Port-Hamiltonian modelling of fluid dynamics models with variable cross-section ⋆

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Abstract: Many single- and multi-phase fluid dynamical systems are governed by non-linear evolutionary equations. A key aspect of these systems is that the fluid typically flows across spatially and temporally varying cross-sections. We, first, show that not any choice of state-variables may be apt for obtaining a port-Hamiltonian realization under spatially varying cross-section. We propose a modified choice of the state-variables and then represent fluid dynamical systems in port-Hamiltonian representations. We define these port-Hamiltonian representations under spatial variation in the cross-section with respect to a new proposed state-dependent and extended Stokes- Dirac structure. Finally, we also account for temporal variations in the cross-section and obtain a suitable structure that respects key properties, such as, for instance, the property of dissipation inequality.

Keywords: multi-phase, non-linear, evolutionary equations, varying cross-sections, port-Hamiltonian, Stokes-Dirac structure, dissipation inequality.

1. INTRODUCTION

Port-Hamiltonian (pH) representations and corresponding structure-preserving discretization and model order reduction have been gaining a lot of momentum recently. Some relevant works include Altmann and Schulze (2017); Martins et al. (2010); Zhou et al. (2015); Farle et al. (2013); Chaturantabut et al. (2016); Macchelli et al. (2004); Maschke and van der Schaft (2005); Cardoso-Ribeiro et al. (2015); Trang VU et al. (2012); van der Schaft and Maschke (2002); Maschke and van der Schaft (1992); Badlyan et al. (2018); Kotyczka et al. (2018), and van der Schaft (2020). This is mainly owed to the favourable properties of such a formalism for both finite- and infinite-dimensional dynamical systems that are characterized by differential, algebraic or mixture of differential and algebraic equations; refer to van der Schaft (2020), and van der Schaft and Maschke (2019, 2018). This paradigm has helped to integrate finite- and infinite-dimensional components and preserve key system properties such as compositionality (Pasumarthy and van der Schaft (2007)).

A pH formulation for single-phase models for flows across constant cross-section already exists with several different choices for the equation of state for the phase of interest; see de Wilde (2015). Moreover, in Bansal et al. (2020), a pH formulation has been presented for two-phase models with fluid(s)/ phases(s) flowing across constant cross-section. However, to the best of our knowledge, no works have considered a pH representation of single- and two-phase flow models across spatially and temporally varying cross-sections.

This aspect is relevant and encountered in many practical applications. For instance, fluid (single- or multi-phase) flows across components with different cross-sections in the drilling application as shown in Naderi Lordejani et al. (2019), in blood flow through a stenosis as shown in Sankar (2010), and many more. Moreover, in general, most of the research has not dealt with the spatial and temporal variations in the parameters of the mathematical model, such as, the cross-sectional area. Additionally, most of the research in the field of pH systems has been done for quadratic Hamiltonian functionals. However, single-phase or multi-phase flow models possess non-quadratic Hamiltonian functional. It is of great interest to investigate

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whether these aspects require mathematical modifications to the existing theory of pH systems, which is quite rich for linear problems (see Jacob and Zwart (2012)) and promises a lot of interesting research in the scope of non-linear problems with non-quadratic Hamiltonian functionals.

In this paper, we develop a (dissipative) Hamiltonian representation of single-phase and two-phase flow models for flows across spatially (and temporally) varying cross-sections. We start with non-quadratic Hamiltonian functionals and then extend the resulting formal Hamiltonian operators to a new infinite-dimensional Stokes-Dirac structure. Moreover, we define the pH systems with respect to this state dependent extended (Stokes-) Dirac structure thus accounting for the dissipation effects and the exchange of energy via the boundaries. Such a representation yields port-variables which are important for practical control purposes. This modeling effort is a stepping stone towards simulation and control for a Managed Pressure Drilling set-up as introduced in Naderi Lordejani et al. (2019).

The structure of this paper is as follows. The mathematical models governing single- and two-phase flow across a variable cross-section are introduced in Section 2. We, then, consider only spatially varying geometry and present a (dissipative-) Hamiltonian representation, propose a state-dependent (extended) Stokes-Dirac structure and define pH representations with respect to this structure in Section 3 for both mathematical models of interest. Section 4 discusses the corresponding pH structure under both spatial and temporal variations in the area of cross-section. We finally end the paper with the conclusions and potential future works.

2. MODEL INTRODUCTION

Let us consider a managed pressure drilling (MPD) system as a motivating application for this work. In MPD, the drill string and the Bottomhole Assembly (BHA) are part of such systems through, and around, which single-phase and multi-phase fluids flows take place. These flow paths have different geometrical specifications. Hence, the flow area in the annular section of the well varies along the spatial location in the well. In addition, the flow area changes dynamically due to the axial movements of the integrated drill string and the BHA system. This dynamical change depends on the position of the drill string and the BHA inside the well. Hence, the dynamic model must take into account cross-sectional area variations. Cross-section variation affects the downhole pressure. Namely, it alters the pressure transmission between the top and down-hole parts of the well, because part of the pressure wave is reflected and part of it is transmitted at the point where the cross-section changes. Oscillatory pressures profiles may be induced more frequently compared to the case where there are no cross-sectional changes along the well. The convergence to a steady-state situation may become slower with the inclusion of cross sectional change. This motivates us to investigate single- and multi-phase models with time-dependent and spatially varying cross-sections of the flow path.

A single-phase flow is mathematically modeled by isothermal Euler equations as in LeVeque (2002):

$$\begin{cases} \partial_t(A\rho) + \partial_x(A\rho v) = 0, \\ \partial_t(A\rho v) + \partial_x(A\rho v^2 + AP) = AS + P\partial_x A, \\ \rho = \rho_0 + \dfrac{P}{c_\ell^2}, \\ S = -\rho g \sin\theta - \dfrac{32\mu v}{d^2}, \end{cases} \tag{1}$$

where $t \in \mathbf{R}_{\geq 0}$ and $x \in [a, b]$ are respectively the time and the spatial domain. Here, variables ρ, v, P, A, g, μ, d and θ respectively, refer to density, velocity, pressure, cross-section area, gravitational constant, fluid viscosity, the diameter of the well/ pipe, and, the (constant) pipe inclination.

