Transformation of Reduction Systems

a view on proving correctness

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Preface

My graduation project, carried out at the Programming Research Group of the Department of Computer Science, was centered around two questions:

1. What are good notions for proving correctness of transformations for Reduction Systems?

2. Does Thatte's transformation preserve confluence for the class of weakly persistent systems?

Jan Bergstra helped me with the first question and was my supervisor in the first part of the project. We had long talks about computer science, university, and other things. Those talks were extremely instructive.
At a particular moment, Jan led me into the room of Piet Rodenburg. Piet would become my supervisor for the second part of the project. He literally taught me how read and write mathematics and meticulously read various drafts of this thesis. Moreover, he fixed my attention on the second question and provided the counterexample to a claim in Verma's proof. We spent a lot of time searching for proofs and counterexamples of Verma's theorem, but did not find the answer yet. Nonetheless, I still haven't lost hope that we will eventually solve the problem and I hope Piet hasn't either.
Thank you Jan and Piet.
I would also like to thank Wan Fokkink, Jasper Kamperman and Jaco van de Pol for the discussions we had about the first question. They helped me to acquire an opinion on the subject. Furthermore, acknowledgements are due to Jan Willem Klop for explaining to me some things about Term Rewriting and stimulating me to investigate simulation relations.
Finally, I thank my friends of the Faculty and Anna for support in various ways.

Bas Luttik

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1 Introduction

With the emergence of the use of Reduction Systems as semantical basis of programming paradigms, the use of transformation methods for Reduction Systems is growing. Some systems are more suitable for execution on a machine than others, for instance because they have a normalizing reduction strategy that is easily decidable, or because they are in a specific format. Transformation methods are then used to translate an arbitrary system into such a suitable system.

Evidently, such a transformation should not be applied if it does not induce some sensible connection between the transform and its original. Only if a transformation fulfills a number of predefined criteria, it should be called correct. Of course, the answer to the question of whether the connection between transform and original is a sensible one is rather subjective. It depends largely on the application and little can be said in general. However, if the transformation is part of an implementation of Reduction Systems on automata, then there are some widely accepted intuitions:

- In Concurrency Theory and Automata Theory much research has been done towards satisfactory connections, there called "simulation relations" (see for instance [LV95]). These relations induce a preorder on the class of automata, such that if automaton A simulates automaton B, then A can be seen as an implementation of B.

- In papers dealing with transformations for Reduction Systems and their correctness, it is usually proved that the normal form relations of the original and its transform are related [Tha85, Tha88, Ver95a, Ver95b]. This is not surprising, because the normal form relation of a Reduction System is its computational content; it relates input and result.

Kamperman and Walters initiated a new approach towards correctness proofs of transformations for Reduction Systems [KW95, KW96]. In order to make correctness of their transformation for Term Rewriting Systems plausible, they prove that transform "simulates" original. Fokkink and Van de Pol prove that the simulation relation of Kamperman and Walters implies preservation of the normal form relation. They define a notion of *correct transformation* in which a connection between the normal form relations of original and transform figures as a principal component.

One of the virtues of the theory of Reduction Systems is that general notions such as 'termination' and 'uniqueness of normal forms' are well-defined. Powerful techniques have been developed to prove that a Reduction System has such properties (see for instance [Zan93, Oos94]). Therefore, many authors of papers dealing with transformations for Reduction Systems prove that their transformation preserves a number of such properties.

In this paper we will set up a general framework in which correctness criteria can be expressed such that all the notions we encountered in literature fit into this framework (section 3). As a derivative of the notion of Kamperman and Walters, we will define a notion of simulation that we will call ω-simulation. We will carefully study the implications of this notion in terms of the preservation of general properties in section 4 and we will conclude that an ω-simulation in combination with preservation of termination implies a correct transformation as defined by Fokkink and Van de Pol.

