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Structure-preserving Spatial Discretization of a Two-Fluid Model*

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***Abstract*— We present a structure-preserving spatial discretization method for infinite-dimensional non-linear port-Hamiltonian representations of a commonly used one-dimensional two-phase flow model: the Two-Fluid Model. We introduce the port-Hamiltonian representation of this two-phase flow model and then invoke a mixed-finite-element method to perform a structure-preserving spatial discretization. Consequently, we obtain a finite-dimensional realization of a recently proposed novel Stokes-Dirac structure for this model. The properties of the resulting finite-dimensional realization are assessed and the conditions under which it is known to respect the properties of a finite-dimensional Dirac structure are discussed. Moreover, we derive the complete finite-dimensional interconnected port-Hamiltonian model by invoking the notion of power-preserving interconnection.**

I. INTRODUCTION

Structure preservation in models involving differential equations arising in different domains of science and engineering has received considerable attention, see [1], [2]. Structure-preserving numerical methods are crucial for simulation purposes and design of passivity-based and energy shaping control techniques [3]. Such numerical methods include geometric or symplectic integration of ordinary differential equations. It is well known that discretization techniques or model reduction methods can destroy, for instance, the Hamiltonian structure of the model [4]. The focus of this work is to obtain structure-preserving numerical methods for infinite-dimensional port-Hamiltonian (pH) representations of non-linear partial differential equations. In particular, we are interested in developing numerical methods for pH representations of a commonly used one-dimensional multi-phase flow model, namely, the Two-Fluid Model (TFM) [5].

Several methodologies exist for structure-preserving discretization of finite-dimensional pH systems, i.e., explicit discrete-time pH systems [6] and discrete-time pH descriptor systems [7]. In the scope of structure-preserving discretization of infinite-dimensional pH systems, the spatial discretization of the underlying geometry of the model forms a key step. Such a spatial discretization essentially reduces the infinite-dimensional pH representation to a finite-dimensional pH system described by ordinary differential equations (ODEs). The latter can be effectively dealt with by using existing (time) integration methods.

In the past, spatial discretization methods preserving the pH structure have been proposed in the scope of one- and multi-dimensional problems [8]-[18]. Several well-known spatial discretization methods include pseudo-spectral methods [8], mixed-finite-element methods [9], [11], partitioned finite-element methods [15], [16], staggered finite-difference and finite-volume methods [17], and explicit simplicial discretization methods [10].

However, most of these existing works on structure-preserving spatial discretization of infinite-dimensional pH representations have focused on the approximation of a constant Stokes-Dirac structure arising from a *state-independent* (and extended) skew-adjoint Hamiltonian operator. For instance, in [19], non-canonical but physically relevant Hamiltonian functionals lead to non-linear pH systems with an underlying constant and an extended *state-independent* Stokes-Dirac structure. Such a geometric structure has subsequently been discretized in a structure-preserving manner using partitioned finite-element method in [18]. Structure-preserving spatial discretization has also been performed for first- and higher-order state-independent Hamiltonian operators with distributed inputs, see [20]. However, for these methods, the Hamiltonian operators were again *state-independent*. As for applications, the (lossless) transmission line, wave equations, Maxwell's equations, rendering *state-independent* Hamiltonian operators, have generally been used as test-beds for the numerical validation of the structure-preserving discretization framework [9]. Additionally, most of the existing spatial discretization methods have been developed and tested for models with quadratic Hamiltonian functionals.

Non-quadratic Hamiltonian functionals and *state-dependent* skew-adjoint Hamiltonian operators (under a certain choice of the state-variables) are specific features of the TFM. One of the existing works on structure-preserving spatial discretization of mathematical models with non-quadratic Hamiltonian functionals includes [21]. This work deals with the structure-preserving discretization of a pH representation of a compressible single-phase flow model. It focuses on discretizing *state-independent* Hamiltonian operators or Stokes-Dirac structures. In contrast, the focus of our work is to spatially discretize *state-dependent* Stokes-Dirac structures (or pH representations).

Relevant works in the direction of structure-preserving discretization of a *state-dependent* Stokes-Dirac structure include [14], [22] and [23]. These works are built upon the mixed-finite-element method (mFEM), which was initially proposed for canonical pH systems in [9]. The methodol-

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ogy is based on the concept of using mixed or different finite elements for the approximation of flow and effort variables. This methodology has been used to deal with the spatial discretization of linear shallow water equations (accounting for Coriolis forces and topography) with a non-constant Stokes-Dirac structure in [23]. Spatial discretization essentially translates a Stokes-Dirac structure into a finite-dimensional Dirac structure [15]. In view of our interest to develop structure-preserving discretization techniques for a pH representation of the TFM, we recall that the corresponding Stokes-Dirac structure, recently introduced in [5], is novel and has not been spatially-discretized or both spatially- and temporally-discretized in a structure-preserving manner.

