An Invariance Theorem for Helices

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Definitions Helix $H(\psi, \phi)$ is defined by the helix turtle program (see [1, 2]) with unit step, roll angle ψ, and turn angle ϕ, that is, an indefinite repetition of $\mathit{Move}(1)\,;\mathit{Roll}(\psi)\,;\mathit{Turn}(\phi)$. This figure starts at the origin, and extends into infinity, winding around its axis. Figure 1 shows examples. $H(\psi, \phi)$

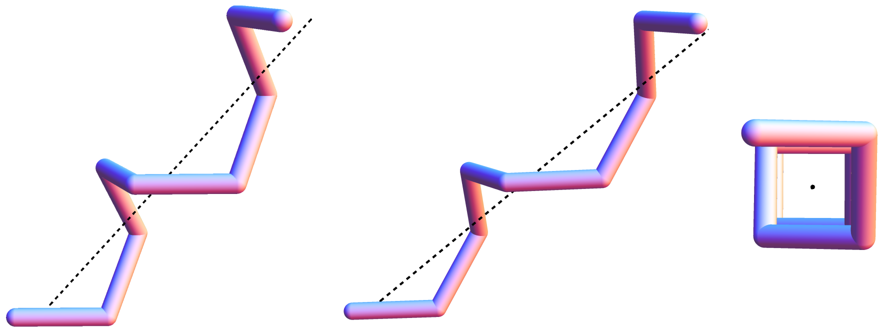

Figure 1: Helices $H(60°, \arccos(1/3))$ (left); $H(\arccos(1/3), 60°)$ (middle); $H(60°, \arccos(1/3))$ viewed along its axis ($\arccos(1/3) \approx 70.5°$)

Consider the parallel projection of $H(\psi, \phi)$ along its axis. This is a regular polygon (possibly infinite). The exterior angle of this polygon is denoted by $\theta(\psi, \phi)$. Figure 1, shows the square projection of $H(60°, \arccos(1/3))$. $\theta(\psi, \phi)$

Invariance Theorem

$$\theta(\psi, \phi) \quad = \quad \theta(\phi, \psi) \tag{1}$$

That is, the exterior angle of the projection is invariant under swapping the roll angle and the turn angle of the helix.

Proof of Invariance Theorem Let P_i be the i-th vertex of helix $H(\psi, \phi)$, and let $\theta = \theta(\psi, \phi)$. Also see Figure 2. Observe that

- the exterior angle $P_i P_{i+1} P_{i+2}$ (the supplement of the angle at P_{i+1} in $\triangle P_i P_{i+1} P_{i+2}$) equals ϕ;
- the angle between the planes $P_i P_{i+1} P_{i+2}$ and $P_{i+1} P_{i+2} P_{i+3}$ equals ψ;

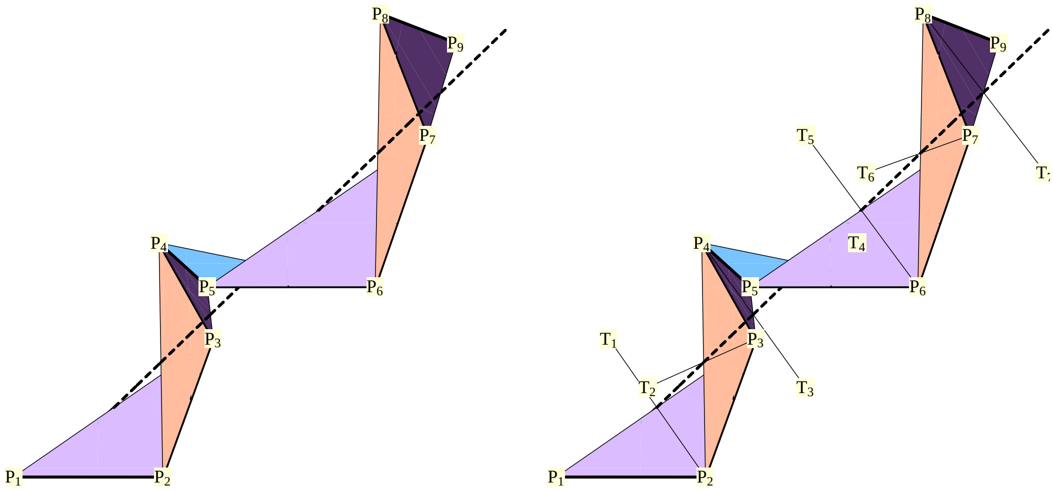

Figure 2: Helix with labels and filled angles (left), and angle bisectors (right)

- the interior angle bisectors $P_{i+1}T_i$ at the vertices of the helix are perpendicular to the axis of the helix;
- the angle between adjacent angle bisectors equals θ, the exterior angle of the polygon obtained by projection.