In sections 5 and 6 we study the applicability of our notion of ω-simulation and the notion of correct transformation of Fokkink and Van de Pol by means of a notorious transformation of Thatte. This transformation was introduced in 1985 in [Tha85] for the class of regular Term Rewriting Systems. In [Tha88], Thatte claimed that his transformation preserved confluence and (up to some translation on terms) the normal form relation for the class of nonoverlapping systems, but it was shown by Verma in [Ver95b] that this claim was false.

Verma, for his part, tried to fix Thatte's proof by imposing some extra conditions. However, as we will demonstrate, his proof is incorrect. Unfortunately, as of yet, we have not been able to find sufficient evidence that Verma's claim is either true or untrue, so this remains an open question. However, using the correctness criteria of sections 3 and 4, we will repair Verma's proof for a number of significant subclasses of the class that Verma considers, as well as for some other important classes of Term Rewriting Systems.

In appendix A we will improve Verma's definition of the class of weakly persistent systems, as well as his proof that a number of important classes contain only weakly persistent systems.

This paper starts with a section in which we will introduce all the mathematical notions that we will need in the rest. Many of these notions belong to the standard theory of Reduction Systems. We decided to include this section mainly for reasons of self-containedness. However, sections 2.3.2 and 2.3.3 contain some definitions that are new.

2 Preliminaries

This section introduces all the basic theory needed in the rest of the paper. The parts about Abstract Reduction Systems and Term Rewriting Systems are loosely based on [Klo92], with some extensions, changes and reformulations. It is merely meant to fix terminology and notation for the rest of the paper and to make it self-contained. The most important extensions and diversions can be found in sections 2.3.2 and 2.3.3. The definition of paths in terms is adapted from [KM91].

2.1 Relations and Functions

A *relation* over sets X and Y is defined to be any subset of $X \times Y$. If $\mathcal{R}$ is a relation over X and Y, then we define the *domain* of $\mathcal{R}$ by $\mathsf{domain}(\mathcal{R}) = \{x \in X \mid \exists y \in Y(\langle x, y\rangle \in \mathcal{R})\}$ and the *range* of $\mathcal{R}$ by $\mathsf{range}(\mathcal{R}) = \{y \in Y \mid \exists x \in X(\langle x, y\rangle \in \mathcal{R})\}$. If $\mathcal{R}$ is a relation over X and X, then $\mathcal{R}$ is said to be a relation *on* X. Moreover, the *identity relation* on X is defined by $\mathsf{id}(X) = \{\langle x, x\rangle \mid x \in X\}$. If $\mathcal{R}$ is a relation over X and Y and $x \in X$, then we define $\mathcal{R}[x] = \{y \in Y \mid \langle x, y\rangle \in \mathcal{R}\}$; for $A \subseteq X$, we also write $\mathcal{R}[A]$ to denote $\bigcup_{a \in A} \mathcal{R}[a]$. If $|\mathcal{R}[x]| \geq 1$ for all $x \in X$, then $\mathcal{R}$ is said to be *total*; otherwise, $\mathcal{R}$ is said to be *partial*. Furthermore, if $|\mathcal{R}[x]| = 1$, then we use $\mathcal{R}(x)$ to denote the single element of $\mathcal{R}[x]$.

On relations $\mathcal{R} \subseteq X \times Y$ and $\mathcal{S} \subseteq Y \times Z$ we define *composition* ($\circ$) in the usual way, that is, $\mathcal{R} \circ \mathcal{S}$ denotes the relation $\mathcal{R} \circ \mathcal{S} = \{\langle x, z\rangle \mid \exists y \in Y(\langle x, y\rangle \in \mathcal{R} \wedge \langle y, z\rangle \in \mathcal{S}\}$ over X and Z. Composition is associative, that is, $\mathcal{T} \circ (\mathcal{S} \circ \mathcal{R}) = (\mathcal{T} \circ \mathcal{S}) \circ \mathcal{R}$. The *inverse* of a relation $\mathcal{R}$ over X and Y is the relation $\mathcal{R}^{-1}$ over Y and X such that $\langle y, x\rangle \in \mathcal{R}^{-1}$ iff $\langle x, y\rangle \in \mathcal{R}$.

