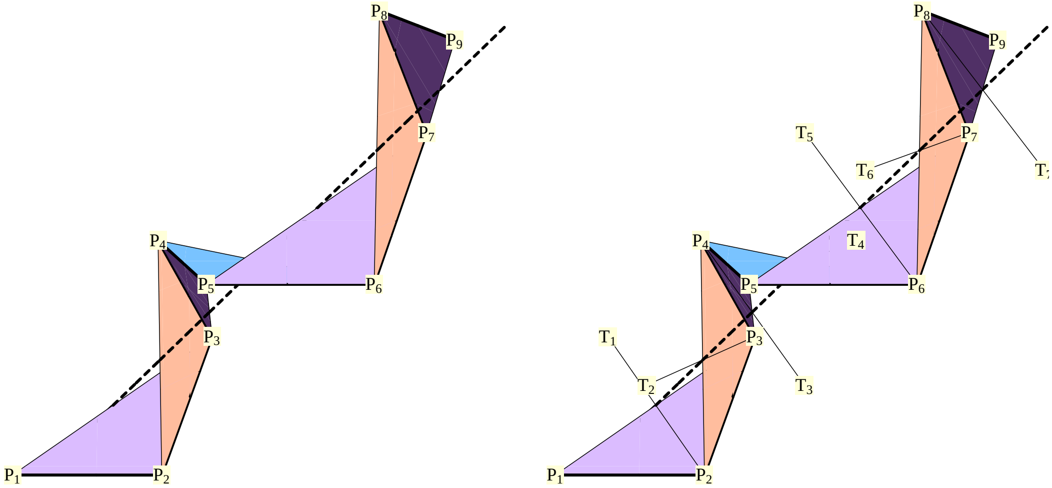

Figure 2: Helix with labels and filled angles (left), and angle bisectors (right)

- the interior angle bisectors $P_{i+1}T_i$ at the vertices of the helix are perpendicular to the axis of the helix;
- the angle between adjacent angle bisectors equals θ, the exterior angle of the polygon obtained by projection.

Now consider Figure 3 (left). It consists of three segments $P_1P_2P_3P_4$ of helix $H(\psi, \phi)$. The angles $\angle P_1P_2P_3$ and $\angle P_2P_3P_4$ have been filled with rhombi. Thus, $P_{i+1}T_i$ is an interior angle bisector at P_i, and it is perpendicular to the other rhombus diagonal $P_{i-1}P_{i+1}$. M_1 is the midpoint of P_2P_3.

M_1S_1 is the translation of P_2T_1 along P_2P_3. Similarly, M_1S_2' is the translation of P_3T_2 along P_2P_3. Consider a plane through M_1 perpendicular to P_2P_3. This plane intersects rhombus $P_1P_2P_3T_1$ at M_1Q_1, and rhombus $P_2P_3P_4T_2$ at M_1Q_1'. Hence, both M_1Q_1 and M_1Q_1' are perpendicular to P_2P_3. M_1R_1 is the interior angle bisector of $\angle Q_1M_1Q_1'$.

From this construction we know (also see Figure 3, right) that

- $\angle P_2P_1T_1 = \phi$ and, hence, $\angle P_1P_3M_1 = \phi/2$;
- M_1S_1 is perpendicular to P_1P_3 (rhombus diagonal), and $\angle P_3M_1Q_1$ is a right angle; hence, $\angle S_1M_1Q_1 = \angle P_1P_3M_1 = \phi/2$;
- $\angle Q_1M_1Q_1' = \psi$ and, hence $\angle Q_1M_1R_1 = \psi/2$;
- $\angle S_1M_1S_2' = \theta$ and, hence $\angle R_1M_1S_1 = \theta/2$.

In Figure 3 (right), we have included x, y, z-axes with

- the origin at M_1,
- the x^+-axis along M_1P_3,
- the y^+-axis along M_1Q_1, and
- the z^+-axis perpendicular to $P_3M_1Q_1$.

