Appendix to "Port-Hamiltonian Formulation of Two-phase Flow Models", H. Bansal, P. Schulze, M.H. Abbasi, H. Zwart, L. Iapichino, W.H.A. Schilders and N. van de Wouw

Appendix I: Check for the satisfaction of Jacobi Identity

Operator arising in the Hamiltonian formulation of the TFM:
The Poisson bracket $\{F,G\}$, under the consideration of vanishing variational derivatives at the boundary, is defined as follows:

$$\begin{aligned}\{F,G\} = -\int_\Omega \Big[q_1\Big(\frac{\delta F}{\delta q_3}\partial_x\frac{\delta G}{\delta q_1} - \frac{\delta G}{\delta q_3}\partial_x\frac{\delta F}{\delta q_1}\Big) + q_3\Big(\frac{\delta F}{\delta q_3}\partial_x\frac{\delta G}{\delta q_3} - \frac{\delta G}{\delta q_3}\partial_x\frac{\delta F}{\delta q_3}\Big) + \\ q_2\Big(\frac{\delta F}{\delta q_4}\partial_x\frac{\delta G}{\delta q_2} - \frac{\delta G}{\delta q_4}\partial_x\frac{\delta F}{\delta q_2}\Big) + q_4\Big(\frac{\delta F}{\delta q_4}\partial_x\frac{\delta G}{\delta q_4} - \frac{\delta G}{\delta q_4}\partial_x\frac{\delta F}{\delta q_4}\Big)\Big]\mathrm{d}x. \quad (1)\end{aligned}$$

Consider the term: $\int_\Omega q_1 \frac{\delta F}{\delta q_3}\partial_x\frac{\delta G}{\delta q_1}\mathrm{d}x$. Its variational derivative with respect to q_1 is given by:

$$\frac{\delta}{\delta q_1}\Big(\int_\Omega q_1\frac{\delta F}{\delta q_3}\partial_x\frac{\delta G}{\delta q_1}\mathrm{d}x\Big) = \frac{\delta F}{\delta q_3}\partial_x\frac{\delta G}{\delta q_1} + q_1\frac{\delta^2 F}{\delta q_1\delta q_3}\partial_x\frac{\delta G}{\delta q_1} - \frac{\delta^2 G}{\delta q_1^2}\partial_x\Big(q_1\frac{\delta F}{\delta q_3}\Big), \quad (2)$$

where we have used the product rule and invoked the notion of the integration by parts.
Using the idea illustrated in (2), the variational derivative of $\{F,G\}$ with respect to the state variables q_1 is given by:

$$\begin{aligned}\frac{\delta\{F,G\}}{\delta q_1} = &-\frac{\delta F}{\delta q_3}\partial_x\frac{\delta G}{\delta q_1} + \frac{\delta G}{\delta q_3}\partial_x\frac{\delta F}{\delta q_1} - q_1\frac{\delta^2 F}{\delta q_1\delta q_3}\partial_x\frac{\delta G}{\delta q_1} + \frac{\delta^2 G}{\delta q_1^2}\partial_x\Big(q_1\frac{\delta F}{\delta q_3}\Big) + q_1\frac{\delta^2 G}{\delta q_1\delta q_3}\partial_x\frac{\delta F}{\delta q_1} - \\ &\frac{\delta^2 F}{\delta q_1^2}\partial_x\Big(q_1\frac{\delta G}{\delta q_3}\Big) - q_3\frac{\delta^2 F}{\delta q_1\delta q_3}\partial_x\frac{\delta G}{\delta q_3} + \frac{\delta^2 G}{\delta q_1\delta q_3}\partial_x\Big(q_3\frac{\delta F}{\delta q_3}\Big) + q_3\frac{\delta^2 G}{\delta q_1\delta q_3}\partial_x\frac{\delta F}{\delta q_3} - \frac{\delta^2 F}{\delta q_1\delta q_3}\partial_x\Big(q_3\frac{\delta G}{\delta q_3}\Big) - \\ &q_2\frac{\delta^2 F}{\delta q_1\delta q_4}\partial_x\frac{\delta G}{\delta q_2} + q_2\frac{\delta^2 G}{\delta q_1\delta q_4}\partial_x\frac{\delta F}{\delta q_2} + \frac{\delta^2 G}{\delta q_1\delta q_2}\partial_x\Big(q_2\frac{\delta F}{\delta q_4}\Big) - \frac{\delta^2 F}{\delta q_1\delta q_2}\partial_x\Big(q_2\frac{\delta G}{\delta q_4}\Big) - q_4\frac{\delta^2 F}{\delta q_1\delta q_4}\partial_x\frac{\delta G}{\delta q_4} \\ &+ q_4\frac{\delta^2 G}{\delta q_1\delta q_4}\partial_x\frac{\delta F}{\delta q_4} + \frac{\delta^2 G}{\delta q_1\delta q_4}\partial_x\Big(q_4\frac{\delta F}{\delta q_4}\Big) - \frac{\delta^2 F}{\delta q_1\delta q_4}\partial_x\Big(q_4\frac{\delta G}{\delta q_4}\Big). \quad (3)\end{aligned}$$