The *restriction* to a set $Z \subseteq X$ of a relation $\mathcal{R}$ over X and Y is the relation $\mathcal{R} \restriction Z$ over $X \cap Z$ and Y defined by $\mathcal{R} \restriction Z = \mathcal{R} \cap (Z \times Y)$.

A relation $\mathcal{R}$ on X is *reflexive* if $\mathsf{id}(X) \subseteq \mathcal{R}$ and *transitive* if $\forall x, y, z \in X(\langle x, y\rangle, \langle y, z\rangle \in \mathcal{R} \Longrightarrow \langle x, z\rangle \in \mathcal{R})$. A relation that is reflexive and transitive is called a *preorder*.

If $\mathcal{R}$ is a relation on X, then the *reflexive closure* (*transitive closure*, *reflexive transitive closure*) of $\mathcal{R}$ is the smallest relation that contains $\mathcal{R}$ and is reflexive (transitive, reflexive transitive).

We say that a relation f over X and Y is a *function (map) from X to Y* (notation: $f : X \to Y$) if $|f[x]| = 1$ for all $x \in \mathsf{domain}(f)$. If $\mathsf{domain}(f) = X$ then f is a *total function*; otherwise it is a *partial function*. If f is a function and $x \in \mathsf{domain}(f)$, then we say that f is *defined for x* and we let $f(x)$ denote the unique element in $f[x]$. On the other hand, if $|f[x]| = 0$, then f is said to be *undefined for x* and we write $f(x)\!\uparrow$. Note that if $f : X \to Y$ and $g : Y \to Z$ are functions, then $f \circ g$ is again a function and $(f \circ g)(x) = g(f(x))$.

If $f : X \to Y$ is a total function and $\forall x_1, x_2 \in \mathsf{domain}(f)(f(x_1) = f(x_2) \Longrightarrow x_1 = x_2)$, then it is called *injective* (an injection). It is *surjective* (a surjection) if $\mathsf{range}(f) = Y$. Finally, it is *bijective* (a bijection) if it is both injective and surjective.

2.2 Abstract Reduction Systems

Definition 2.1 i. An *Abstract Reduction System* (ARS) is a pair $\langle A, (\to_\alpha)_{\alpha \in I}\rangle$ consisting of a set A and a sequence of binary relations $\to_\alpha$ on A (indexed by some set I), also called *reduction* or *rewrite relations*. We reserve $a, b, c, \ldots$ (possibly subscripted) for elements of A, and we use $\to_\alpha, \to_\beta, \to_\gamma, \ldots$ or simply $\to$ to refer to rewrite relations. $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$ (sometimes primed or subscripted) will be used to denote ARSs.

If for $a, b \in A$ we have $\langle a, b\rangle \in\to_\alpha$ then we write $a \to_\alpha b$ and call b a 'one-step (α-)reduct of a'.

ii. The transitive closure of $\to_\alpha$ is written as $\to_\alpha^+$. So $a \to_\alpha^+ b$ if there is a finite sequence of 'reduction steps' (a reduction) $a = a_0 \to_\alpha a_1 \to_\alpha \cdots \to_\alpha a_n = b$, where $n \geq 1$. We also write $a \to_\alpha^n b$ if we want to be explicit about the number of reduction steps.

We write $a \to_\alpha^0 b$ if $a = b$ and denote by $\twoheadrightarrow_\alpha$ the transitive reflexive closure of $\to_\alpha$. That is, $a \twoheadrightarrow_\alpha b$ iff $a \to_\alpha^n b$ for some $n \in \omega$.[1] If $a \twoheadrightarrow_\alpha b$, then b is called a *reduct* of a, and a is said to *reduce to* b.