A two-phase flow across a geometry with variable cross-section can be modelled by the Drift Flux Model as in Aarsnes et al. (2014), which consists of a combined set of differential equations and algebraic closure laws. The differential equations read as follows:

$$\begin{cases} \partial_t(Am_\ell) + \partial_x(Am_\ell v) = 0, \\ \partial_t(Am_g) + \partial_x(Am_g v) = 0, \\ \partial_t(A(m_\ell v + m_g v)) + \partial_x(A(m_\ell + m_g)v^2) + A\partial_x P = \tilde{S}. \end{cases} \tag{2}$$

Here the abbreviations $m_\ell := \alpha_\ell \rho_\ell$ and $m_g := \alpha_g \rho_g$ have been used. The model is completed via the following algebraic closure laws:

$$\begin{cases} \alpha_g + \alpha_\ell = 1, \\ \rho_g = \dfrac{P}{c_g^2}, \\ \rho_l = \rho_0 + \dfrac{P}{c_\ell^2}, \\ \tilde{S} = -A\Big(g(m_g + m_\ell)\sin\theta - \dfrac{32\mu_m v}{d^2}\Big). \end{cases} \tag{3}$$

The variables α_ℓ and α_g respectively denote liquid and gas void fraction. Variables ρ_ℓ and ρ_g refer to the density of the liquid and the gaseous phasered, respectively. v is the velocity of the phases (no slip assumed), μ_m is the mixture viscosity, and, c_g and c_ℓ, respectively, are the speed of sound in the gaseous and the liquid phase.

Remark 2.1. Using elimination of variables, the system (1) *can be rewritten in terms of two partial differential equations in two unknowns. Similarly, the set of equations* (2) *and* (3) *can be expressed in terms of three partial differential equations in three unknowns. We omit this model reformulation in this work and instead refer to Bansal et al. (2020) for further insights on similar models.*

Remark 2.2. We only consider smooth spatial area variations in this work. Non-smooth (discontinuous) area variations will be considered in future works.

3. PH MODELING - SPATIAL AREA VARIATIONS

We focus on accounting for only spatial cross-section variations and developing a corresponding pH model representation(s) in this section. We, first, introduce (dissipative) Hamiltonian representations i.e., without boundary effects/ under the assumption of zero boundary conditions for the mathematical models under consideration. The resulting formal skew-adjoint operator(s) and the resistive matrix are used as a tool to define a candidate geometrical structure, which is later shown to be a non-canonical/ extended (Stokes-)Dirac structure. This geometric object yields a way to describe the boundary port-variables ultimately leading to the port-Hamiltonian representation of the models of interest. These pH representations inherit properties from the (Stokes-) Dirac structures.

3.1 Dissipative Hamiltonian Representation

Considering the total energy of the system, the Hamiltonian functional, consisting of kinetic, internal and potential energy, given by:

$$\mathcal{H}_s = \int_\Omega A(\rho\frac{v^2}{2} + \rho c_l^2 \ln\rho + c_\ell^2\rho_0 + \rho g x \sin\theta)\mathrm{d}x, \tag{4}$$

where $\Omega = [a, b]$ refers to the spatial domain.

Remark 3.1. The above functional is similar to the functional used in de Wilde (2015). However, here $\mathcal{H}_s$ is distinct as it accounts for the effects of area (A). Moreover, the equation of state (an algebraic relation relating density and pressure) is also different.

We, first, chose a state coordinate vector comprised of non-conservative variables i.e., ρ and v, and aim to develop a port-Hamiltonian framework for Isothermal Euler equations (governed by the set of equations (1)) across a variable cross-section. This case is used as a test-bed to emphasize that not any choice of state-variables may be apt to obtain a structure with the required properties.

The isothermal Euler equations in (1) without gravitational effects can be re-written as follows:

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}\begin{pmatrix} \partial_t\rho \\ \partial_t v \end{pmatrix} = \left[\underbrace{\begin{pmatrix} 0 & -\partial_x \\ -\partial_x + \frac{1}{A}\partial_x A & 0 \end{pmatrix}}_{M} + \begin{pmatrix} 0 & 0 \\ 0 & -\frac{32\mu}{\rho^2 d^2} \end{pmatrix}\right]\begin{pmatrix} \frac{\delta\mathcal{H}_s}{\delta\rho} \\ \frac{\delta\mathcal{H}_s}{\delta v} \end{pmatrix}. \tag{5}$$

Remark 3.2. We have ignored the gravitational effect in the representation (5) as its role is not key in discussing the critical issue that arises due to the variation in the cross-sectional area. Then, the Hamiltonian functional only consists of kinetic and internal energy. Hence, the above representation will hold for a horizontal pipe setting.

We omit the derivation as the above formulation can be obtained in a straightforward manner.

We decompose the operator M, introduced in (5), as follows:

$$M := \begin{pmatrix} 0 & -\partial_x \\ -\partial_x & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \frac{1}{A}\partial_x A & 0 \end{pmatrix}. \tag{6}$$

It is trivial to see that the first term in the right-hand side of (6) is formally skew-adjoint. However, the second term in the above equation is not formally skew-adjoint (under spatial cross-section variation). As a result, the operator M is not formally skew-adjoint. It is also clearly observable that the second term in the right-hand side of (6) would have been the zero matrix (which is trivially formally skew-adjoint) under constant cross-section. Hence, the operator M would be formally skew-adjoint in that case.

The above observations illustrate that the primitive state variables may not always have the desired properties attributed to general (port-) Hamiltonian representations. However, the conservative state variables (generally) yield relevant structural properties. We now define the state vector in terms of conservative variables. In addition, we extend the reduced version of (1) (obtained upon elimination of variable P) by an extra equation $\partial_t A = 0$, which means that only spatial variations of A are allowed. Finally, by invoking these proposed modifications, we demonstrate the dissipative Hamiltonian representation for the single-phase flow model while accounting for (smooth) spatial cross-sectional area variations.

We re-write the Hamiltonian functional in terms of the chosen set of state-variables $q = [q_1,\ q_2,\ q_3]^T := [A,\ A\rho,\ A\rho v]^T$. This yields

$$\mathcal{H}_s = \int_\Omega \frac{q_3^2}{2q_2} + q_2 c_\ell^2 \ln(\frac{q_2}{q_1}) + q_1 c_\ell^2\rho_0 + q_2 g x \sin\theta \mathrm{d}x. \tag{7}$$

We now present the dissipative Hamiltonian representation[1] for the single-phase model.

Theorem 1. Considering the governing equations (1)*, the associated dissipative Hamiltonian representation is given by*

$$\partial_t q = (\mathcal{J}_s(q) - \mathcal{R}_s(q))\delta_q\mathcal{H}_s(q), \tag{8}$$

with the Hamiltonian functional (7)*, where*

$$\mathcal{J}_s = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\partial_x(q_2\cdot) \\ 0 & -q_2\partial_x(\cdot) & -q_3\partial_x(\cdot) - \partial_x(q_3\cdot) \end{pmatrix}, \tag{9}$$

is a formally skew-adjoint operator with respect to the $\mathcal{L}^2$ inner product, and,

$$\mathcal{R}_s = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & q_1\frac{32\mu}{d^2} \end{pmatrix}, \tag{10}$$

is symmetric and positive semi-definite matrix.