Analogously, $\frac{\delta\{F,G\}}{\delta q_1}$, $\frac{\delta\{F,G\}}{\delta q_2}$ and $\frac{\delta\{F,G\}}{\delta q_4}$ can also be computed and, subsequently, used to evaluate $\{\{F,G\},H\}, \{\{G,H\},F\}$ and $\{\{H,F\},G\}$.

In order to simplify an extensively lengthy computation, we consider the following bracket:

$$\{F,G\}_{q_1,q_3} = -\int_\Omega \Big[q_1\Big(\frac{\delta F}{\delta q_3}\partial_x\frac{\delta G}{\delta q_1} - \frac{\delta G}{\delta q_3}\partial_x\frac{\delta F}{\delta q_1}\Big) + q_3\Big(\frac{\delta F}{\delta q_3}\partial_x\frac{\delta G}{\delta q_3} - \frac{\delta G}{\delta q_3}\partial_x\frac{\delta F}{\delta q_3}\Big)\Big]\mathrm{d}x, \tag{4}$$

in the pursuit of checking whether the Jacobi Identity holds. The bracket $\{F,G\}_{q_1,q_3}$ is of a similar type as that of the one depicted in (1). It should also be remarked that one would mathematically obtain the aforementioned bracket in the scope of the single-phase flow models formulated in a port-Hamiltonian representation using the conservative state variables.
In order to prove that the Jacobi Identity holds, we need to show that:

$$\{\{F,G\},H\}_{q_1,q_3} + \{\{G,H\},F\}_{q_1,q_3} + +\{\{H,F\},G\}_{q_1,q_3} = 0. \tag{5}$$

To this end, we need to compute the variational derivative of $\{F,G\}_{q_1,q_3}$ with respect to the state variables q_1 and q_3. These are given by:

$$\begin{aligned}\frac{\delta\{F,G\}}{\delta q_1} &= -\frac{\delta F}{\delta q_3}\partial_x\frac{\delta G}{\delta q_1} + \frac{\delta G}{\delta q_3}\partial_x\frac{\delta F}{\delta q_1} - q_1\frac{\delta^2 F}{\delta q_1\delta q_3}\partial_x\frac{\delta G}{\delta q_1} + \frac{\delta^2 G}{\delta q_1^2}\partial_x\Big(q_1\frac{\delta F}{\delta q_3}\Big) + q_1\frac{\delta^2 G}{\delta q_1\delta q_3}\partial_x\frac{\delta F}{\delta q_1} - \\ &\frac{\delta^2 F}{\delta q_1^2}\partial_x\Big(q_1\frac{\delta G}{\delta q_3}\Big) - q_3\frac{\delta^2 F}{\delta q_1\delta q_3}\partial_x\frac{\delta G}{\delta q_3} + \frac{\delta^2 G}{\delta q_1\delta q_3}\partial_x\Big(q_3\frac{\delta F}{\delta q_3}\Big) + q_3\frac{\delta^2 G}{\delta q_1\delta q_3}\partial_x\frac{\delta F}{\delta q_3} - \frac{\delta^2 F}{\delta q_1\delta q_3}\partial_x\Big(q_3\frac{\delta G}{\delta q_3}\Big),\end{aligned} \tag{6}$$

$$\begin{aligned}\frac{\delta\{F,G\}}{\delta q_3} &= -\frac{\delta F}{\delta q_3}\partial_x\frac{\delta G}{\delta q_3} + \frac{\delta G}{\delta q_3}\partial_x\frac{\delta F}{\delta q_3} - q_1\frac{\delta^2 F}{\delta q_3^2}\partial_x\frac{\delta G}{\delta q_1} + \frac{\delta^2 G}{\delta q_3\delta q_1}\partial_x\Big(q_1\frac{\delta F}{\delta q_3}\Big) + q_1\frac{\delta^2 G}{\delta q_3^2}\partial_x\frac{\delta F}{\delta q_1} - \\ &\frac{\delta^2 F}{\delta q_3\delta q_1}\partial_x\Big(q_1\frac{\delta G}{\delta q_3}\Big) - q_3\frac{\delta^2 F}{\delta q_3^2}\partial_x\frac{\delta G}{\delta q_3} + \frac{\delta^2 G}{\delta q_3^2}\partial_x\Big(q_3\frac{\delta F}{\delta q_3}\Big) + q_3\frac{\delta^2 G}{\delta q_3^2}\partial_x\frac{\delta F}{\delta q_3} - \frac{\delta^2 F}{\delta q_3^2}\partial_x\Big(q_3\frac{\delta G}{\delta q_3}\Big).\end{aligned} \tag{7}$$