We write $\leftarrow_\alpha$ for the converse relation of $\to_\alpha$ and this same convention ports to all other 'reduction relations' that we will define. So, for instance, $\twoheadleftarrow_\alpha$ and $\leftarrow_\alpha^+$ denote the converses of $\twoheadrightarrow_\alpha$ and $\to_\alpha^+$.

iii. If an ARS is of the form $\langle A, \to \rangle$, i.e. it involves only one reduction relation ($|I| = 1$), then we call it *simple*. Each ARS $\mathfrak{A} = \langle A, (\to_\alpha)_{\alpha \in I} \rangle$ is straightforwardly mapped onto a simple ARS $\langle A, \bigcup_{\alpha \in I} \to_\alpha \rangle$, where the latter is called the *simple ARS underlying* $\mathfrak{A}$. ⊣

In the sequel we will define some properties for simple ARSs. These properties generalize to arbitrary ARSs as follows:

$\langle A, (\to_\alpha)_{\alpha \in I} \rangle$ has property X if its underlying simple ARS $\langle A, \bigcup_{\alpha \in I} \to_\alpha \rangle$ has property X.

Definition 2.2 Let $\mathfrak{A} = \langle A, \to \rangle$ be an ARS.

i. $a \in A$ is *weakly confluent* (or: *weakly Church-Rosser*) (notation: $\mathfrak{A} \models \mathsf{WCR}(a)$) iff

$$\forall b, c \in A\, \exists d \in A\ (c \leftarrow a \to b \Longrightarrow c \twoheadrightarrow d \twoheadleftarrow b)$$

$\mathfrak{A}$ (or $\to$) is *weakly confluent* ($\mathfrak{A} \models \mathsf{WCR}$) iff $\forall a \in A (\mathfrak{A} \models \mathsf{WCR}(a))$.

ii. $a \in A$ is *confluent* (or: *Church-Rosser*) (notation: $\mathfrak{A} \models \mathsf{CR}(a)$) iff

$$\forall b, c \in A\, \exists d \in A\ (c \twoheadleftarrow a \twoheadrightarrow b \Longrightarrow c \twoheadrightarrow d \twoheadleftarrow b)$$

$\mathfrak{A}$ (or $\to$) is *confluent* ($\mathfrak{A} \models \mathsf{CR}$) iff $\forall a \in A (\mathfrak{A} \models \mathsf{CR}(a))$.

iii. We say that $a \in A$ is a *normal form* if there is no $b \in A$ such that $a \to b$. Furthermore, $b \in A$ *has a normal form* if $b \twoheadrightarrow a$ for some normal form $a \in A$.

iv. $a \in A$ is *weakly terminating* (or: *weakly normalizing*) (notation: $\mathfrak{A} \models \mathsf{WN}(a)$) iff it has a normal form.

$\mathfrak{A}$ (or $\to$) is *weakly normalizing* ($\mathfrak{A} \models \mathsf{WN}$) iff $\forall a \in A (\mathfrak{A} \models \mathsf{WN}(a))$.

v. $a \in A$ is *terminating* (or: *strongly normalizing, noetherian*) (notation: $\mathfrak{A} \models \mathsf{SN}(a)$) iff there is no sequence $(a_n)_{n \in \omega}$, such that $a_0 = a$ and $a_n \to a_{n+1}$.

$\mathfrak{A}$ (or $\to$) is *strongly normalizing* ($\mathfrak{A} \models \mathsf{SN}$) iff $\forall a \in A (\mathfrak{A} \models \mathsf{SN}(a))$.

vi. $a \in A$ has the *unique normal form property* (notation: $\mathfrak{A} \models \mathsf{UN}(a)$) if

$$\forall b, c \in A (a \twoheadrightarrow b \ \&\ a \twoheadrightarrow c \ \&\ b, c \text{ are normal forms} \Longrightarrow b = c)$$

$\mathfrak{A}$ (or $\to$) has the *unique normal form property* ($\mathfrak{A} \models \mathsf{UN}$) if $\forall a \in A (\mathfrak{A} \models \mathsf{UN}(a))$. ⊣

The following results are well-known:

[1] $\to_\alpha^*$ is more customary notation, but in rewriting $\twoheadrightarrow_\alpha$ is fairly common too.