Proof. We evaluate the variational derivatives with respect to the states. These are

$$\frac{\delta\mathcal{H}_s}{\delta q_1} = -\frac{q_2}{q_1}c_\ell^2 + \rho_0 c_\ell^2, \tag{11}$$

$$\frac{\delta\mathcal{H}_s}{\delta q_2} = -\frac{q_3^2}{2q_2^2} + c_\ell^2\ln(\frac{q_2}{q_1}) + c_\ell^2 + g x \sin\theta, \tag{12}$$

$$\frac{\delta\mathcal{H}_s}{\delta q_3} = \frac{q_3}{q_2}. \tag{13}$$

Using these variational derivatives, the claim that (8) is equivalent to a reformulated version of (1) (with additional $\partial_t A = 0$) follows in a manner similar to the derivation discussed in-depth in Theorem 2. Hence, we omit the derivation here.

The positive semi-definiteness and symmetric nature of $\mathcal{R}_s$ follows immediately from the positivity of q_1, μ and d and the structure of the matrix. The formal skew-adjointness of $\mathcal{J}_s$ essentially follows from integration by parts and neglecting the boundary conditions. The operator $\mathcal{J}_s$ has terms similar to the skew-adjoint operator in Bansal et al. (2020). For the sake of brevity, we omit the proof and instead refer to Bansal et al. (2020) for a similar derivation.

[1] *The dissipative Hamiltonian representation refers to the model representation abiding by the non-increasing behavior of the Hamiltonian functional along the solutions of the model.*

Using the properties of $\mathcal{J}_s$ and $\mathcal{R}_s$, the following dissipation inequality holds:

$$\begin{aligned}\frac{d\mathcal{H}_s}{dt} &= \int_\Omega (\delta_q \mathcal{H}_s(q))^T \partial_t q \ \mathrm{d}x \\ &= \int_\Omega (\delta_q \mathcal{H}_s(q))^T (\mathcal{J}_s(q) - \mathcal{R}_s(q)) \delta_q \mathcal{H}_s(q) \ \mathrm{d}x \\ &= \int_\Omega (\delta_q \mathcal{H}_s(q))^T (-\mathcal{R}_s(q)) \delta_q \mathcal{H}_s(q) \ \mathrm{d}x \leq 0.\end{aligned} \tag{14}$$

This completes the proof.

We now consider a two-phase Drift Flux Model without slip i.e., (2) and (3), and show the corresponding dissipative Hamiltonian representation under the choice of conservative state-variables. Following the choice of candidate Hamiltonian functional in Bansal et al. (2020), we now choose the Hamiltonian functional in the following manner:

$$\mathcal{H}_t = \int_\Omega A(m_g \frac{v^2}{2} + m_\ell \frac{v^2}{2} + m_g c_g^2 \ln \rho_g + m_\ell c_\ell^2 \ln \rho_\ell + (1-\alpha_g)\beta + (m_g + m_\ell) g x \sin\theta) \mathrm{d}x,$$

where $\beta = \rho_0 c_\ell^2$. The above functional can be expressed in terms of the following choice of state-variables $\tilde{q} = [\tilde{q}_1,\ \tilde{q}_2,\ \tilde{q}_3,\ \tilde{q}_4]^T := [A,\ Am_g,\ Am_\ell,\ A(m_g+m_\ell)v]^T$ as follows:

$$\mathcal{H}_t = \int_\Omega \Big(\tilde{q}_1 (\frac{\tilde{q}_2}{2\tilde{q}_1} v^2 + \frac{\tilde{q}_3}{2\tilde{q}_1} v^2) + \tilde{q}_2 c_g^2 \ln(\frac{P}{c_g^2}) + \tilde{q}_3 c_\ell^2 \ln(\frac{P+\beta}{c_\ell^2}) + \tilde{q}_1 (1-\alpha_g)\beta + (\tilde{q}_2+\tilde{q}_3) g x \sin\theta \Big) \mathrm{d}x, \tag{15}$$

where v can be expressed in terms of the chosen state-variables by a relation $v = \frac{\tilde{q}_4}{\tilde{q}_2+\tilde{q}_3}$. Moreover, we use the relations in Aarsnes et al. (2014) to obtain the gas void fraction α_g from the mass variables, which is given by:

$$\alpha_g = -\frac{\tilde{q}_2}{\tilde{q}_1}\frac{c_g^2}{2\beta} - \frac{\tilde{q}_3}{\tilde{q}_1}\frac{c_\ell^2}{2\beta} + \frac{1}{2} + \sqrt{\Delta}, \tag{16}$$

where

$$\Delta = \Big(\big(\frac{\tilde{q}_2}{\tilde{q}_1}\frac{c_g^2}{2\beta} + \frac{\tilde{q}_3}{\tilde{q}_1}\frac{c_\ell^2}{2\beta} - \frac{1}{2} \big)^2 + \frac{\tilde{q}_2}{\tilde{q}_1}\frac{c_g^2}{\beta} \Big). \tag{17}$$

The pressure P can be computed in the following way:

$$P = \frac{\tilde{q}_2}{\tilde{q}_1} c_g^2 + \frac{\tilde{q}_3}{\tilde{q}_1} c_\ell^2 - \beta(1-\alpha_g). \tag{18}$$

Next, we discuss the dissipative Hamiltonian representation for the two-phase model. We consider a model reformulation of the governing equations (2) along with the closure equations (3), and, express these as a system composed of three equations in three unknowns (state-variables). Moreover, as before, we consider an additional equation $\partial_t A = 0$. We refer to the resulting model as Σ in the sequel.