We employ the mFEM for the structure-preserving spatial discretization of the TFM. We demonstrate that a finite-dimensional representation preserves the finite-dimensional Dirac structure, i.e., the geometric interconnection structure and the power balance equation. We also discuss the role of the boundary port-variables in guaranteeing a finite-dimensional Dirac structure. It is of interest to investigate whether the mFEM, which has so far been tested on linear(ized) problems or *state-independent* Stokes-Dirac structures, requires fundamental modifications to deal with the non-linearity of the mathematical model or the *state-dependent* nature of the Stokes-Dirac structure. Hence, we provide insights on the influence of the model non-linearity or *state-dependent* Stokes-Dirac structure in obtaining a structure-preserving spatial discretization method. We subsequently reason that it is not necessary to eliminate the appearance of feedthrough terms from the finite-dimensional model approximation and that a special structure in the feedthrough matrix makes it amenable to use the mFEM for spatially discretizing hyperbolic systems, such as, the TFM.

The main scientific contributions of this work are as follows: *(i)* we propose spatial discretization of the TFM using the mFEM, *(ii)* we prove the preservation of finite-dimensional Dirac structure, and *(iii)* we derive a finite-dimensional interconnected pH model.

The paper is organized as follows. In Section II, we recall the basic definition of finite-dimensional Dirac structures. We present a pH formulation of the TFM in Section II-A. We, then, invoke the mixed-finite-element method for structure-preserving spatial discretization of the TFM in Section II-B. The power-preserving interconnection of two discretized lumps is performed in Section II-C. Finally, Section III ends with conclusions and future works.

II. Structure-preserving spatial discretization

In the scope of the current work, we only recall the definition of Dirac structures in a finite-dimensional setting. For the definition of Stokes-Dirac structures in an infinite-dimensional setting, we refer to [24], [25] for canonical Hamiltonian operators and to [5], [26] for non-canonical state-dependent Hamiltonian operators.

A (finite-dimensional) Dirac structure $\mathcal{D}$ is defined next.

Definition 2.1: Consider $e \in \mathcal{E}$ and $f \in \mathcal{F}$, where $\mathcal{E}$ and $\mathcal{F}$ are real finite-dimensional normed vector spaces and where $\mathcal{E} = \mathcal{F}^*$, the dual space of $\mathcal{F}$. We define

$$e(f) := \langle e, f\rangle := f^T e = \frac{1}{2} f^T e + \frac{1}{2} e^T f. \tag{1}$$

Moreover, we define the non-degenerate bilinear pairing $d : (\mathcal{E}\times\mathcal{F})\times(\mathcal{E}\times\mathcal{F}) \to \mathbb{R}$ in the following way:

$$d\Big((e_1, f_1), (e_2, f_2)\Big) := \langle e_2, f_1\rangle + \langle e_1, f_2\rangle. \tag{2}$$

The subspace $\mathcal{D} \subset \mathcal{E}\times\mathcal{F}$ is a Dirac structure if $\mathcal{D} = \mathcal{D}^\perp$, where $\mathcal{D}^\perp$ is defined as follows:

$$\mathcal{D}^\perp := \{(e,f) \in \mathcal{E}\times\mathcal{F} \,|\, d\Big((e_0,f_0),(e,f)\Big) = 0 \quad \forall (e_0,f_0) \in \mathcal{D}\}.$$

We, first, recall a pH representation introduced in [5] in the scope of the TFM.

A. Port-Hamiltonian formulation of the Two-Fluid Model

The one-dimensional TFM is governed by the following partial differential equations:

$$\partial_t m_g + \partial_z (m_g v_g) = 0, \tag{3a}$$
$$\partial_t m_\ell + \partial_z (m_\ell v_\ell) = 0, \tag{3b}$$
$$\partial_t (m_g v_g) + \partial_z \left(m_g v_g^2\right) = -\alpha_g \partial_z p + b_g^M (v_\ell - v_g), \tag{3c}$$
$$\partial_t (m_\ell v_\ell) + \partial_z \left(m_\ell v_\ell^2 + p\right) = \alpha_g \partial_z p - b_g^M (v_\ell - v_g), \tag{3d}$$

where

$$p\left(m_g, m_\ell, \alpha_g\right) = m_g c_g^2 + m_\ell c_\ell^2 - \beta\left(1 - \alpha_{\mathrm{g}}\right), \tag{4}$$

$$\alpha_g\left(m_g, m_\ell\right) = -m_g \frac{c_g^2}{2\beta} - m_\ell \frac{c_\ell^2}{2\beta} + \frac{1}{2} + \sqrt{\left(m_g \frac{c_g^2}{2\beta} + m_\ell \frac{c_\ell^2}{2\beta} - \frac{1}{2}\right)^2 + m_g \frac{c_g^2}{\beta}}, \tag{5}$$

with $\beta = \rho_{\ell 0} c_\ell^2 - p_{\ell 0}$. Here, $t \in \mathbb{R}_{\geq 0}$ and $z \in \Omega = [0, L]$ are, respectively, the temporal and spatial variables with L denoting the length of the spatial domain. Moreover, m_g is the mass of the gaseous phase per unit length, m_ℓ is the mass of the liquid phase per unit length, v_g is the velocity of the gaseous phase, v_ℓ is the velocity of the liquid phase, p is the common pressure and b_g^M is a positive constant that accounts for the interaction between the phases. Here, variables c_g and c_ℓ, respectively, represent the constant speed of sound in the gaseous and the liquid phase, $\rho_{\ell 0}$ is the reference density of the liquid phase, and $p_{\ell 0}$ is the reference pressure.