Lemma 2.3 i. $\mathsf{CR} \Longrightarrow \mathsf{UN}$;

ii. $\mathsf{SN}\,\&\,\mathsf{WCR} \Longrightarrow \mathsf{CR}$ (Newman's Lemma);

iii. $\mathsf{UN}\,\&\,\mathsf{WN} \Longrightarrow \mathsf{CR}$. □

$\mathsf{NF}(\mathfrak{A})$ denotes the set of normal forms of $\mathfrak{A}$, i.e. $\mathsf{NF}(\mathfrak{A}) = \{a \in A \mid a \text{ is a normal form}\}$. Furthermore, $\mathsf{nf}_{\mathfrak{A}}$ (the *normal form relation of* $\mathfrak{A}$) denotes the relation over A and $\mathsf{NF}(\mathfrak{A})$ with $\langle a, b\rangle \in \mathsf{nf}_{\mathfrak{A}}$ iff b is a normal form of a. If $a \in A$ reduces to a unique normal form n, then we denote this n by $\mathsf{nf}_{\mathfrak{A}}(a)$; if $\mathfrak{A}$ is UN then $\mathsf{nf}_{\mathfrak{A}}$ is a (partial) function. If $\mathfrak{A}$ is clear from the context, then we just write nf instead of $\mathsf{nf}_{\mathfrak{A}}$.

Sometimes we would like to speak about the normal forms of some particular reduction relation $\to_\alpha$ of an ARS $\mathfrak{A} = \langle A, (\to_\alpha)_{\alpha \in I}\rangle$. Therefore, we denote by NF_α the set of normal forms w.r.t. the reduction relation $\to_\alpha$, i.e. $\mathsf{NF}_\alpha(\mathfrak{A}) = \{a \in A \mid \neg\exists b \in A\,(a \to_\alpha b)\}$. Similarly, if $n \in \mathsf{NF}_\alpha(\mathfrak{A})$ and $a \twoheadrightarrow_\alpha n$, then we have $\mathsf{nf}_\alpha(a) = n$.

2.3 Term Rewriting Systems

A Term Rewriting System (TRS) $\mathfrak{R}$ is a pair $\langle \Sigma, \mathcal{R}\rangle$ of a signature Σ and a set of reduction rules (rewrite rules) $\mathcal{R}$. A signature is a nonempty set of *function symbols*. Associated with every function symbol $f \in \Sigma$ is its arity $n \in \omega$ and we define a map $\mathsf{arity} : \Sigma \to \omega$. If, for $f \in \Sigma$, $\mathsf{arity}(f) = 0$ then we sometimes call f a *constant symbol* (or 'constant' for short).

Furthermore, $\mathcal{V}$ is a countably infinite set of variables, denoted by $x_1, x_2, x_3, \ldots$ or x, y, z.

2.3.1 Terms

The set of terms over Σ $T_{\Sigma,\mathcal{V}}$ is the smallest set satisfying

1. $\mathcal{V} \subseteq T_{\Sigma,\mathcal{V}}$;
2. $f \in \Sigma$, $\mathsf{arity}(f) = 0 \Longrightarrow f \in T_{\Sigma,\mathcal{V}}$;
3. $f \in \Sigma$, $\mathsf{arity}(f) = n$ $(n \geq 1)$ and $t_1, \ldots, t_n \in T_{\Sigma,\mathcal{V}} \Longrightarrow f(t_1, \ldots, t_n) \in T_{\Sigma,\mathcal{V}}$.