Theorem 2. The dissipative Hamiltonian representation of the reformed model Σ in the scope of two-phase flow models takes the following form:

$$\partial_t \tilde{q} = (\mathcal{J}_t(\tilde{q}) - \mathcal{R}_t(\tilde{q})) \delta_{\tilde{q}} \mathcal{H}_t(\tilde{q}), \tag{19}$$

with the Hamiltonian functional (15)*, and where*

$$\mathcal{J}_t = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\partial_x(\tilde{q}_2 \cdot) \\ 0 & 0 & 0 & -\partial_x(\tilde{q}_3 \cdot) \\ 0 & -\tilde{q}_2\partial_x(\cdot) & -\tilde{q}_3\partial_x(\cdot) & -\partial_x(\tilde{q}_4 \cdot) - \tilde{q}_4\partial_x(\cdot) \end{pmatrix}, \tag{20}$$

and,

$$\mathcal{R}_t = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{q}_1 \frac{32\mu_m}{d^2} \end{pmatrix}. \tag{21}$$

Proof. We first compute the variational derivatives. The variational derivatives [2] are:

$$\frac{\delta \mathcal{H}_t}{\delta \tilde{q}_2} = -\frac{\tilde{q}_4^2}{2(\tilde{q}_2+\tilde{q}_3)^2} + c_g^2 \ln(\frac{P}{c_g^2}) + c_g^2 + g x \sin\theta, \tag{22}$$

$$\frac{\delta \mathcal{H}_t}{\delta \tilde{q}_3} = -\frac{\tilde{q}_4^2}{2(\tilde{q}_2+\tilde{q}_3)^2} + c_\ell^2 \ln(\frac{P+\beta}{c_\ell^2}) + c_\ell^2 + g x \sin\theta, \tag{23}$$

$$\frac{\delta \mathcal{H}_t}{\delta \tilde{q}_4} = \frac{\tilde{q}_4}{\tilde{q}_2+\tilde{q}_3} = v. \tag{24}$$

We now prove the claim equation by equation. The first line holds trivially as we assume that the cross-sectional area only varies spatially. The second line reads

$$\partial_t(Am_g) = -\partial_x(\tilde{q}_2 \frac{\delta \mathcal{H}_t}{\delta \tilde{q}_4}) = -\partial_x(Am_g v). \tag{25}$$

Similarly, the third line results in

$$\partial_t(Am_\ell) = -\partial_x(\tilde{q}_3 \frac{\delta \mathcal{H}_t}{\delta \tilde{q}_4}) = -\partial_x(Am_\ell v). \tag{26}$$

Finally, the fourth line yields

$$\partial_t(\tilde{q}_4) = -\tilde{q}_2 \partial_x(\frac{\delta \mathcal{H}_t}{\delta \tilde{q}_2}) - \tilde{q}_3 \partial_x(\frac{\delta \mathcal{H}_t}{\delta \tilde{q}_3}) - \partial_x(\tilde{q}_4 \frac{\delta \mathcal{H}_t}{\delta \tilde{q}_4}) - \tilde{q}_4 \partial_x(\frac{\delta \mathcal{H}_t}{\delta \tilde{q}_4}) - \tilde{q}_1 \frac{32\mu}{d^2} \frac{\delta \mathcal{H}_t}{\delta \tilde{q}_4}.$$

Substituting the variational derivatives, we have

$$\begin{aligned}\partial_t(\tilde{q}_4) = &-Am_g \partial_x(-\frac{v^2}{2} + c_g^2 \ln(\frac{P}{c_g^2}) + c_g^2) \\ &- Am_\ell \partial_x(-\frac{v^2}{2} + c_\ell^2 \ln(\frac{P+\beta}{c_\ell^2}) + c_\ell^2) \\ &- \partial_x(A(m_g+m_\ell)v^2) - A(m_g+m_\ell) v \partial_x v \\ &- A(m_g+m_\ell) g \sin\theta - A\frac{32\mu_m v}{d^2}.\end{aligned} \tag{27}$$

This simplifies to:

$$\partial_t(A(m_g+m_\ell)v) = -\partial_x(A(m_g+m_\ell)v^2) - A\partial_x P - A(m_g+m_\ell) g \sin\theta - A\frac{32\mu_m v}{d^2}, \tag{28}$$

where we have used the identity

$$-Am_g c_g^2 \partial_x(\ln \frac{P}{c_g^2}) - Am_\ell c_\ell^2 \partial_x(\ln \frac{P+\beta}{c_\ell^2}) =: -A\partial_x P.$$

This completes the proof.

Remark 3.3. We have only used constant pipe-inclination θ in this work. However, it is straightforward to account for spatially varying pipe inclinations; see Bansal et al. (2020).

The formal skew-adjointness of $\mathcal{J}_t$ with respect to the $\mathcal{L}^2$ inner product and the symmetric positive semi-definiteness of $\mathcal{R}_t$ can directly be recognized in (20), (21) by following the line of reasoning as outlined in earlier proofs.

[2] *The variational derivative with respect to q_1 can also be computed. However, we omit its computation as the corresponding elements in the operator $\mathcal{J}_t$ and the matrix $\mathcal{R}_t$ are zero.*

3.2 Stokes-Dirac Structures

Infinite-dimensional port-Hamiltonian systems are described through a geometric structure known as Stokes-Dirac structure; refer to Le Gorrec et al. (2005), and Duindam et al. (2009). Such a structure has been associated to canonical skew-symmetric differential operator in Le Gorrec et al. (2005). Moreover, in Le Gorrec et al. (2005), the notion of Stokes-Dirac structures has been extended to skew-symmetric differential operator of any order. This geometric object helps to gain insight in describing the consistent boundary port-variables. Existing works have focused on the state-independent operators and have also considered an extended structure to account for dissipative effects (which may include differential terms), while mostly dealing with quadratic Hamiltonian functionals.

The properties of the Stokes-Dirac structure can be exploited in the development of energy-based boundary control laws for distributed port-Hamiltonian systems. We do not recall the formal definition of infinite-dimensional (Stokes-) Dirac structure and instead refer to Le Gorrec et al. (2005), and Duindam et al. (2009).

Next, we propose two variants of extended (Stokes-)Dirac structures. The pH representation for the two-phase model (See Section 3.3) will be defined with respect to the structure in Proposition 3. Secondly, the Stokes-Dirac structure in Proposition 4 will be used to define pH representation for the single-phase model.

We, first, show the (Stokes-)Dirac structure representation that will be useful in the scope of the Drift Flux Model without slip. Hereto, we introduce the following notations

$$\mathbf{f}_t = \begin{bmatrix} f_1 & f_2 & f_3 & f_4 & f_R & f_a^B & f_b^B \end{bmatrix}^T,$$
$$\mathbf{e}_t = \begin{bmatrix} e_1 & e_2 & e_3 & e_4 & e_R & e_a^B & e_b^B \end{bmatrix}^T,$$
$$\mathbf{f}_{tr} = \begin{bmatrix} f_1 & f_2 & f_3 & f_4 & f_R \end{bmatrix}^T,$$

and,

$$\mathbf{e}_{tr} = \begin{bmatrix} e_1 & e_2 & e_3 & e_4 & e_R \end{bmatrix}^T,$$

and define the space of flow variables in the following manner:

$$\mathcal{F}_t = \mathcal{L}^2(\Omega)^5 \times \mathcal{L}^2(\partial\Omega)^2, \tag{29}$$

where $\mathcal{L}^2(\Omega)$ is the space of square-integrable functions and

$$\mathcal{L}^2(\Omega)^p = \mathcal{L}^2(\Omega) \times \mathcal{L}^2(\Omega) \times ... \times \mathcal{L}^2(\Omega) \quad (p-times). \tag{30}$$

The space of effort variables can be analogously defined as follows:

$$\mathcal{E}_t = \mathcal{L}^2(\Omega)^5 \times \mathcal{L}^2(\partial\Omega)^2. \tag{31}$$

Functions in $H^1(\Omega)$ and $H_0^1(\Omega)$ are also considered in the sequel. Mathematically, $H^1(\Omega)$ denotes the Sobolev space of functions that also possess a weak derivative. $H_0^1(\Omega)$ denotes the functions in $H^1(\Omega)$ that have zero boundary values.