As per the definition of the Poisson bracket, the term $\{\{F,G\},H\}_{q_1,q_3}$ is given by:

$$\{\{F,G\},H\}_{q_1,q_3} = -\int_\Omega \Big[q_1\Big(\frac{\delta\{F,G\}}{\delta q_3}\partial_x\frac{\delta H}{\delta q_1} - \frac{\delta H}{\delta q_3}\partial_x\frac{\delta\{F,G\}}{\delta q_1}\Big) + q_3\Big(\frac{\delta\{F,G\}}{\delta q_3}\partial_x\frac{\delta H}{\delta q_3} - \frac{\delta H}{\delta q_3}\partial_x\frac{\delta\{F,G\}}{\delta q_3}\Big)\Big]\mathrm{d}x. \tag{8}$$

Using (6) and (7), (8) takes the following form:

$$
\begin{aligned}
\{\{F,G\},H\}_{q_1,q_3} = -\int_\Omega \Bigg[& -\underbrace{q_1 \frac{\delta F}{\delta q_3}\partial_x \frac{\delta G}{\delta q_3}\partial_x\frac{\delta H}{\delta q_1}}_{(1)} + \underbrace{q_1 \frac{\delta G}{\delta q_3}\partial_x\frac{\delta F}{\delta q_3}\partial_x\frac{\delta H}{\delta q_1}}_{(1)} - \underbrace{q_1^2\frac{\delta^2 F}{\delta q_3^2}\partial_x\frac{\delta G}{\delta q_1}\partial_x\frac{\delta H}{\delta q_1}}_{(2b)} + \\
& \underbrace{q_1\frac{\delta^2 G}{\delta q_3\delta q_1}\partial_x\left(q_1\frac{\delta F}{\delta q_3}\right)\partial_x\frac{\delta H}{\delta q_1}}_{(3b)} + \underbrace{q_1^2\frac{\delta^2G}{\delta q_3^2}\partial_x\frac{\delta F}{\delta q_1}\partial_x\frac{\delta H}{\delta q_1}}_{(2d)} - \underbrace{q_1\frac{\delta^2F}{\delta q_3\delta q_1}\partial_x\left(q_1\frac{\delta G}{\delta q_3}\right)\partial_x\frac{\delta H}{\delta q_1}}_{(3a)} - \underbrace{q_1q_3\frac{\delta^2F}{\delta q_3^2}\partial_x\frac{\delta G}{\delta q_3}\partial_x\frac{\delta H}{\delta q_1}}_{(2b)} + \\
& \underbrace{q_1\frac{\delta^2G}{\delta q_3^2}\partial_x\left(q_3\frac{\delta F}{\delta q_3}\right)\partial_x\frac{\delta H}{\delta q_1}}_{(2d)} + \underbrace{q_1q_3\frac{\delta^2G}{\delta q_3^2}\partial_x\frac{\delta F}{\delta q_3}\partial_x\frac{\delta H}{\delta q_1}}_{(2d)} - \underbrace{q_1\frac{\delta^2F}{\delta q_3^2}\partial_x\left(q_3\frac{\delta G}{\delta q_3}\right)\partial_x\frac{\delta H}{\delta q_1}}_{(2b)} - \underbrace{\frac{\delta F}{\delta q_3}\partial_x\frac{\delta G}{\delta q_1}\partial_x\left(q_1\frac{\delta H}{\delta q_3}\right)}_{(1)} + \\
& \underbrace{\frac{\delta G}{\delta q_3}\partial_x\frac{\delta F}{\delta q_1}\partial_x\left(q_1\frac{\delta H}{\delta q_3}\right)}_{(1)} - \underbrace{q_1\frac{\delta^2F}{\delta q_1\delta q_3}\partial_x\frac{\delta G}{\delta q_1}\partial_x\left(q_1\frac{\delta H}{\delta q_3}\right)}_{(3a)} + \underbrace{\frac{\delta^2G}{\delta q_1^2}\partial_x\left(q_1\frac{\delta F}{\delta q_3}\right)\partial_x\left(q_1\frac{\delta H}{\delta q_3}\right)}_{(2c)} + \underbrace{q_1\frac{\delta^2G}{\delta q_1\delta q_3}\partial_x\frac{\delta F}{\delta q_1}\partial_x\left(q_1\frac{\delta H}{\delta q_3}\right)}_{(3b)} - \\
& \underbrace{\frac{\delta^2F}{\delta q_1^2}\partial_x\left(q_1\frac{\delta G}{\delta q_3}\right)\partial_x\left(q_1\frac{\delta H}{\delta q_3}\right)}_{(2a)} - \underbrace{q_3\frac{\delta^2F}{\delta q_1\delta q_3}\partial_x\frac{\delta G}{\delta q_3}\partial_x\left(q_1\frac{\delta H}{\delta q_3}\right)}_{(3a)} + \underbrace{\frac{\delta^2G}{\delta q_1\delta q_3}\partial_x\left(q_3\frac{\delta F}{\delta q_3}\right)\partial_x\left(q_1\frac{\delta H}{\delta q_3}\right)}_{(3b)} + \\