Terms not containing variables are called *ground terms* and we write T_Σ to denote the set of ground terms over signature Σ (obtained by leaving out the first condition in the inductive definition above). Terms that contain each variable $x \in \mathcal{V}$ at most once are called *linear*. We define a map $\mathsf{var} : T_{\Sigma,\mathcal{V}} \to \mathcal{P}(\mathcal{V})$, with the property that variable x occurs in term t iff $x \in \mathsf{var}(t)$. In order to make explicit which variables occur in a term t we will sometimes write $t(x_1, \ldots, x_n)$ if $\mathsf{var}(t) \subseteq \{x_1, \ldots, x_n\}$.

Contexts are 'terms' containing an occurrence of a special symbol $\Box$, denoting an empty place. A context is generally denoted by $C[\ ]$. If $t \in T_{\Sigma,\mathcal{V}}$ and t is substituted for $\Box$, then the result is $C[t] \in T_{\Sigma,\mathcal{V}}$ and t is said to be a subterm of $C[t]$ (notation: $t \subseteq C[t]$).

The notion of a *path* (or *occurrence*, or *position*) is used to refer to subterms in a term t. For any term $t \in T_{\Sigma,\mathcal{V}}$ the set of occurrences $O(t)$ in t is inductively defined as follows:

1. $x \in \mathcal{V} \Longrightarrow O(x) = \{\lambda\}$;
2. $f \in \Sigma$, $\mathsf{arity}(f) = 0 \Longrightarrow O(f) = \{\lambda\}$;
3. $f \in \Sigma$, $\mathsf{arity}(f) = n$ $(n \geq 1)$ and $t_1, \ldots, t_n \in T_{\Sigma,\mathcal{V}} \Longrightarrow$
   $O(f(t_1, \ldots, t_n)) = \{\lambda\} \cup \{i.u \mid 1 \leq i \leq n \text{ and } u \in O(t_i)\}$.

Intuitively, a path is either the empty string λ that reaches the root or a sequence $i.p$ (p is a path and i a positive integer) which reaches the subterm at position p in the ith argument of the root. t/p refers to the subterm of t reached by p. We will also use the notation $p.q$ (p and q both paths), where $t/p.q$ denotes the subterm reached by q in the subterm t/p. $o < q$ whenever there is a nonempty path p, such that $o.p = q$; $o \leq q$ whenever $o < q$ or $o = q$ and we say that o and q are *disjoint* if $o \not\leq q$ and $q \not\leq o$. We moreover divide $O(t)$ into a set $O^{\mathcal{F}}(t)$ of nonvariable occurrences and a set $O^{\mathcal{V}}(t)$ of occurrences of variables:

$$O^{\mathcal{F}}(t) = \{o \in O(t) \mid t/o \text{ is a nonvariable subterm of } t\}$$

and

$$O^{\mathcal{V}}(t) = \{o \in O(t) \mid t/o \text{ is a variable}\}$$

If $o \in O^{\mathcal{F}}(t)$, then we will denote by $t(o)$ the outermost symbol of t/o, i.e. if $t/o = f(t_1, \ldots, t_n)$ for some $n \geq 1$ then $t(o) = f$ and if $t/o = a$ (a a constant symbol) then $t(o) = a$. We will say that o is an occurrence of symbol f in t and if $o \neq \lambda$, then o is called an *inner* occurrence of f in t.

A *substitution* is a map $\sigma : T_{\Sigma,\mathcal{V}} \to T_{\Sigma,\mathcal{V}}$ satisfying $\sigma(f(t_1, \ldots, t_n)) = f(\sigma(t_1), \ldots, \sigma(t_n))$ for every n-ary function symbol $f \in \Sigma$ ($n \geq 0$). If σ maps all terms into T_Σ, then it is called a *ground substitution*. For $t \in T_{\Sigma,\mathcal{V}}$ and σ a substitution it is common to write t^σ for $\sigma(t)$. Moreover, if $t \in T_{\Sigma,\mathcal{V}}$, $t = t(x_1, \ldots, x_n)$ and $\sigma(x_i) = t_i$ for $1 \leq i \leq n$, then $t^\sigma = t(x_1, \ldots, x_n)^\sigma = t(t_1, \ldots, t_n)$.