The non-degenerated bilinear product on $\mathcal{F}_t \times \mathcal{E}_t$ is defined in the following way:

$$< \mathbf{f}_t \mid \mathbf{e}_t >= \int_\Omega (f_1 e_1 + f_2 e_2 + f_3 e_3 + f_4 e_4 + f_R e_R)\mathrm{d}x + f_b^B e_b^B + f_a^B e_a^B. \tag{32}$$

Proposition 3. We consider $\mathcal{F}_t$ and $\mathcal{E}_t$ as given in (29) *and* (31)*. We assume that $\tilde{q}_1, \tilde{q}_2, \tilde{q}_3, \tilde{q}_4 \in H^1(\Omega)$ and that $\tilde{q}_2 + \tilde{q}_3 > 0$ on Ω. Then, the linear subset $\mathcal{D}_t \subset \mathcal{F}_t \times \mathcal{E}_t$ given by:*

$$\mathcal{D}_t = \Bigg\{ (\mathbf{f}_t, \mathbf{e}_t) \in \mathcal{F}_t \times \mathcal{E}_t \mid \begin{pmatrix} \tilde{q}_2 e_2 + \tilde{q}_3 e_3 \\ \tilde{q}_2 e_4 \\ e_4 \end{pmatrix} \in H^1(\Omega)^3,$$
$$\mathbf{f}_{tr} = \mathcal{J}_{ext}\mathbf{e}_{tr},$$
$$\begin{pmatrix} f_a^B \\ e_a^B \end{pmatrix} = \begin{pmatrix} -\tilde{q}_2 & -\tilde{q}_3 & -\tilde{q}_4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e_2 \\ e_3 \\ e_4 \end{pmatrix} |_a,$$
$$\begin{pmatrix} f_b^B \\ e_b^B \end{pmatrix} = \begin{pmatrix} \tilde{q}_2 & \tilde{q}_3 & \tilde{q}_4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e_2 \\ e_3 \\ e_4 \end{pmatrix} |_b \Bigg\}, \tag{33}$$

with

$$\mathcal{J}_{ext} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\partial_x(\tilde{q}_2 \cdot) & 0 \\ 0 & 0 & 0 & -\partial_x(\tilde{q}_3 \cdot) & 0 \\ 0 & -D(\tilde{q}_2 \cdot) \& D(\tilde{q}_3 \cdot) & & -\partial_x(\tilde{q}_4 \cdot) - \tilde{q}_4 \partial_x & -I \\ 0 & 0 & 0 & I & 0 \end{pmatrix}, \tag{34}$$

is a (Stokes-)Dirac structure with respect to the symmetric pairing given by:

$$\ll \begin{bmatrix} \mathbf{f}_t \\ \mathbf{e}_t \end{bmatrix}, \begin{bmatrix} \tilde{\mathbf{f}}_t \\ \tilde{\mathbf{e}}_t \end{bmatrix} \gg = < \mathbf{f}_t \mid \tilde{\mathbf{e}}_t > + < \tilde{\mathbf{f}}_t \mid \mathbf{e}_t >,$$
$$\begin{bmatrix} \mathbf{f}_t \\ \mathbf{e}_t \end{bmatrix}, \begin{bmatrix} \tilde{\mathbf{f}}_t \\ \tilde{\mathbf{e}}_t \end{bmatrix} \in \mathcal{F}_t \times \mathcal{E}_t, \tag{35}$$

where the pairing $< \cdot \mid \cdot >$ is given in (32)*. Furthermore, the notation $(\cdot) \mid_a$ (similarly for $(\cdot) \mid_b$) refers to the function value evaluated at the boundary $x = a$ (similarly for $x = b$). Moreover, $D(\tilde{q}_2 \cdot)\&D(\tilde{q}_3 \cdot)$ is the operator with domain all $e_2, e_3 \in \mathcal{L}^2(\Omega)$ such that $\tilde{q}_2 e_2 + \tilde{q}_3 e_3 \in H^1(\Omega)$ and the action of this operator is*

$$D(\tilde{q}_2 e_2)\&D(\tilde{q}_3 e_3) = \partial_x(\tilde{q}_2 e_2 + \tilde{q}_3 e_3) - e_2 \partial_x \tilde{q}_2 - e_3 \partial_x \tilde{q}_3. \tag{36}$$

The above action is an extension of the normal action of the operator, which for all $e_2, e_3 \in H^1$ will take the following form:

$$D(\tilde{q}_2 e_2)\&D(\tilde{q}_3 e_3) = \tilde{q}_2 \partial_x e_2 + \tilde{q}_3 \partial_x e_3.$$

Proof. The proof consists of two parts. The first part comprises of the proof $\mathcal{D}_t \subset \mathcal{D}_t^\perp$. And, the second part comprises of the proof $\mathcal{D}_t^\perp \subset \mathcal{D}_t$. For the first part of the proof, we begin with considering two pairs of flow and effort variables belonging to the Dirac structure i.e., $(\mathbf{f}_t, \mathbf{e}_t) \in \mathcal{D}_t$ and $(\tilde{\mathbf{f}}_t, \tilde{\mathbf{e}}_t) \in \mathcal{D}_t$. Using the earlier introduced notations, the pairing (35) gives:

$$\int_\Omega (f_1 \tilde{e}_1 + f_2 \tilde{e}_2 + f_3 \tilde{e}_3 + f_4 \tilde{e}_4 + f_R \tilde{e}_R)\mathrm{d}x + \int_\Omega (\tilde{f}_1 e_1 + \tilde{f}_2 e_2 + \tilde{f}_3 e_3 + \tilde{f}_4 e_4 + \tilde{f}_R e_R)\mathrm{d}x + f_a^B \tilde{e}_a^B + f_b^B \tilde{e}_b^B + \tilde{f}_a^B e_a^B + \tilde{f}_b^B e_b^B. \tag{37}$$