& \underbrace{q_3\frac{\delta^2G}{\delta q_1\delta q_3}\partial_x\frac{\delta F}{\delta q_3}\partial_x\left(q_1\frac{\delta H}{\delta q_3}\right)}_{(3b)} - \underbrace{\frac{\delta^2F}{\delta q_1\delta q_3}\partial_x\left(q_3\frac{\delta G}{\delta q_3}\right)\partial_x\left(q_1\frac{\delta H}{\delta q_3}\right)}_{(3a)} - \underbrace{q_3\frac{\delta F}{\delta q_3}\partial_x\frac{\delta G}{\delta q_3}\partial_x\frac{\delta H}{\delta q_3}}_{(1)} + \underbrace{q_3\frac{\delta G}{\delta q_3}\partial_x\frac{\delta F}{\delta q_3}\partial_x\frac{\delta H}{\delta q_3}}_{(1)} - \\
& \underbrace{q_1q_3\frac{\delta^2F}{\delta q_3^2}\partial_x\frac{\delta G}{\delta q_1}\partial_x\frac{\delta H}{\delta q_3}}_{(2b)} + \underbrace{q_3\frac{\delta^2G}{\delta q_3\delta q_1}\partial_x\left(q_1\frac{\delta F}{\delta q_3}\right)\partial_x\frac{\delta H}{\delta q_3}}_{(3b)} + \underbrace{q_1q_3\frac{\delta^2G}{\delta q_3^2}\partial_x\frac{\delta F}{\delta q_1}\partial_x\frac{\delta H}{\delta q_3}}_{(2d)} - \underbrace{q_3\frac{\delta^2F}{\delta q_3\delta q_1}\partial_x\left(q_1\frac{\delta G}{\delta q_3}\right)\partial_x\frac{\delta H}{\delta q_3}}_{(3a)} - \\
& \underbrace{q_3^2\frac{\delta^2F}{\delta q_3^2}\partial_x\frac{\delta G}{\delta q_3}\partial_x\frac{\delta H}{\delta q_3}}_{(2b)} + \underbrace{q_3\frac{\delta^2G}{\delta q_3^2}\partial_x\left(q_3\frac{\delta F}{\delta q_3}\right)\partial_x\frac{\delta H}{\delta q_3}}_{(2d)} + \underbrace{q_3^2\frac{\delta^2G}{\delta q_3^2}\partial_x\frac{\delta F}{\delta q_3}\partial_x\frac{\delta H}{\delta q_3}}_{(2d)} - \underbrace{q_3\frac{\delta^2F}{\delta q_3^2}\partial_x\left(q_3\frac{\delta G}{\delta q_3}\right)\partial_x\frac{\delta H}{\delta q_3}}_{(2b)} - \\
& \underbrace{\frac{\delta F}{\delta q_3}\partial_x\frac{\delta G}{\delta q_3}\partial_x\left(q_3\frac{\delta H}{\delta q_3}\right)}_{(1)} + \underbrace{\frac{\delta G}{\delta q_3}\partial_x\frac{\delta F}{\delta q_3}\partial_x\left(q_3\frac{\delta H}{\delta q_3}\right)}_{(1)} - \underbrace{q_1\frac{\delta^2F}{\delta q_3^2}\partial_x\frac{\delta G}{\delta q_1}\partial_x\left(q_3\frac{\delta H}{\delta q_3}\right)}_{(2b)} + \\
& \underbrace{\frac{\delta^2G}{\delta q_3\delta q_1}\partial_x\left(q_1\frac{\delta F}{\delta q_3}\right)\partial_x\left(q_3\frac{\delta H}{\delta q_3}\right)}_{(3b)} + \underbrace{q_1\frac{\delta^2G}{\delta q_3^2}\partial_x\frac{\delta F}{\delta q_1}\partial_x\left(q_3\frac{\delta H}{\delta q_3}\right)}_{(2d)} - \underbrace{\frac{\delta^2F}{\delta q_3\delta q_1}\partial_x\left(q_1\frac{\delta G}{\delta q_3}\right)\partial_x\left(q_3\frac{\delta H}{\delta q_3}\right)}_{(3a)} - \\
& \underbrace{q_3\frac{\delta^2F}{\delta q_3^2}\partial_x\frac{\delta G}{\delta q_3}\partial_x\left(q_3\frac{\delta H}{\delta q_3}\right)}_{(2b)} + \underbrace{\frac{\delta^2G}{\delta q_3^2}\partial_x\left(q_3\frac{\delta F}{\delta q_3}\right)\partial_x\left(q_3\frac{\delta H}{\delta q_3}\right)}_{(2d)} + \underbrace{q_3\frac{\delta^2G}{\delta q_3^2}\partial_x\frac{\delta F}{\delta q_3}\partial_x\left(q_3\frac{\delta H}{\delta q_3}\right)}_{(2d)} - \underbrace{\frac{\delta^2F}{\delta q_3^2}\partial_x\left(q_3\frac{\delta G}{\delta q_3}\right)\partial_x\left(q_3\frac{\delta H}{\delta q_3}\right)}_{(2b)} \Bigg] \mathrm{d}x,
\end{aligned}
\tag{9}
$$