For $s, t \in T_{\Sigma,\mathcal{V}}$ we say that *s overlaps t* if there are substitutions σ, τ and an occurrence $o \in O^{\mathcal{F}}(t)$, such that $\sigma(t/o) = \tau(s)$. Sometimes we need to be explicit about where the overlap takes place. Therefore we also say that *s overlaps t at o*. If $o = \lambda$ then *s root overlaps t* and if $o \neq \lambda$ then *s nonroot overlaps t*.

We extend these notions of overlap to sets of terms. A set $X \subseteq T_{\Sigma,\mathcal{V}}$ is said to be *nonoverlapping* if $\neg\exists s, t \in X$ (s nonroot overlaps t). If $\forall s, t \in X (s \text{ overlaps } t \Longrightarrow s = t)$ then X is said to have *no overlap*.

2.3.2 Rules

$\mathcal{R}$ is a set of *reduction rules*, pairs $\langle l, r\rangle$ with $l, r \in T_{\Sigma,\mathcal{V}}$, usually written as $l \to r$. For convenience we will denote by $\mathcal{L}^{\mathcal{R}}$ the set of lefthand sides of the rules in $\mathcal{R}$, i.e. $\{l \mid \langle l, r\rangle \in \mathcal{R}\}$, also called the *(rule-)base* of $\mathcal{R}$.

The following two conditions are imposed on all $l \to r$ in $\mathcal{R}$:

1. $l \notin \mathcal{V}$;
2. $\mathsf{var}(r) \subseteq \mathsf{var}(l)$.

$\mathcal{R}$ generates a reduction relation $\to_{\mathcal{R}}$ on $T_{\Sigma,\mathcal{V}}$ as follows. For $s, t \in T_{\Sigma,\mathcal{V}}$ we write $s \to_{\mathcal{R}} t$ if there is an occurrence $o \in O(s)$, a substitution σ and a rule $l \to r$ in $\mathcal{R}$, such that $s/o = l^\sigma$ and $t = s[o \leftarrow r^\sigma]$. Here $s[o \leftarrow r^\sigma]$ denotes the result of replacing the term s/o in s at occurrence o by r^σ. To be specific about the position in s at which the reduction takes place, we sometimes write $s \xrightarrow{o}_{\mathcal{R}} t$. In case $o = \lambda$, then we call this a *root reduction* (notation: $s \xrightarrow{r}_{\mathcal{R}} t$), while if $o \neq \lambda$ we call it a *nonroot reduction* (notation: $s \xrightarrow{nr}_{\mathcal{R}} t$).[2] A root reduction is also called a *contraction*.

An instance l^σ of a lefthand side l of a rule $l \to r$ in $\mathcal{R}$ is called a *redex*[3] (or *l-redex*).

[2] $\xrightarrow{r}^{+}_{\mathcal{R}}$, $\xrightarrow{r}^{*}_{\mathcal{R}}$, $\xrightarrow{nr}^{+}_{\mathcal{R}}$ and $\xrightarrow{nr}^{*}_{\mathcal{R}}$ are used to denote transitive and transitive-reflexive closures of $\xrightarrow{r}_{\mathcal{R}}$ and $\xrightarrow{nr}_{\mathcal{R}}$, respectively.

[3] **red**ucible **ex**pression

A *subtemplate* of $\mathcal{R}$ is a nonvariable proper subterm of a lefthand side of a rule in $\mathcal{R}$. An instance of a subtemplate l' of $\mathcal{R}$ is called an *l'-pseudo-redex*, or *pseudo-redex* for short. If s is an l'-pseudo-redex and $s \xrightarrow{o}_{\mathcal{R}} t$ with $o \notin O^{\mathcal{F}}(l')$, then we write $s \xrightarrow[l']{nt}_{\mathcal{R}} t$.