Using (33), (34) and (36) in (37), we obtain:

$$\int_\Omega \Big((-\partial_x\tilde q_2 e_4)\tilde e_2 + (-\partial_x\tilde q_3 e_4)\tilde e_3 + \Big(-\partial_x(\tilde q_2 e_2 + \tilde q_3 e_3) + e_2\partial_x\tilde q_2 + e_3\partial_x\tilde q_3\Big)\tilde e_4 - \partial_x(\tilde q_4 e_4)\tilde e_4 - \tilde q_4(\partial_x e_4)\tilde e_4 - e_R\tilde e_4 + e_4\tilde e_R\Big)\mathrm{d}x + \int_\Omega \Big((-\partial_x\tilde q_2\tilde e_4)e_2 + (-\partial_x\tilde q_3\tilde e_4)e_3 + \Big(-\partial_x(\tilde q_2\tilde e_2 + \tilde q_3\tilde e_3) + \tilde e_2\partial_x\tilde q_2 + \tilde e_3\partial_x\tilde q_3\Big)e_4 - \partial_x(\tilde q_4\tilde e_4)e_4 - \tilde q_4(\partial_x\tilde e_4)e_4 - \tilde e_R e_4 + \tilde e_4 e_R\Big)\mathrm{d}x + f_a^B\tilde e_a^B + f_b^B\tilde e_b^B + \tilde f_a^B e_a^B + \tilde f_b^B e_b^B. \quad (38)$$

Performing integration by parts on few terms in the above equation, (38) can be easily shown to be zero and hence, $\mathcal{D}_t \subset \mathcal{D}_t^\perp$. This concludes the first part of the proof, which carries the symbolism of power-conserving structure.

We now prove the converse part: $\mathcal{D}_t^\perp \subset \mathcal{D}_t$. The proof consists of several small but repeated steps. Hence, we summarize the key steps that are followed in each step while proving the converse part. We take $(\tilde{\mathbf{f}}_t, \tilde{\mathbf{e}}_t) \in \mathcal{D}_t^\perp$ i.e., $(\tilde{\mathbf{f}}_t, \tilde{\mathbf{e}}_t) \in \mathcal{F}_t \times \mathcal{E}_t$ such that $\ll \begin{bmatrix}\mathbf{f}_t\\ \mathbf{e}_t\end{bmatrix}, \begin{bmatrix}\tilde{\mathbf{f}}_t\\ \tilde{\mathbf{e}}_t\end{bmatrix} \gg = 0$ for all $(\mathbf{f}_t, \mathbf{e}_t) \in \mathcal{D}_t$. Furthermore, we make a certain choice on the effort variables (which can be freely chosen as per the definition of the (Stokes-)Dirac structure) in each step. We also exploit the fundamental lemma of calculus of variations to obtain several identities. Each step (and the associated choices) is described below:

Step 1: Let $(\mathbf{f}_t, \mathbf{e}_t) \in \mathcal{D}_t$ with $e_2, e_3, e_4, e_R = 0$ and $e_1(a) = e_1(b) = 0$. Following the procedure leads to:

$$\int_\Omega \tilde f_1 e_1 \mathrm{d}x = 0 \quad \forall e_1 \in \mathcal{L}^2(\Omega). \quad (39)$$

Thus $\tilde f_1 = 0$.

Step 2: We now consider $(\mathbf{f}_t, \mathbf{e}_t) \in \mathcal{D}_t$ with $e_1, e_3, e_4, e_R = 0$ and $e_2(a) = e_2(b) = 0$. Plugging the flow-effort relations (33) in (37) under the aforementioned considerations gives:

$$\int_\Omega \Big((-\tilde q_2\partial_x e_2)\tilde e_4 + \tilde f_2 e_2\Big)\mathrm{d}x + b.c. = 0 \quad \forall e_2 \in H^1(\Omega). \quad (40)$$

Using the fundamental lemma of calculus of variations gives

$$\tilde q_2\tilde e_4 \in H^1(\Omega) \quad \text{and} \quad \tilde f_2 = -\partial_x(\tilde q_2\tilde e_4). \quad (41)$$

Considering $(\mathbf{f}_t, \mathbf{e}_t) \in \mathcal{D}_t$ with e_1, e_2, e_4 and $e_R = 0$ along with $e_3(a) = e_3(b) = 0$ gives by a similar argument that

$$\tilde q_3\tilde e_4 \in H^1(\Omega) \quad \text{and} \quad \tilde f_3 = -\partial_x(\tilde q_3\tilde e_4). \quad (42)$$

Now using $\tilde q_2\tilde e_4 \in H^1(\Omega)$ and $\tilde q_3\tilde e_4 \in H^1(\Omega)$, we have that $(\tilde q_2 + \tilde q_3)\tilde e_4 \in H^1(\Omega)$. Furthermore, using $\tilde q_2, \tilde q_3 \in H^1(\Omega)$ along with $\tilde q_2 + \tilde q_3 > 0$ on Ω, we have that $\tilde e_4 \in H^1(\Omega)$.

Step 3: Now choosing $(\mathbf{f}_t, \mathbf{e}_t) \in \mathcal{D}_t$ with $e_1, e_2, e_3, e_R = 0$ and $e_4 \in H_0^1(\Omega)$ gives

$$\int_\Omega \Big(-\partial_x(\tilde q_2 e_4)\tilde e_2 - \partial_x(\tilde q_3 e_4)\tilde e_3 - \partial_x(\tilde q_4 e_4)\tilde e_4 - (\tilde q_4\partial_x e_4)\tilde e_4 + e_4\tilde e_R + \tilde f_4 e_4\Big)\mathrm{d}x = 0. \quad (43)$$

We rewrite (43) as

$$\int_\Omega \Big(-e_4\tilde e_2\partial_x\tilde q_2 - e_4\tilde e_3\partial_x\tilde q_3 - (\tilde q_2\tilde e_2 + \tilde q_3\tilde e_3)\partial_x e_4 - \partial_x(\tilde q_4 e_4)\tilde e_4 - (\tilde q_4\partial_x e_4)\tilde e_4 + e_4\tilde e_R + \tilde f_4 e_4\Big)\mathrm{d}x = 0 \quad \forall e_4 \in H_0^1(\Omega). \quad (44)$$

As a result of the fundamental lemma of calculus of variations, we obtain the following identity:

$$\tilde f_4 = -\partial_x(\tilde q_2\tilde e_2 + \tilde q_3\tilde e_3) + \tilde e_2\partial_x\tilde q_2 + \tilde e_3\partial_x\tilde q_3 - \partial_x(\tilde q_4\tilde e_4) - \tilde q_4\partial_x\tilde e_4 - \tilde e_R. \quad (45)$$