Next, we continue with the evaluation of the terms appearing in (5). $\{\{G,H\},F\}_{q_1,q_3}$ is given by:

$$
\begin{aligned}
\{\{G,H\},F\}_{q_1,q_3} = -\int_\Omega \Bigg[& -q_1\frac{\delta G}{\delta q_3}\partial_x\frac{\delta H}{\delta q_3}\partial_x\frac{\delta F}{\delta q_1} + q_1\frac{\delta H}{\delta q_3}\partial_x\frac{\delta G}{\delta q_3}\partial_x\frac{\delta F}{\delta q_1} - q_1^2\frac{\delta^2 G}{\delta q_3^2}\partial_x\frac{\delta H}{\delta q_1}\partial_x\frac{\delta F}{\delta q_1} + \\
& q_1\frac{\delta^2 H}{\delta q_3\delta q_1}\partial_x\Big(q_1\frac{\delta G}{\delta q_3}\Big)\partial_x\frac{\delta F}{\delta q_1} + q_1^2\frac{\delta^2 H}{\delta q_3^2}\partial_x\frac{\delta G}{\delta q_1}\partial_x\frac{\delta F}{\delta q_1} - q_1\frac{\delta^2 G}{\delta q_3\delta q_1}\partial_x\Big(q_1\frac{\delta H}{\delta q_3}\Big)\partial_x\frac{\delta F}{\delta q_1} - \\
& q_1q_3\frac{\delta^2 G}{\delta q_3^2}\partial_x\frac{\delta H}{\delta q_3}\partial_x\frac{\delta F}{\delta q_1} + q_1\frac{\delta^2 H}{\delta q_3^2}\partial_x\Big(q_3\frac{\delta G}{\delta q_3}\Big)\partial_x\frac{\delta F}{\delta q_1} + q_1q_3\frac{\delta^2 H}{\delta q_3^2}\partial_x\frac{\delta G}{\delta q_3}\partial_x\frac{\delta F}{\delta q_1} - \\
& q_1\frac{\delta^2 G}{\delta q_3^2}\partial_x\Big(q_3\frac{\delta H}{\delta q_3}\Big)\partial_x\frac{\delta F}{\delta q_1} - \frac{\delta G}{\delta q_3}\partial_x\frac{\delta H}{\delta q_1}\partial_x\Big(q_1\frac{\delta F}{\delta q_3}\Big) + \frac{\delta H}{\delta q_3}\partial_x\frac{\delta G}{\delta q_1}\partial_x\Big(q_1\frac{\delta F}{\delta q_3}\Big) - \\
& q_1\frac{\delta^2 G}{\delta q_1\delta q_3}\partial_x\frac{\delta H}{\delta q_1}\partial_x\Big(q_1\frac{\delta F}{\delta q_3}\Big) + \frac{\delta^2 H}{\delta q_1^2}\partial_x\Big(q_1\frac{\delta G}{\delta q_3}\Big)\partial_x\Big(q_1\frac{\delta F}{\delta q_3}\Big) + q_1\frac{\delta^2 H}{\delta q_1\delta q_3}\partial_x\frac{\delta G}{\delta q_1}\partial_x\Big(q_1\frac{\delta F}{\delta q_3}\Big) - \\
& \frac{\delta^2 G}{\delta q_1^2}\partial_x\Big(q_1\frac{\delta H}{\delta q_3}\Big)\partial_x\Big(q_1\frac{\delta F}{\delta q_3}\Big) - q_3\frac{\delta^2 G}{\delta q_1\delta q_3}\partial_x\frac{\delta H}{\delta q_3}\partial_x\Big(q_1\frac{\delta F}{\delta q_3}\Big) + \frac{\delta^2 H}{\delta q_1\delta q_3}\partial_x\Big(q_3\frac{\delta G}{\delta q_3}\Big)\partial_x\Big(q_1\frac{\delta F}{\delta q_3}\Big) + \\
& q_3\frac{\delta^2 H}{\delta q_1\delta q_3}\partial_x\frac{\delta G}{\delta q_3}\partial_x\Big(q_1\frac{\delta F}{\delta q_3}\Big) - \frac{\delta^2 G}{\delta q_1\delta q_3}\partial_x\Big(q_3\frac{\delta H}{\delta q_3}\Big)\partial_x\Big(q_1\frac{\delta F}{\delta q_3}\Big) - q_3\frac{\delta G}{\delta q_3}\partial_x\frac{\delta H}{\delta q_3}\partial_x\frac{\delta F}{\delta q_3} + \\