Remark 1: It is assumed throughout the paper that both gas-void fraction and liquid-void fraction are positive. Moreover, we do not consider gravitational and frictional effects in the description of the TFM for the sake of simplicity.

The non-quadratic Hamiltonian functional $\mathcal{H}$ is given by

$$\mathcal{H}(q_1, q_2, q_3, q_4) := \int_\Omega \frac{q_3^2}{2q_1} + \frac{q_4^2}{2q_2} + q_1 c_g^2 \ln\left(\frac{p}{c_g^2}\right) + q_2 c_\ell^2 \ln\left(\frac{p+\beta}{c_\ell^2}\right) + (1 - \alpha_g)\,\beta \,\mathrm{d}z, \tag{6}$$

where p and α_g can be replaced by the relations (4) and (5), respectively, and where $q = [q_1,\ q_2,\ q_3,\ q_4]^T := [m_g,\ m_\ell,\ m_g v_g,\ m_\ell v_l]^T$ denotes the state of the system.

Theorem 2.1: The governing equations of the Two-Fluid Model can be written in a pH representation as follows:

$$\partial_t q = (\mathcal{J}(q) - \mathcal{R})\,\delta_q \mathcal{H}(q) \text{ or } f = (\mathcal{J}(q) - \mathcal{R})\,e, \quad (7)$$

with state-variable q, the Hamiltonian functional $\mathcal{H}$ (6),

$$\mathcal{J}(q) = -\begin{bmatrix} 0 & 0 & \partial_z(q_1\cdot) & 0 \\ 0 & 0 & 0 & \partial_z(q_2\cdot) \\ q_1\partial_z(\cdot) & 0 & \partial_z(q_3\cdot)+q_3\partial_z(\cdot) & 0 \\ 0 & q_2\partial_z(\cdot) & 0 & \partial_z(q_4\cdot)+q_4\partial_z(\cdot) \end{bmatrix},$$

$$\mathcal{R} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b_g^M & -b_g^M \\ 0 & 0 & -b_g^M & b_g^M \end{bmatrix},$$

and the boundary port flow and effort variables

$$\begin{cases} e_{b1}^B = e_{q_3}|_{z=L}, \quad f_{b1}^B = \Big(q_1 e_{q_1} + q_3 e_{q_3}\Big)|_{z=L}, \\ e_{b2}^B = e_{q_4}|_{z=L}, \quad f_{b2}^B = \Big(q_2 e_{q_2} + q_4 e_{q_4}\Big)|_{z=L}, \\ f_{a1}^B = -e_{q_3}|_{z=0}, \quad e_{a1}^B = \Big(q_1 e_{q_1} + q_3 e_{q_3}\Big)|_{z=0}, \\ f_{a2}^B = -e_{q_4}|_{z=0}, \quad e_{a2}^B = \Big(q_2 e_{q_2} + q_4 e_{q_4}\Big)|_{z=0}, \end{cases} \quad (8)$$

where the flow and effort variables have been introduced in (7)*. The flow variables f are defined as the derivative of the state-variables with respect to time i.e., $f = [f_{q_1},\ f_{q_2},\ f_{q_3},\ f_{q_4}]^T := \partial_t q$. Moreover, the effort variables e are defined as the variational derivative of the Hamiltonian functional $\mathcal{H}(q)$ with respect to the state-variable, i.e., $e = [e_{q_1},\ e_{q_2},\ e_{q_3},\ e_{q_4}]^T := \delta_q \mathcal{H}(q)$.*

Using the formally skew-adjointness of $\mathcal{J}(q)$, the behavior of the Hamiltonian along the solutions of the mathematical model is governed by the following power balance equation:

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \int_\Omega (\delta_q \mathcal{H}(q))^T(-\mathcal{R}(q))\delta_q \mathcal{H}(q)\ \mathrm{d}z + \Big(e_{q_3}\Big(q_1 e_{q_1} + q_3 e_{q_3}\Big)\Big)\,|_{z=0} + \Big(e_{q_4}\Big(q_2 e_{q_2} + q_4 e_{q_4}\Big)\Big)\,|_{z=0} - \Big(e_{q_3}\Big(q_1 e_{q_1} + q_3 e_{q_3}\Big)\Big)\,|_{z=L} - \Big(e_{q_4}\Big(q_2 e_{q_2} + q_4 e_{q_4}\Big)\Big)\,|_{z=L}\ .$$

We can clearly observe that $\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t}$, in the absence of dissipation, is governed by the product of boundary port variables.