Step 4: Let us consider $(\mathbf{f}_t, \mathbf{e}_t) \in \mathcal{D}_t$ with $e_1, e_2, e_3, e_4 = 0$. The identity that follows under these considerations is:

$$-e_R\tilde e_4 + \tilde f_R e_R = 0 \implies \tilde f_R = \tilde e_4. \quad (46)$$

Step 5: Let $(\mathbf{f}_t, \mathbf{e}_t) \in \mathcal{D}_t$ with $e_1, e_3, e_4, e_R = 0$ and $e_2(a) = 0$ and $e_2(b) \neq 0$. Performing similar steps now gives:

$$-\tilde q_2 e_2\tilde e_4 \mid_b + \tilde e_b^B(\tilde q_2 e_2) \mid_b = 0. \quad (47)$$

The identity that follows is:

$$\tilde e_b^B = \tilde e_4 \mid_b. \quad (48)$$

Step 6: We now let $(\mathbf{f}_t, \mathbf{e}_t) \in \mathcal{D}_t$ with $e_1, e_3, e_4, e_R = 0$ and $e_2(b) = 0$ and $e_2(a) \neq 0$. We follow the procedure similar to Step 5 and obtain the following identity:

$$\tilde e_a^B = \tilde e_4 \mid_a. \quad (49)$$

Step 7: Consider $(\mathbf{f}_t, \mathbf{e}_t) \in \mathcal{D}_t$ with $e_1, e_2, e_3, e_R = 0$ and $e_4(a) = 0$ and $e_4(b) \neq 0$. Following the outlined procedure and using $\tilde e_b^B$ from (48), we have:

$$\tilde f_b^B e_4 \mid_b + \tilde e_4 \mid_b (\tilde q_4 e_4) \mid_b - \tilde q_2 e_4\tilde e_2 \mid_b - \tilde q_3 e_4\tilde e_3 \mid_b - \tilde q_4 e_4\tilde e_4 \mid_b - \tilde q_4 e_4\tilde e_4 \mid_b = 0. \quad (50)$$

This results in the following identity:

$$\tilde f_b^B = \Big(\tilde q_2\tilde e_2 + \tilde q_3\tilde e_3\Big) \mid_b + \Big(\tilde q_4\tilde e_4\Big) \mid_b. \quad (51)$$

Step 8: We now consider $(\mathbf{f}_t, \mathbf{e}_t) \in \mathcal{D}_t$ with $e_1, e_2, e_3, e_R = 0$ and $e_4(b) = 0$ and $e_4(a) \neq 0$. Under these considerations, we follow the procedure similar to Step 7 and also use $\tilde e_a^B$ from (49) to obtain the following identity:

$$\tilde f_a^B = -\Big(q_2\tilde e_2 + q_3\tilde e_3\Big) \mid_a - \Big(q_4\tilde e_4\Big) \mid_a. \quad (52)$$

Thus, we have shown that $\mathcal{D}_t^\perp \subset \mathcal{D}_t$ and, hence, $\mathcal{D}_t$ is a (Stokes-) Dirac structure. This completes the proof.

We now discuss the (Stokes-)Dirac structure representation that will be useful in the scope of the single-phase model. We introduce $\mathbf{f}_s = \begin{bmatrix} f_1 & f_2 & f_3 & f_R & f_a^B & f_b^B\end{bmatrix}^T$ and $\mathbf{e}_s = \begin{bmatrix} e_1 & e_2 & e_3 & e_R & e_a^B & e_b^B\end{bmatrix}^T$. Using these notations, we define the space of flow variables as follows:

$$\mathcal{F}_s = \mathcal{L}^2(\Omega)^4 \times \mathcal{L}^2(\partial\Omega)^2. \quad (53)$$

Similarly, the space of effort variables is defined as follows:

$$\mathcal{E}_s = \mathcal{L}^2(\Omega)^4 \times \mathcal{L}^2(\partial\Omega)^2. \quad (54)$$

The non-degenerated bilinear product on $\mathcal{F}_s \times \mathcal{E}_s$ is defined as:

$$< \mathbf{f}_s \mid \mathbf{e}_s >= \int_\Omega (f_1 e_1 + f_2 e_2 + f_3 e_3 + f_R e_R)\mathrm{d}x + f_b^B e_b^B + f_a^B e_a^B. \quad (55)$$

Proposition 4. *Consider* $\mathcal{F}_s$ *and* $\mathcal{E}_s$ *as given in* (53) *and* (54). *Additionally, we consider* $q_1, q_2, q_3 \in H^1(\Omega)$ *and* q_2 *(or* $A\rho$*) is invertible. The linear subset* $\mathcal{D}_s \subset \mathcal{F}_s \times \mathcal{E}_s$ *defined as:*

$$\mathcal{D}_s = \Bigg\{ (\mathbf{f}_s, \mathbf{e}_s) \in \mathcal{F}_s \times \mathcal{E}_s \mid \begin{pmatrix} q_2 e_2 \\ e_3 \end{pmatrix} \in H^1(\Omega)^2,$$
$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_R \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\partial_x(q_2 \cdot) & 0 \\ 0 & -D(q_2\cdot) & -\partial_x(q_3\cdot) - q_3\partial_x & -I \\ 0 & 0 & I & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_R \end{pmatrix},$$
$$\begin{pmatrix} f_a^B \\ e_a^B \end{pmatrix} = \begin{pmatrix} -q_2 & -q_3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e_2 \\ e_3 \end{pmatrix} |_a,$$
$$\begin{pmatrix} f_b^B \\ e_b^B \end{pmatrix} = \begin{pmatrix} q_2 & q_3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e_2 \\ e_3 \end{pmatrix} |_b \Bigg\}, \quad (56)$$

is a (Stokes-)Dirac structure with respect to the symmetric pairing given by:

$$\ll \begin{bmatrix} \mathbf{f}_s \\ \mathbf{e}_s \end{bmatrix}, \begin{bmatrix} \tilde{\mathbf{f}}_s \\ \tilde{\mathbf{e}}_s \end{bmatrix} \gg = < \mathbf{f}_s \mid \tilde{\mathbf{e}}_s > + < \tilde{\mathbf{f}}_s \mid \mathbf{e}_s >,$$
$$\begin{bmatrix} \mathbf{f}_s \\ \mathbf{e}_s \end{bmatrix}, \begin{bmatrix} \tilde{\mathbf{f}}_s \\ \tilde{\mathbf{e}}_s \end{bmatrix} \in \mathcal{F}_s \times \mathcal{E}_s, \quad (57)$$

where the pairing $< \cdot \mid \cdot >$ *is given in* (55). *Moreover,* $D(q_2\cdot)$ *is the operator with domain all* $e_2 \in \mathcal{L}^2(\Omega)$ *such that* $q_2 e_2 \in H^1(\Omega)$ *and the extended action of the operator is*

$$D(q_2 e_2) = \partial_x(q_2 e_2) - e_2 \partial_x q_2.$$

Remark 3.4. We do not prove the Proposition 4. The proof of the corresponding (extended) Stokes-Dirac structure can be easily demonstrated by following the similar lines of reasoning as in the proof of the Proposition 3.