& q_3\frac{\delta H}{\delta q_3}\partial_x\frac{\delta G}{\delta q_3}\partial_x\frac{\delta F}{\delta q_3} - q_1q_3\frac{\delta^2 G}{\delta q_3^2}\partial_x\frac{\delta H}{\delta q_1}\partial_x\frac{\delta F}{\delta q_3} + q_3\frac{\delta^2 H}{\delta q_3\delta q_1}\partial_x\Big(q_1\frac{\delta G}{\delta q_3}\Big)\partial_x\frac{\delta F}{\delta q_3} + \\
& q_1q_3\frac{\delta^2 H}{\delta q_3^2}\partial_x\frac{\delta G}{\delta q_1}\partial_x\frac{\delta F}{\delta q_3} - q_3\frac{\delta^2 G}{\delta q_3\delta q_1}\partial_x\Big(q_1\frac{\delta H}{\delta q_3}\Big)\partial_x\frac{\delta F}{\delta q_3} - q_3^2\frac{\delta^2 G}{\delta q_3^2}\partial_x\frac{\delta H}{\delta q_3}\partial_x\frac{\delta F}{\delta q_3} + \\
& q_3\frac{\delta^2 H}{\delta q_3^2}\partial_x\Big(q_3\frac{\delta G}{\delta q_3}\Big)\partial_x\frac{\delta F}{\delta q_3} + q_3^2\frac{\delta^2 H}{\delta q_3^2}\partial_x\frac{\delta G}{\delta q_3}\partial_x\frac{\delta F}{\delta q_3} - q_3\frac{\delta^2 G}{\delta q_3^2}\partial_x\Big(q_3\frac{\delta H}{\delta q_3}\Big)\partial_x\frac{\delta F}{\delta q_3} - \\
& \frac{\delta G}{\delta q_3}\partial_x\frac{\delta H}{\delta q_3}\partial_x\Big(q_3\frac{\delta F}{\delta q_3}\Big) + \frac{\delta H}{\delta q_3}\partial_x\frac{\delta G}{\delta q_3}\partial_x\Big(q_3\frac{\delta F}{\delta q_3}\Big) - q_1\frac{\delta^2 G}{\delta q_3^2}\partial_x\frac{\delta H}{\delta q_1}\partial_x\Big(q_3\frac{\delta F}{\delta q_3}\Big) + \\
& \frac{\delta^2 H}{\delta q_3\delta q_1}\partial_x\Big(q_1\frac{\delta G}{\delta q_3}\Big)\partial_x\Big(q_3\frac{\delta F}{\delta q_3}\Big) + q_1\frac{\delta^2 H}{\delta q_3^2}\partial_x\frac{\delta G}{\delta q_1}\partial_x\Big(q_3\frac{\delta F}{\delta q_3}\Big) - \frac{\delta^2 G}{\delta q_3\delta q_1}\partial_x\Big(q_1\frac{\delta H}{\delta q_3}\Big)\partial_x\Big(q_3\frac{\delta F}{\delta q_3}\Big) - \\
& q_3\frac{\delta^2 G}{\delta q_3^2}\partial_x\frac{\delta H}{\delta q_3}\partial_x\Big(q_3\frac{\delta F}{\delta q_3}\Big) + \frac{\delta^2 H}{\delta q_3^2}\partial_x\Big(q_3\frac{\delta G}{\delta q_3}\Big)\partial_x\Big(q_3\frac{\delta F}{\delta q_3}\Big) + q_3\frac{\delta^2 H}{\delta q_3^2}\partial_x\frac{\delta G}{\delta q_3}\partial_x\Big(q_3\frac{\delta F}{\delta q_3}\Big) - \frac{\delta^2 G}{\delta q_3^2}\partial_x\Big(q_3\frac{\delta H}{\delta q_3}\Big)\partial_x\Big(q_3\frac{\delta F}{\delta q_3}\Big)\Bigg]\,\mathrm{d}x.
\end{aligned}
\tag{10}
$$