Remark 2: We have considered a non-unique way to define the boundary port flow and effort variables. Such boundary port variables can be elegantly parametrized following the principles laid down in [25].

The boundary flow and effort variables can be interpreted physically. Ignoring the sign associated to the boundary flow (and effort) variables; f_{a1}^B and f_{a2}^B (e_{b1}^B and e_{b2}^B) can be interpreted as gas and liquid volumetric flow rate, respectively, at the left end (and at the right end) of the spatial domain. The flow variables f_{b1}^B and f_{b2}^B have physical dimensions of energy per unit mass. Similarly, the effort variables e_{a1}^B and e_{a2}^B have the physical dimensions of energy per unit mass.

Remark 3: Given the nature of $\mathcal{R}$, this work basically considers the setting which abides by the strict conservation of energy in the absence of such dissipative terms. So, $b_g^M = 0$ and therefore $\mathcal{R} = 0$ in the sequel.

B. Spatial discretization of the Two-Fluid Model

The structure-preserving spatial discretization procedure essentially consists of the following steps. We approximate the states, and the flow and effort variables by suitable discrete functions. We then use such an approximation to obtain finite-dimensional equations and ensure power balance. In addition to preserving the power balance equation, we assess the existence of the finite-dimensional Dirac structure.

Numerical discretization entails discretizing the spatial domain $\Omega = [0, L]$ into several finite elements. We apply the procedure to a single discretized lump $Z_{ab} = [a, b]$, with $0 \le a < b \le L$, in the scope of the TFM.

Each state variable is discretized using one spatial basis function as follows:

$$q_i(t, z) = q_i^{ab}(t)\omega_{q_i}^{ab}(z), \text{ for all } i = 1, 2, 3, 4, \quad (9)$$

where

$$\omega_{q_i}^{ab}(z) = \frac{1}{b-a}, \text{ for all } i = 1, 2, 3, 4, \quad (10)$$

with $z \in Z_{ab}$. The normalization assumption on $\omega_{q_i}^{ab}(z)$ yields

$$\int_a^b \omega_{q_i}^{ab}(z)dz = 1, \text{ for all } i \in \{1, 2, 3, 4\}. \quad (11)$$

Recall that we have:

$$f_{q_i} = \partial_t q_i, \quad e_{q_i} = \delta_{q_i}\mathcal{H}(q), \text{ for } i = 1, 2, 3, 4. \quad (12)$$

Each flow variable f_{q_i} is discretized in a manner similar to the state approximation, i.e.,

$$f_{q_i}(t, z) = f_{q_i}^{ab}(t)\omega_{q_i}^{ab}(z), \quad \text{for all } i \in \{1, 2, 3, 4\}. \quad (13)$$

Each effort variable e_{q_i} is spatially discretized using two different basis functions $\omega_{q_i}^a(z)$ and $\omega_{q_i}^b(z)$ (for $i = 1, 2, 3, 4$) in the following manner:

$$e_{q_i}(t, z) = e_{q_i}^a(t)\omega_{q_i}^a(z) + e_{q_i}^b(t)\omega_{q_i}^b(z), \text{ for } i = 1, 2, 3, 4, \quad (14)$$

where $\omega_{q_i}^a(z) = \frac{b-z}{b-a}$, and $\omega_{q_i}^b(z) = \frac{z-a}{b-a}$, and the following boundary conditions hold:

$$\omega_{q_i}^a(a) = 1, \omega_{q_i}^a(b) = 0, \omega_{q_i}^b(a) = 0, \omega_{q_i}^b(b) = 1, i = 1, 2, 3, 4,$$

and

$$e_{q_i}^a(t) = e_{q_i}(t, a), \quad e_{q_i}^b(t) = e_{q_i}(t, b), \text{ for } i = 1, 2, 3, 4. \quad (15)$$

Here, $e_{q_i}(t, a)$ (and $e_{q_i}(t, b)$) is the value of the effort variable at the left (and the right) boundary of Z_{ab}.

Remark 4: We impose suitable constraints on the finite-element (FE) spaces such that the geometric properties are preserved while associating the lowest order FE spaces.

Remark 5: The time-dependence is omitted in the sequel. We focus on obtaining a (point-wise in time) finite-dimensional Dirac structure.