Now consider Figure 3 (left). It consists of three segments $P_1P_2P_3P_4$ of helix $H(\psi, \phi)$. The angles $\angle P_1P_2P_3$ and $\angle P_2P_3P_4$ have been filled with rhombi. Thus, $P_{i+1}T_i$ is an interior angle bisector at P_i, and it is perpendicular to the other rhombus diagonal $P_{i-1}P_{i+1}$. M_1 is the midpoint of P_2P_3.

M_1S_1 is the translation of P_2T_1 along P_2P_3. Similarly, M_1S_2' is the translation of P_3T_2 along P_2P_3. Consider a plane through M_1 perpendicular to P_2P_3. This plane intersects rhombus $P_1P_2P_3T_1$ at M_1Q_1, and rhombus $P_2P_3P_4T_2$ at M_1Q_1'. Hence, both M_1Q_1 and M_1Q_1' are perpendicular to P_2P_3. M_1R_1 is the interior angle bisector of $\angle Q_1M_1Q_1'$.

From this construction we know (also see Figure 3, right) that

- $\angle P_2P_1T_1 = \phi$ and, hence, $\angle P_1P_3M_1 = \phi/2$;
- M_1S_1 is perpendicular to P_1P_3 (rhombus diagonal), and $\angle P_3M_1Q_1$ is a right angle; hence, $\angle S_1M_1Q_1 = \angle P_1P_3M_1 = \phi/2$;
- $\angle Q_1M_1Q_1' = \psi$ and, hence $\angle Q_1M_1R_1 = \psi/2$;
- $\angle S_1M_1S_2' = \theta$ and, hence $\angle R_1M_1S_1 = \theta/2$.

In Figure 3 (right), we have included x, y, z-axes with

- the origin at M_1,
- the x^+-axis along M_1P_3,
- the y^+-axis along M_1Q_1, and
- the z^+-axis perpendicular to $P_3M_1Q_1$.

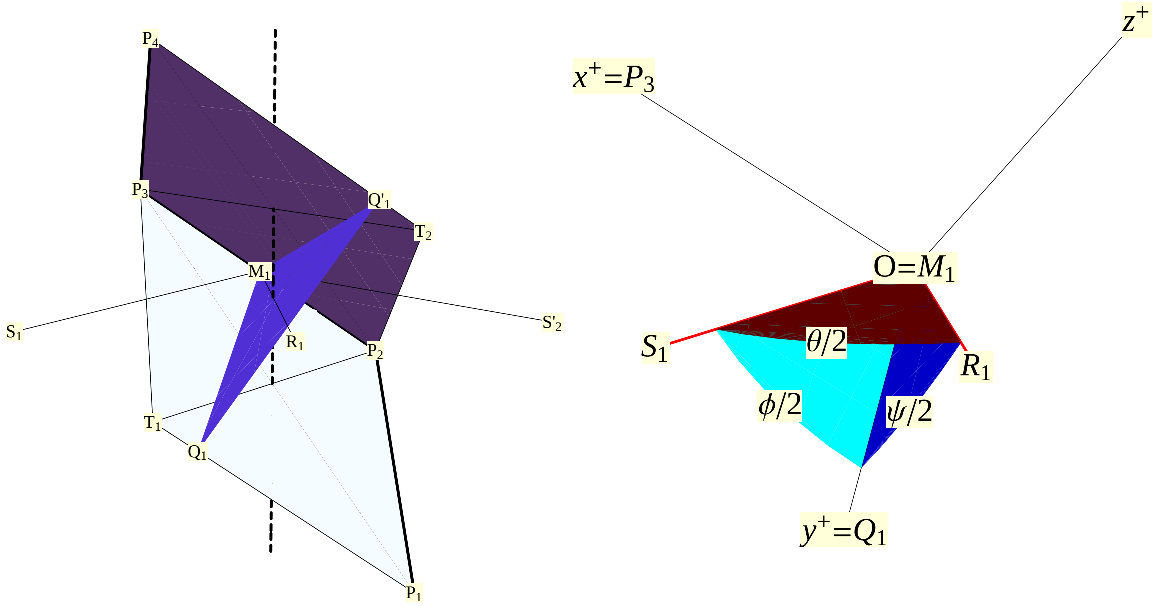

Figure 3: Three segments $P_1P_2P_3P_4$ of a helix with various auxiliary points (point N_1 not shown, see text), lines, and polygons (left); the three relevant half angles and their relationship (right)

M_1R_1 can be obtained from M_1Q_1 by rotating it over $\psi/2$ about the x^+-axis. M_1S_1 can be obtained from M_1Q_1 by rotating it over $-\phi/2$ about the z^+-axis. The angle between M_1R_1 and M_1S_1 is $\theta/2$. Exchanging angles ψ and ϕ is equivalent to reflecting the configuration in the plane $x = z$. Hence, the angle $\theta/2$, and thus also the angle θ, is invariant under exchanging the angles ψ and ϕ. (End of Proof)