The term $\{\{H,F\},G\}_{q_1,q_3}$ is given by:

$$
\begin{aligned}
\{\{H,F\},G\}_{q_1,q_3} = -\int_{\Omega}\Bigg[& -q_1\frac{\delta H}{\delta q_3}\partial_x\frac{\delta F}{\delta q_3}\partial_x\frac{\delta G}{\delta q_1} + q_1\frac{\delta F}{\delta q_3}\partial_x\frac{\delta H}{\delta q_3}\partial_x\frac{\delta G}{\delta q_1} - q_1^2\frac{\delta^2 H}{\delta q_3^2}\partial_x\frac{\delta F}{\delta q_1}\partial_x\frac{\delta G}{\delta q_1} + \\
& q_1\frac{\delta^2 F}{\delta q_3\delta q_1}\partial_x\Big(q_1\frac{\delta H}{\delta q_3}\Big)\partial_x\frac{\delta G}{\delta q_1} + q_1^2\frac{\delta^2 F}{\delta q_3^2}\partial_x\frac{\delta H}{\delta q_1}\partial_x\frac{\delta G}{\delta q_1} - q_1\frac{\delta^2 H}{\delta q_3\delta q_1}\partial_x\Big(q_1\frac{\delta F}{\delta q_3}\Big)\partial_x\frac{\delta G}{\delta q_1} - \\
& q_1 q_3\frac{\delta^2 H}{\delta q_3^2}\partial_x\frac{\delta F}{\delta q_3}\partial_x\frac{\delta G}{\delta q_1} + q_1\frac{\delta^2 F}{\delta q_3^2}\partial_x\Big(q_3\frac{\delta H}{\delta q_3}\Big)\partial_x\frac{\delta G}{\delta q_1} + q_1 q_3\frac{\delta^2 F}{\delta q_3^2}\partial_x\frac{\delta H}{\delta q_3}\partial_x\frac{\delta G}{\delta q_1} - \\
& q_1\frac{\delta^2 H}{\delta q_3^2}\partial_x\Big(q_3\frac{\delta F}{\delta q_3}\Big)\partial_x\frac{\delta G}{\delta q_1} - \frac{\delta H}{\delta q_3}\partial_x\frac{\delta F}{\delta q_1}\partial_x\Big(q_1\frac{\delta G}{\delta q_3}\Big) + \frac{\delta F}{\delta q_3}\partial_x\frac{\delta H}{\delta q_1}\partial_x\Big(q_1\frac{\delta G}{\delta q_3}\Big) - \\
& q_1\frac{\delta^2 H}{\delta q_1\delta q_3}\partial_x\frac{\delta F}{\delta q_1}\partial_x\Big(q_1\frac{\delta G}{\delta q_3}\Big) + \frac{\delta^2 F}{\delta q_1^2}\partial_x\Big(q_1\frac{\delta H}{\delta q_3}\Big)\partial_x\Big(q_1\frac{\delta G}{\delta q_3}\Big) + q_1\frac{\delta^2 F}{\delta q_1\delta q_3}\partial_x\frac{\delta H}{\delta q_1}\partial_x\Big(q_1\frac{\delta G}{\delta q_3}\Big) - \\
& \frac{\delta^2 H}{\delta q_1^2}\partial_x\Big(q_1\frac{\delta F}{\delta q_3}\Big)\partial_x\Big(q_1\frac{\delta G}{\delta q_3}\Big) - q_3\frac{\delta^2 H}{\delta q_1\delta q_3}\partial_x\frac{\delta F}{\delta q_3}\partial_x\Big(q_1\frac{\delta G}{\delta q_3}\Big) + \frac{\delta^2 F}{\delta q_1\delta q_3}\partial_x\Big(q_3\frac{\delta H}{\delta q_3}\Big)\partial_x\Big(q_1\frac{\delta G}{\delta q_3}\Big) + \\
& q_3\frac{\delta^2 F}{\delta q_1\delta q_3}\partial_x\frac{\delta H}{\delta q_3}\partial_x\Big(q_1\frac{\delta G}{\delta q_3}\Big) - \frac{\delta^2 H}{\delta q_1\delta q_3}\partial_x\Big(q_3\frac{\delta F}{\delta q_3}\Big)\partial_x\Big(q_1\frac{\delta G}{\delta q_3}\Big) - q_3\frac{\delta H}{\delta q_3}\partial_x\frac{\delta F}{\delta q_3}\partial_x\frac{\delta G}{\delta q_3} + \\