The basis functions for the flow and the effort variables should satisfy the system model in (7). Under the assumptions of linear spline basis functions, we obtain following

finite-dimensional equations:

$$\begin{aligned} f_{q_1}^{ab} &= \frac{1}{b-a}\Big(q_1^{ab}e_{q_3}^a - q_1^{ab}e_{q_3}^b\Big),\\ f_{q_2}^{ab} &= \frac{1}{b-a}\Big(q_2^{ab}e_{q_4}^a - q_2^{ab}e_{q_4}^b\Big),\\ f_{q_3}^{ab} &= \frac{1}{b-a}\Big(q_1^{ab}e_{q_1}^a - q_1^{ab}e_{q_1}^b + 2q_3^{ab}e_{q_3}^a - 2q_3^{ab}e_{q_3}^b\Big),\\ f_{q_4}^{ab} &= \frac{1}{b-a}\Big(q_2^{ab}e_{q_2}^a - q_2^{ab}e_{q_2}^b + 2q_4^{ab}e_{q_4}^a - 2q_4^{ab}e_{q_4}^b\Big). \end{aligned} \tag{16}$$

The above system (16) cannot be used for defining the Dirac structure as we have four flow-type variables and eight effort-type variables. Following [20], we motivate an alternative definition of (discrete) effort variables to ultimately obtain a Dirac structure at the finite-dimensional level. Alternative effort-type variables are defined as the average values of the efforts on the boundary of the discretization interval. Such a definition of effort-type variables is in the same spirit as that used in [20] (see Section 8.2.3), and can be viewed as a special case of the definition in [20] with $\alpha = 0.5$. As shown, such a definition essentially guarantees that the energy is preserved after (spatial) discretization. Mathematically, these alternative effort-type variables take the following form:

$$e_{q_i}^{ab} = \frac{1}{2}(e_{q_i}^a + e_{q_i}^b), \quad \text{for all } i \in \{1,2,3,4\}. \tag{17}$$

Let us define $\mathbf{f} = [f_{q_1}^{ab},\ f_{q_2}^{ab},\ f_{q_3}^{ab},\ f_{q_4}^{ab},\ f_{a1}^{B},\ f_{a2}^{B},\ f_{b1}^{B},\ f_{b2}^{B}]^T$, $\mathbf{e} = [e_{q_1}^{ab},\ e_{q_2}^{ab},\ e_{q_3}^{ab},\ e_{q_4}^{ab},\ e_{a1}^{B},\ e_{a2}^{B},\ e_{b1}^{B},\ e_{b2}^{B}]^T$, and $\Delta z := b - a$. Using (16), (17) and discrete approximation of (8), we then have

$$\mathbf{f} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & \frac{q_1^{ab}}{\Delta z} & -\frac{q_1^{ab}}{\Delta z} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{q_2^{ab}}{\Delta z} & -\frac{q_2^{ab}}{\Delta z}\\ \frac{q_1^{ab}}{\Delta z} & -\frac{q_1^{ab}}{\Delta z} & 0 & 0 & \frac{2q_3^{ab}}{\Delta z} & -\frac{2q_3^{ab}}{\Delta z} & 0 & 0\\ 0 & 0 & \frac{q_2^{ab}}{\Delta z} & -\frac{q_2^{ab}}{\Delta z} & 0 & 0 & \frac{2q_4^{ab}}{\Delta z} & -\frac{2q_4^{ab}}{\Delta z}\\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0\\ 0 & \frac{q_1^{ab}}{\Delta z} & 0 & 0 & 0 & \frac{q_3^{ab}}{\Delta z} & 0 & 0\\ 0 & 0 & 0 & \frac{q_2^{ab}}{\Delta z} & 0 & 0 & 0 & \frac{q_4^{ab}}{\Delta z} \end{bmatrix}}_{F^T} \begin{bmatrix} e_{q_1}^a\\ e_{q_1}^b\\ e_{q_2}^a\\ e_{q_2}^b\\ e_{q_3}^a\\ e_{q_3}^b\\ e_{q_4}^a\\ e_{q_4}^b \end{bmatrix}, \tag{18}$$

$$\mathbf{e} = \underbrace{\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2}\\ \frac{q_1^{ab}}{\Delta z} & 0 & 0 & 0 & \frac{q_3^{ab}}{\Delta z} & 0 & 0 & 0\\ 0 & 0 & \frac{q_2^{ab}}{\Delta z} & 0 & 0 & 0 & \frac{q_4^{ab}}{\Delta z} & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{E^T} \begin{bmatrix} e_{q_1}^a\\ e_{q_1}^b\\ e_{q_2}^a\\ e_{q_2}^b\\ e_{q_3}^a\\ e_{q_3}^b\\ e_{q_4}^a\\ e_{q_4}^b \end{bmatrix}. \tag{19}$$

Theorem 2.2: Consider $\mathcal{E} \times \mathcal{F} = \mathbb{R}^8 \times \mathbb{R}^8$. The subspace

$$\begin{aligned} \mathcal{D}_{TFM} = \{(\mathbf{e},\mathbf{f}) \in \mathcal{E}\times\mathcal{F} \mid\ & \exists\ \lambda \in \mathbb{R}^8\\ & s.t.\ \mathbf{e} = E^T\lambda, \mathbf{f} = F^T\lambda\}, \end{aligned} \tag{20}$$

with E and F defined in (19) *and* (18)*, respectively, is a Dirac structure in $\mathcal{E}_{TFM} \times \mathcal{F}_{TFM} = \mathbb{R}^8 \times \mathbb{R}^8$.*