The configuration in Figure 3 (right) allows us to express angle θ in terms of ψ and ϕ. Consider the unit vector $(0, 1, 0)$, which points in the direction M_1Q_1. Rotation by $\psi/2$ about the x^+-axis yields the vector $(0, \cos(\psi/2), \sin(\psi/2))$, pointing in the direction of M_1R_1. Rotating $(0, 1, 0)$ by $\phi/2$ about the z^+-axis results in the vector $(\sin(\phi/2), \cos(\phi/2), 0)$, pointing in the direction of M_1S_1. Taking the inner product of these unit vectors gives

$$(0, \cos(\psi/2), \sin(\psi/2)) \cdot (\sin(\phi/2), \cos(\phi/2), 0) \quad = \quad \cos(\psi/2)\cos(\phi/2)$$

So, by the cosine theorem, we have:

Exterior Projected Angle Theorem

$$\cos(\theta/2) \quad = \quad \cos(\psi/2)\cos(\phi/2) \tag{2}$$

where $\theta = \theta(\psi, \phi)$. Alternative:

$$\cos\theta \quad = \quad (1 + \cos\psi)(1 + \cos\phi)/2 - 1 \tag{3}$$

Note that the right-hand side is symmetric in ϕ and ψ.

Only recently, did it occur to me that (2) immediately follows from applying the Pythagorean Theorem for Spherical Geomertry [3] to Figure 3 (right), where $Q_1R_1S_1$ defines a triangle on the unit sphere with center O, in which the dihedral angle between the planes Q_1OR_1 and Q_1OS_1 is a right angle.

Dual Helix

Let us call $H(-\phi, -\psi)$ the *dual helix* of $H(\psi, \phi)$. It has reverse handedness. From Figure 3 (right) we see that the dual helix can be so positioned with respect to the original helix that the axes are parallel and two segments are perpendicular, when P_2P_3 of the dual helix lies along the z^+-axis.

Does the invariance theorem for discrete helices have a counterpart for continuous helices? Is something invariant under exchanging curvature and torsion? Yes: arc length of one period.

How Was This Discovered?

My father, Koos Verhoeff, had told me about some helix weavings, and had communicated the roll and turn angles by phone ($60°$ and $\arccos(1/3) \approx 70.5°$). While constructing the helix in Mathematica, I had accidentally swapped the two values. But I still got a helix winding around a cylinder with the expected cross section, viz. a square, and therefore I thought everything was okay.

When I found out about the mistake, I was surprised that the projection was also a square, and asked myself why that was the case. Was this something accidental for this particular helix, or a general property of these turtle helices?

Note that $H(60°, \arccos(1/3))$ has the (accidental) property that its axis passes through the center of the angle spanning rhombi, as you can see in Figure 3. This is not a general helix property, as you can see in Figure 4.

References

[1] Tom Verhoeff. "3D Turtle Geometry: Artwork, Theory, Program Equivalence and Symmetry". *Int. J. of Arts and Technology*, **3**(2/3):288–319 (2010).

[2] Tom Verhoeff, Koos Verhoeff. "From Chain-link Fence to Space-Spanning Mathematical Structures", *Proceedings of Bridges 2011: Mathematics, Music, Art, Architecture, Culture*, Reza Sarhangi and Carlo H. Séquin (Eds.), pp.73–80. Tessellations Publishing, 2011.

[3] Wikipedia, *Pythagorean Theorem, for Spherical Geomertry.* `http://en.wikipedia.org/wiki/Pythagorean_theorem#Spherical_geometry` (accessed 12 August 2013).

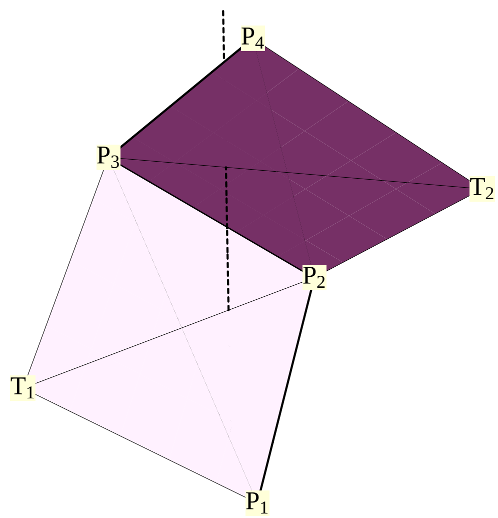

Figure 4: Three segments $P_1P_2P_3P_4$ of $H(90°, 90°)$ with angle bisectors, angle-filling rhombi, and (dashed) axis