Remark 3.5. We have associated a particular choice of boundary-port variables with a (Stokes-) Dirac structure. In principle, it would be ideal to derive an admissible set of boundary conditions in a parametrized way similar to Le Gorrec et al. (2005), where a parametrization was derived for a canonical skew-symmetric differential operator. However, the structures derived in our current paper are non-canonical and eventually have a non-invertible matrix (Q as per the notation in Le Gorrec et al. (2005)), that hinders the elegant parametrization for the class of systems under discussion. An elegant parametrization of boundary port-variables will be considered in future works for the class of non-linear pH systems with non-quadratic Hamiltonian functionals.

3.3 Port-Hamiltonian Representation

The preceding discussion showed that interconnection relations associated with the conservation laws define a so-called (Stokes-)Dirac structure. The interconnection structure, given by a (Stokes-)Dirac structure, together with the Hamiltonian functional representing the total energy of the system constitute a pH representation. Following Duindam et al. (2009), we define the resulting pH system as follows.

Definition 5. A port-Hamiltonian system for the Drift Flux Model (without slip) with state-variables

$$\tilde{q}(t) = [\tilde{q}_1(t),\ \tilde{q}_2(t),\ \tilde{q}_3(t),\ \tilde{q}_4(t)]^T, \quad (58)$$

and port-variables

$$\left[f_a^B(t),\ f_b^B(t),\ e_a^B(t),\ e_b^B(t)\right]^T \in \mathcal{L}^2(\partial\Omega)^4, \quad (59)$$

generated by the Hamiltonian functional (with smooth integrand) (15), and, with respect to the (Stokes-) Dirac structure (33) is defined by:

$$\left(\begin{bmatrix} \partial_t \tilde{q}_1(t) \\ \partial_t \tilde{q}_2(t) \\ \partial_t \tilde{q}_3(t) \\ \partial_t \tilde{q}_4(t) \\ f_a^B(t) \\ f_b^B(t) \\ f_R \end{bmatrix}, \begin{bmatrix} \delta_{\tilde{q}_1}\mathcal{H}_t \\ \delta_{\tilde{q}_2}\mathcal{H}_t \\ \delta_{\tilde{q}_3}\mathcal{H}_t \\ \delta_{\tilde{q}_4}\mathcal{H}_t \\ e_a^B(t) \\ e_b^B(t) \\ \dfrac{32\mu_m \tilde{q}_1}{d^2} f_R \end{bmatrix} \right) \in \mathcal{D}_t. \quad (60)$$

Remark 3.6. Similarly, we can also define a pH system for the single-phase model with respect to the (Stokes-)Dirac structure (56)*. We omit it for the sake of brevity.*

4. MODELING TEMPORAL VARIATIONS

We now briefly consider temporal variations in the geometrical cross-section i.e., $\partial_t A \neq 0$. We consider that the evolution of the area is described as:

$$\partial_t A = r_1(t, z), \quad (61)$$

where r_1 is a function, which say is known a-priori or can be determined via some control law.

Allowing for temporal variations in area can be viewed as the structure (with state-variable z), which contains additional terms (relative to the structure with only spatial variations) that can be perceived as state and time-dependent control inputs. See the following theorem.

Theorem 6. Consider the system Σ_t [3] *governed by the combination of* (2), (3) *and* (61)*. Then, it can be formulated in the dissipative Hamiltonian representation of the following form:*

$$\partial_t z = (\mathcal{J}(z) - \mathcal{R}(z))\delta_z \mathcal{H}(z) - r(t, z). \quad (62)$$

Remark 4.1. In the scope of two-phase models, $z = \tilde{q}$, $\mathcal{J} = \mathcal{J}_t$, $\mathcal{R} = \mathcal{R}_t$ *and* $\mathcal{H} = \mathcal{H}_t$*. Equivalently, the structure holds in the scope of single-phase models with corresponding state-variables, interconnection (formal skew-adjoint) operator, resistive matrix and the Hamiltonian functional.*

The above structure can be viewed as a special case of the representation in Mehrmann and Morandin (2019). If we ignore the boundary ports in the pHDAE definition of Mehrmann and Morandin (2019) and use slightly different notations for the sake of consistency in this paper, then we obtain:

$$\mathcal{E}\dot{z} = (\mathcal{J}(z) - \mathcal{R}(z))s - r(t, z), \quad (63)$$

where $r(t, z)$ is of the form: $[r_1(t, z) \quad 0 \quad 0 \quad 0]^T$. The reasoning behind the choice of this form is apparent from the comment in the footnote.

We consider the mapping $\mathcal{E} = I$, $\partial_z \mathcal{H} = s$ and $\partial_t \mathcal{H} = s^T r$ and follow Mehrmann and Morandin (2019) to obtain the dissipation inequality.

$$\begin{aligned} \frac{d\mathcal{H}}{dt} &= (\partial_z \mathcal{H})^T \dot{z} + \partial_t \mathcal{H} \\ &= s^T \Big((\mathcal{J} - \mathcal{R})s - r \Big) + s^T r \\ &= -s^T \mathcal{R} s \leq 0. \end{aligned} \quad (64)$$

The structural representation as in (63) has already been shown to be a Dirac structure in Mehrmann and Morandin

[3] *The first equation of the composed system* Σ_t *is* (61)*. The rest of the equations in the composed system are the mass and the momentum conservation laws.*

(2019). Hence, we refer the reader to Mehrmann and Morandin (2019) for further details.

Remark 4.2. Structure (62) *or* (63) *has been presented in rather general sense. It is worth mentioning that a desirable structure is realizable for the models governing the single-phase and two-phase fluid flow across variable geometrical cross-section by using specific choice of state-variables and the associated interconnection operator and the dissipation matrix.*

5. CONCLUSION

The main results of this paper are the dissipative Hamiltonian realizations and definition of (extended) state-dependent (Stokes-)Dirac structure consequently leading to port-Hamiltonian representations for both single-phase and two-phase models governing fluid flow across spatially and temporally varying cross-section. Future works will deal with developing structure preserving numerical schemes for the obtained representations.

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