Proof: The two conditions that need to hold in order to ensure that $\mathcal{D}_{TFM}$ is a Dirac structure are as follows [8]: *(i)* $EF^T + FE^T = 0$, and *(ii)* $\text{rank}([E \quad F]) = 8$. Condition *(i)* is a discrete equivalent of a power balance equation (or power-preservation), while, condition *(ii)* is representative of richness in the interconnection structure in order to interconnect discretized lumps in a power-preserving manner. It is straightforward to check that F and E introduced respectively in (18) and (19) fulfill the above two conditions for all discrete states $q_1^{ab}, q_2^{ab}, q_3^{ab}$ and q_4^{ab}. It is also clearly observable that $\mathcal{D}_{TFM}$ is a Dirac structure irrespective of the value of the spatial step Δz. ■

Remark 6: Recently, the notion of power-preserving maps has been introduced in [27] to define a finite-dimensional Dirac structure abiding by the non-degenerate bilinear form.

Remark 7: The approximation space generated by the chosen polynomial basis may not be suitable for resolving moving discontinuities. It is well known that not all choices of finite-element approximation spaces may lead to a stable mixed Galerkin discretization or a well-behaved scheme. Hence, an alternative (discrete) function approximation might be needed to resolve these sharp gradients.

Using (18) and (19), the flow-effort relations are given by:

$$\begin{aligned} \mathbf{f}^{ab} &= J\mathbf{e}^{ab} + B\mathbf{u}_\partial,\\ \mathbf{y}_\partial &= C\mathbf{e}^{ab} + D\mathbf{u}_\partial, \end{aligned} \tag{21}$$

where $\mathbf{f}^{ab} = [f_{q_1}^{ab},\ f_{q_2}^{ab},\ f_{q_3}^{ab},\ f_{q_4}^{ab}]^T$, $\mathbf{e}^{ab} = [e_{q_1}^{ab},\ e_{q_2}^{ab},\ e_{q_3}^{ab},\ e_{q_4}^{ab}]^T$, $\mathbf{y}_\partial = [f_{a1}^{B},\ f_{a2}^{B},\ f_{b1}^{B},\ f_{b2}^{B}]^T$ and $\mathbf{u}_\partial = [e_{a1}^{B},\ e_{a2}^{B},\ e_{b1}^{B},\ e_{b2}^{B}]^T$. The matrices J, B, C and D are as follows:

$$J = \begin{bmatrix} 0 & 0 & \frac{2q_1^{ab}}{\Delta z} & 0\\ 0 & 0 & 0 & \frac{2q_2^{ab}}{\Delta z}\\ -\frac{2q_1^{ab}}{\Delta z} & 0 & 0 & 0\\ 0 & -\frac{2q_2^{ab}}{\Delta z} & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & -\frac{2q_1^{ab}}{\Delta z} & 0\\ 0 & 0 & 0 & -\frac{2q_2^{ab}}{\Delta z}\\ 2 & 0 & -\frac{2q_3^{ab}}{\Delta z} & 0\\ 0 & 2 & 0 & -\frac{2q_4^{ab}}{\Delta z} \end{bmatrix},$$

$$C = -B^T, \quad D = \begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

Remark 8: Invoking the mFEM for the spatial discretization of the TFM is seen to contain feedthrough terms, i.e., $D \neq 0$. However, the matrix D is observed to have a special structure, i.e., it is skew-symmetric. We claim that such a structure is not unnatural for hyperbolic systems and support it by the following reasoning. The instantaneous information that is transferred across each face of the finite-element is non-zero. However, the instantaneous power due to the feedthrough matrix D is zero as $\langle \mathbf{u}_\partial, \mathbf{y}_\partial \rangle = \mathbf{u}_\partial^T D \mathbf{u}_\partial = 0$.

We now invoke few notations in order to define a state-space pH model that represents the TFM locally at the geometry of Z_{ab}. Using (9), (10), (12) and (13), we can straightforwardly deduce that $\mathbf{f}^{ab} = [\frac{\mathrm{d}}{\mathrm{d}t}q_1^{ab},\ \frac{\mathrm{d}}{\mathrm{d}t}q_2^{ab},\ \frac{\mathrm{d}}{\mathrm{d}t}q_3^{ab},\ \frac{\mathrm{d}}{\mathrm{d}t}q_4^{ab}]^T$.