& q_3\frac{\delta F}{\delta q_3}\partial_x\frac{\delta H}{\delta q_3}\partial_x\frac{\delta G}{\delta q_3} - q_1 q_3\frac{\delta^2 H}{\delta q_3^2}\partial_x\frac{\delta F}{\delta q_1}\partial_x\frac{\delta G}{\delta q_3} + q_3\frac{\delta^2 F}{\delta q_3\delta q_1}\partial_x\Big(q_1\frac{\delta H}{\delta q_3}\Big)\partial_x\frac{\delta G}{\delta q_3} + \\
& q_1 q_3\frac{\delta^2 F}{\delta q_3^2}\partial_x\frac{\delta H}{\delta q_1}\partial_x\frac{\delta G}{\delta q_3} - q_3\frac{\delta^2 H}{\delta q_3\delta q_1}\partial_x\Big(q_1\frac{\delta F}{\delta q_3}\Big)\partial_x\frac{\delta G}{\delta q_3} - q_3^2\frac{\delta^2 H}{\delta q_3^2}\partial_x\frac{\delta F}{\delta q_3}\partial_x\frac{\delta G}{\delta q_3} + \\
& q_3\frac{\delta^2 F}{\delta q_3^2}\partial_x\Big(q_3\frac{\delta H}{\delta q_3}\Big)\partial_x\frac{\delta G}{\delta q_3} + q_3^2\frac{\delta^2 F}{\delta q_3^2}\partial_x\frac{\delta H}{\delta q_3}\partial_x\frac{\delta G}{\delta q_3} - q_3\frac{\delta^2 H}{\delta q_3^2}\partial_x\Big(q_3\frac{\delta F}{\delta q_3}\Big)\partial_x\frac{\delta G}{\delta q_3} - \\
& \frac{\delta H}{\delta q_3}\partial_x\frac{\delta F}{\delta q_3}\partial_x\Big(q_3\frac{\delta G}{\delta q_3}\Big) + \frac{\delta F}{\delta q_3}\partial_x\frac{\delta H}{\delta q_3}\partial_x\Big(q_3\frac{\delta G}{\delta q_3}\Big) - q_1\frac{\delta^2 H}{\delta q_3^2}\partial_x\frac{\delta F}{\delta q_1}\partial_x\Big(q_3\frac{\delta G}{\delta q_3}\Big) + \\
& \frac{\delta^2 F}{\delta q_3\delta q_1}\partial_x\Big(q_1\frac{\delta H}{\delta q_3}\Big)\partial_x\Big(q_3\frac{\delta G}{\delta q_3}\Big) + q_1\frac{\delta^2 F}{\delta q_3^2}\partial_x\frac{\delta H}{\delta q_1}\partial_x\Big(q_3\frac{\delta G}{\delta q_3}\Big) - \frac{\delta^2 H}{\delta q_3\delta q_1}\partial_x\Big(q_1\frac{\delta F}{\delta q_3}\Big)\partial_x\Big(q_3\frac{\delta G}{\delta q_3}\Big) - \\
& q_3\frac{\delta^2 H}{\delta q_3^2}\partial_x\frac{\delta F}{\delta q_3}\partial_x\Big(q_3\frac{\delta G}{\delta q_3}\Big) + \frac{\delta^2 F}{\delta q_3^2}\partial_x\Big(q_3\frac{\delta H}{\delta q_3}\Big)\partial_x\Big(q_3\frac{\delta G}{\delta q_3}\Big) + q_3\frac{\delta^2 F}{\delta q_3^2}\partial_x\frac{\delta H}{\delta q_3}\partial_x\Big(q_3\frac{\delta G}{\delta q_3}\Big) - \frac{\delta^2 H}{\delta q_3^2}\partial_x\Big(q_3\frac{\delta F}{\delta q_3}\Big)\partial_x\Big(q_3\frac{\delta G}{\delta q_3}\Big)\Bigg]\,\mathrm{d}x.
\end{aligned}
\tag{11}
$$

Using the terms in (9), (10), and (11), one can identify that Jacobi identity holds. We defer from performing step by step cancellation of the terms clubbed under the number appearing in the underbrace (as in (9)), but can provide even further detailed computations upon the request of the reader.