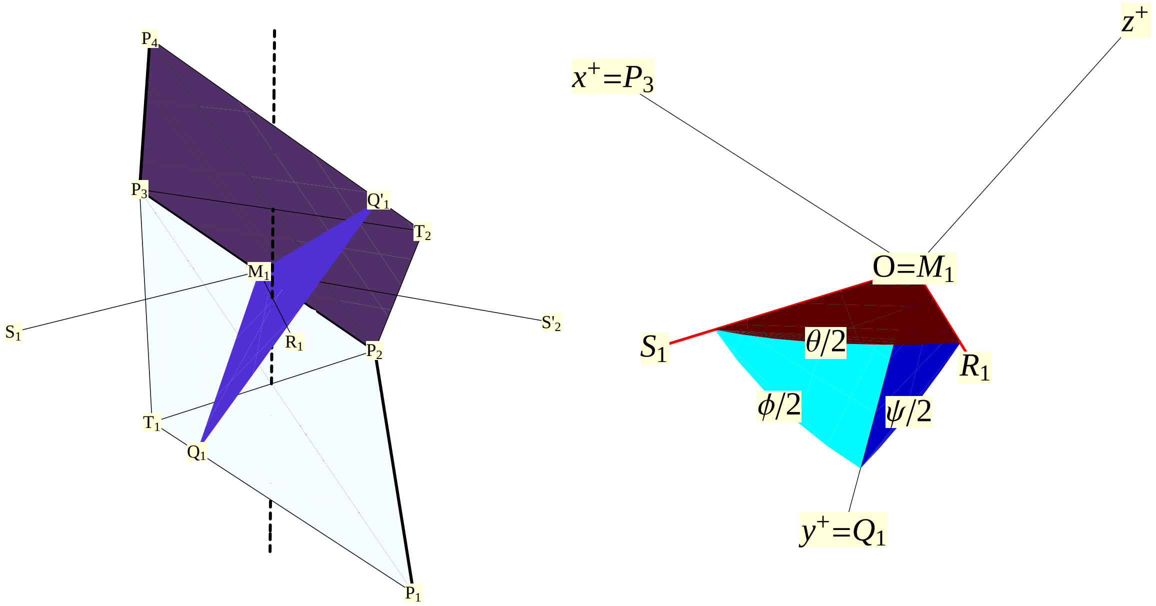

Figure 3: Three segments $P_1P_2P_3P_4$ of a helix with various auxiliary points (point N_1 not shown, see text), lines, and polygons (left); the three relevant half angles and their relationship (right)

M_1R_1 can be obtained from M_1Q_1 by rotating it over $\psi/2$ about the x^+-axis. M_1S_1 can be obtained from M_1Q_1 by rotating it over $-\phi/2$ about the z^+-axis. The angle between M_1R_1 and M_1S_1 is $\theta/2$. Exchanging angles ψ and ϕ is equivalent to reflecting the configuration in the plane $x = z$. Hence, the angle $\theta/2$, and thus also the angle θ, is invariant under exchanging the angles ψ and ϕ. (End of Proof)

The configuration in Figure 3 (right) allows us to express angle θ in terms of ψ and ϕ. Consider the unit vector $(0, 1, 0)$, which points in the direction M_1Q_1. Rotation by $\psi/2$ about the x^+-axis yields the vector $(0, \cos(\psi/2), \sin(\psi/2))$, pointing in the direction of M_1R_1. Rotating $(0, 1, 0)$ by $\phi/2$ about the z^+-axis results in the vector $(\sin(\phi/2), \cos(\phi/2), 0)$, pointing in the direction of M_1S_1. Taking the inner product of these unit vectors gives

$$(0, \cos(\psi/2), \sin(\psi/2)) \cdot (\sin(\phi/2), \cos(\phi/2), 0) \quad = \quad \cos(\psi/2)\cos(\phi/2)$$

So, by the cosine theorem, we have:

Exterior Projected Angle Theorem

$$\cos(\theta/2) \quad = \quad \cos(\psi/2)\cos(\phi/2) \tag{2}$$

where $\theta = \theta(\psi, \phi)$. Note that the right-hand side is symmetric in ϕ and ψ.

Dual Helix

Let us call $H(-\phi, -\psi)$ the *dual helix* of $H(\psi, \phi)$. It has reverse handedness. From Figure 3 (right) we see that the dual helix can be so positioned with respect to the original helix that the axes are parallel and two segments are perpendicular, when P_2P_3 of the dual helix lies along the z^+-axis.

Does the invariance theorem for discrete helices have a counterpart for continuous helices? Is something invariant under exchanging curvature and torsion? Yes: arc length of one period.

How Was This Discovered?

My father, Koos Verhoeff, had told me about some helix weavings, and had communicated the roll and turn angles by phone (60° and $\arccos(1/3) \approx 70.5°$). While constructing the helix in Mathematica, I had accidentally swapped the two values. But I still got a helix winding around a cylinder with the expected cross section, viz. a square, and therefore I thought everything was okay.

When I found out about the mistake, I was surprised that the projection was also a square, and asked myself why that was the case. Was this something accidental for this particular helix, or a general property of these turtle helices?

Note that $H(60°, \arccos(1/3))$ has the (accidental) property that its axis passes through the center of the angle spanning rhombi, as you can see in Figure 3. This is not a general helix property, as you can see in Figure 4.

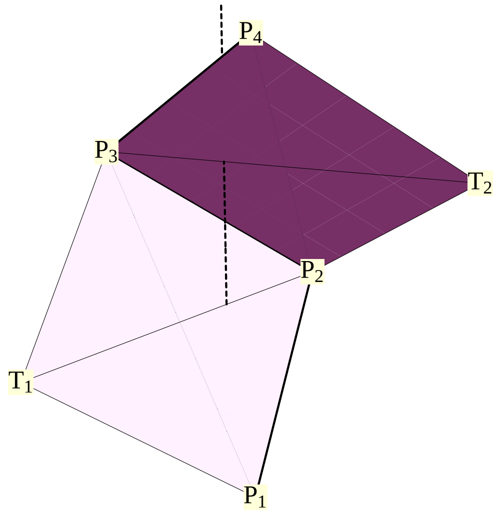

Figure 4: Three segments $P_1P_2P_3P_4$ of $H(90°, 90°)$ with angle bisectors, angle-filling rhombi, and (dashed) axis

References

[1] Tom Verhoeff. “3D Turtle Geometry: Artwork, Theory, Program Equivalence and Symmetry”. *Int. J. of Arts and Technology*, **3**(2/3):288–319 (2010).

[2] Tom Verhoeff, Koos Verhoeff. “From Chain-link Fence to Space-Spanning Mathematical Structures”, *Proceedings of Bridges 2011: Mathematics, Music, Art, Architecture, Culture*, Reza Sarhangi and Carlo H. Séquin (Eds.), pp.73–80. Tessellations Publishing, 2011.