The discrete Hamiltonian H_d^{ab} is expressed as:

$$H_d^{ab} = \int_{Z_{ab}} \Big(\frac{\left(q_3^{ab}\right)^2}{2q_1^{ab}(b-a)} + \frac{\left(q_4^{ab}\right)^2}{2q_2^{ab}(b-a)} + \frac{q_1^{ab}c_g^2}{b-a} \ln\Big(\frac{q_1^{ab}}{b-a} + \frac{c_\ell^2}{c_g^2}\frac{q_2^{ab}}{b-a} - \frac{\beta}{c_g^2}(1-\alpha_g^d)\Big) + \frac{q_2^{ab}c_\ell^2}{b-a} \ln\Big(\frac{q_2^{ab}}{b-a} + \frac{c_g^2}{c_\ell^2}\frac{q_1^{ab}}{b-a} + \frac{\beta}{c_\ell^2}\alpha_g^d\Big) + (1-\alpha_g^d)\beta\Big) \mathrm{d}z, \quad (22)$$

where $\alpha_g^d = -\frac{q_1^{ab}c_g^2}{2\beta(b-a)} - \frac{q_2^{ab}c_\ell^2}{2\beta(b-a)} + \frac{1}{2} + \Delta$, with

$$\Delta = \sqrt{\Big(\frac{q_1^{ab}c_g^2}{2\beta(b-a)} + \frac{q_2^{ab}c_\ell^2}{2\beta(b-a)} - \frac{1}{2}\Big)^2 + \frac{q_1^{ab}c_g^2}{\beta(b-a)}}.$$

Using the discrete Hamiltonian (22) along with (12), (15) and (17), $\mathbf{e}^{ab}$ can also be computed for numerical purposes. For the sake of simplicity, we will use $\dot{Q}^{ab}$ instead of $\mathbf{f}^{ab}$, and ∇H_d^{ab} instead of $\mathbf{e}^{ab}$ in the sequel, i.e.,

$$\dot{Q}^{ab} := \mathbf{f}^{ab}, \quad \nabla H_d^{ab} := \mathbf{e}^{ab}.$$

Using these notations, the finite-dimensional pH model at the geometric domain Z_{ab} is given by

$$\begin{aligned} \dot{Q} &= J\nabla H + BU, \\ Y &= C\nabla H + DU, \end{aligned} \quad (23)$$

where we have omitted the superscript "ab" in the above representation for the sake of generality of the structure. The subscript "d" is also dropped from the Hamiltonian for the sake of clarity in the sequel. Moreover, $U := [U^l,\ U^r]^T = \mathbf{u}_\partial$ and $Y := [Y^l,\ Y^r]^T = \mathbf{y}_\partial$, where we define U^l, U^r, Y^l and Y^r as follows:

$$U^l = [e_{a1}^B,\ e_{a2}^B]^T, \quad U^r = [e_{b1}^B,\ e_{b2}^B]^T, \quad (24)$$

$$Y^l = [f_{a1}^B,\ f_{a2}^B]^T, \quad Y^r = [f_{b1}^B,\ f_{b2}^B]^T. \quad (25)$$

Each discretized lump has been shown to possess the properties of a finite-dimensional Dirac structure. Given this fact, the lumps can be interconnected to obtain an aggregated finite-dimensional pH representation, which is shown next.

C. Power-preserving interconnection of 2 segments

In this section, we show the procedure to interconnect two lumps (in series) in a power-preserving manner.

We consider that the spatial domain $\Omega = [0, L]$ with $0 < a < b < c < L$ is discretized into N lumps. We focus our attention on two discretized lumps, i.e., $Z_{ab} = [a, b]$ and $Z_{bc} = [b, c]$. Using the structure in (23), the first discretized lump across Z_{ab} is governed by the following state ODEs:

$$\begin{aligned} \dot{Q}_1 &= J_1\nabla H_1 + B_1U_1 = J_1\nabla H_1 + B_1^lU_1^l + B_1^rU_1^r, \\ Y_1 &= C_1\nabla H_1 + D_1^lU_1^l + D_1^rU_1^r, \end{aligned} \quad (26)$$

where B_1^l and D_1^l, respectively, refer to the matrix composed from the first 2 columns of B and D introduced earlier. Anologously, B_1^r and D_1^r are composed of the last 2 columns of B and D, respectively. Matrices B_1^l, D_1^l, B_1^r, and D_1^r are each of size 4×2. The subscript "1" refers to the quantities (introduced in (23), (24) and (25)) for the first element Z_{ab}. Similarly, for the second discretized lump Z_{bc}, we have

$$\begin{aligned} \dot{Q}_2 &= J_2\nabla H_2 + B_2U_2 = J_2\nabla H_2 + B_2^lU_2^l + B_2^rU_2^r, \\ Y_2 &= C_2\nabla H_2 + D_2^lU_2^l + D_2^rU_2^r, \end{aligned} \quad (27)$$

where the subscript "2" refers to the quantities corresponding to the second finite-element Z_{bc}.

The following power-preserving interconnection relations hold:

$$\begin{bmatrix} U_1^r \\ U_2^l \end{bmatrix} = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} Y_1^r \\ Y_2^l \end{bmatrix} = \begin{bmatrix} -C_2^{12}\nabla H_2 - D_2^{r,12}U_2^r \\ C_1^{34}\nabla H_1 + D_1^{l,34}U_1^l \end{bmatrix},$$

where the subscripts "1" and "2" carry the same meaning as before. Matrix C_2^{12} refers to the first 2 rows, i.e., row 1 and 2 of the matrix C for the second element Z_{bc}. Similarly, matrix C_1^{34} refers to the last 2 rows, i.e., row 3 and 4 of the matrix C for the first element Z_{ab}. Analogously, $D_2^{r,12}$ refers to the first 2 rows, i.e., row 1 and 2 of the matrix D_2^r for the second element Z_{bc}. Similar explanation holds for $D_1^{l,34}$.

Interconnecting the 2 discretized lumps in a power-preserving manner yields the following structure:

$$\begin{aligned} \begin{bmatrix} \dot{Q}_1 \\ \dot{Q}_2 \end{bmatrix} &= \underbrace{\begin{bmatrix} J_1 & -B_1^rC_2^{12} \\ B_2^lC_1^{34} & J_2 \end{bmatrix}}_{J_{\text{assembled}}} \begin{bmatrix} \nabla H_1 \\ \nabla H_2 \end{bmatrix} + \begin{bmatrix} B_1^l & -B_1^rD_2^{r,12} \\ B_2^lD_1^{l,34} & B_2^r \end{bmatrix} \begin{bmatrix} U_1^l \\ U_2^r \end{bmatrix}, \\ \begin{bmatrix} Y_1^l \\ Y_2^r \end{bmatrix} &= \begin{bmatrix} C_1^{12} & -D_1^{r,12}C_2^{12} \\ D_2^{l,34}C_1^{34} & C_2^{34} \end{bmatrix} \begin{bmatrix} \nabla H_1 \\ \nabla H_2 \end{bmatrix} + \\ &\qquad \begin{bmatrix} \mathbf{O}_{2\times 2} & -D_1^{r,12}D_2^{r,12} \\ D_2^{l,34}D_1^{l,34} & \mathbf{O}_{2\times 2} \end{bmatrix} \begin{bmatrix} U_1^l \\ U_2^r \end{bmatrix}, \end{aligned} \quad (28)$$

where C_1^{12}, $D_1^{r,12}$, C_2^{34} and $D_2^{l,34}$ can be computed by following the notational conventions introduced earlier.

We have described the procedure to interconnect two discretized lumps. Analogously, N discretized lumps can be interconnected in a power-preserving manner to obtain a complete pH model.

Remark 9: The model (28) *is conservative in the sense that* $\frac{dH}{dt} = \frac{dH_1}{dt} + \frac{dH_2}{dt} = -\Big((Y_1^\ell(t))^TU_1^\ell(t) + (Y_2^r(t))^TU_2^r(t)\Big)$ *represents the net power across the left and right ports of the assembled system, and the system has no resistive effects in its dynamic behavior. This behavior is inherent and a representative feature of hyperbolic partial differential equations of the type that is studied here. The conservative (lossless) nature of the approximate mathematical model is therefore physically meaningful and replicates the conservative (lossless) nature of the infinite-dimensional pH model. However, a time-simulation of the behavior of this type of conservative model is numerically quite a challenge. Firstly, a standard implementation of existing symplectic schemes, as in [6], in the scope of our work, does not guarantee the exact preservation of discrete energy balance. Secondly, the structure-preserving temporal discretization framework in [6] needs to be extended to account for state-dependent system matrices and feedthrough terms in the state-space pH representation. Hence, the temporal discretization of the model* (28) *deserves a separate and careful treatment, which is deferred to future works.*

III. Conclusion

We performed spatial discretization on an infinite-dimensional port-Hamiltonian (pH) representation of the Two-Fluid Model using a mixed-finite-element method. We demonstrated that such a discretization preserves the finite-dimensional Dirac structure even for the underlying *state-dependent* Stokes-Dirac structure. We obtained an explicit finite-dimensional state-space representation for each finite-element. Furthermore, we obtained an aggregated finite-dimensional pH model using the notion of power-preserving interconnection. Such a finite-dimensional approximation is amenable for control and observer design.

The resistive effects can be incorporated while developing a state-space pH model by considering the discrete approximation of different variables as introduced in this paper. However, the image representation of a Dirac structure (see Theorem 2.2) will be different, and conditions for the existence of a Dirac structure will need to be modified in accordance. An alternative would be to approximate an extended *state-dependent* Stokes-Dirac structure by using admissible function spaces for different variables. Also, the proposed methodology can be adapted, though not straightforwardly, to consider higher-order finite-element (FE) spaces by using a similar spatial discretization procedure. Moreover, the proposed approach shares similarities with the recently developed mixed Galerkin discretization [11], [27] applicable for multi-dimensional models and, hence, it possesses potential for generalization to $N-$dimensional models. Extensions of the proposed methodology to account for higher-order FE spaces, deal with $N-$dimensional setting, and incorporate resistive effects will be the subject of future works.

Building on the current work, it is natural to develop a temporal discretization methodology. It is worth stating that the temporal discretization should also be structure-preserving. Moreover, it should not give rise to numerical instabilities. Future works will deal with the numerical implementation of spatially- and temporally-discretized scheme, and will encompass the validation of the approach along with the verification of the numerical convergence.

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