If the specific subtemplate is not important, but exists, we usually drop the subscript and write $\xrightarrow{nt}_{\mathcal{R}}$. The transitive and transitive-reflexive closures of this relation are $\xrightarrow{nt}{}^{+}_{\mathcal{R}}$ and $\xrightarrow{nt}{}^{*}_{\mathcal{R}}$.

Let o be some occurrence in a term s. It is useful to have a notion of *set of descendants of $o \in O(s)$ over a reduction $s \xrightarrow{o_1}_{\mathcal{R}} s_1 \xrightarrow{o_2}_{\mathcal{R}} \cdots \xrightarrow{o_n}_{\mathcal{R}} s_n$* ($\mathsf{desc}(o)$). We inductively define this as follows:

1. If $n = 1$ and $s = C[l(t_1, \ldots, t_m)] \xrightarrow{o_1}_{\mathcal{R}} C[r(t'_1, \ldots, t'_m)]$, then

$$\mathsf{desc}(o) = \begin{cases} \emptyset & \text{if } o = o_1.p \text{ and } p \in O^{\mathcal{F}}(l) \\ \{o_1.p'.q \mid r(p') = x_i\} & \text{if } o = o_1.p.q \text{ and } l/p = x_i \\ \{o_1\} & \text{otherwise} \end{cases}$$

   Note that there may be more than one rule $l \to r$ that applies. If this is the case, then we take the union of the sets of descendants we get via the respective rules.

2. If $n > 1$, and D is the set of descendants of o over the reduction $s \xrightarrow{o_1}_{\mathcal{R}} s_1 \xrightarrow{o_2}_{\mathcal{R}} \cdots \xrightarrow{o_{n-1}}_{\mathcal{R}} s_{n-1}$, then $\mathsf{desc}(o)$ is defined to be $\bigcup_{d \in D} \mathsf{desc}(d)$, where $\mathsf{desc}(d)$ is the set of descendants of d over the reduction $s_{n-1} \xrightarrow{o_n}_{\mathcal{R}} s_n$.

Clearly, $\langle T_{\Sigma,\mathcal{V}}, \to_{\mathcal{R}} \rangle$ is an ARS, and it is called the *underlying* ARS of the TRS $\mathfrak{R}$. Also, via this underlying ARS properties that we defined for ARSs, such as CR and SN are defined for TRSs.

$\mathcal{R}$ also generates a reduction relation on the set of ground terms T_Σ, if we restrict to ground substitutions. The ARS $\langle T_\Sigma, \to_{\mathcal{R}} \rangle$ will be called the *underlying ground* ARS of the TRS $\mathfrak{R}$. Via this ARS properties such as CR and SN also port to TRSs. To avoid confusion we will prefix the names of these properties with the word "ground". For example, we will say that a TRS is CR if the underlying ARS is CR and we will say that a TRS is *ground* CR if the underlying *ground* ARS is CR.

2.3.3 Subclasses

By imposing (syntactic) restrictions on the rules we can define subclasses of the class of TRSs. For instance, assume a TRS $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$:

- if the base $\mathcal{L}^{\mathcal{R}}$ of $\mathcal{R}$ is linear then $\mathfrak{R}$ is called *left-linear*;
- if the base $\mathcal{L}^{\mathcal{R}}$ of $\mathcal{R}$ is nonoverlapping then $\mathfrak{R}$ is called *nonoverlapping*;
- if $\mathcal{R}$ is nonoverlapping and in addition it holds that

$$\forall \langle l, r \rangle, \langle l', r' \rangle \in \mathcal{R}\, \forall \sigma, \tau\ (l^\sigma = (l')^\tau \Longrightarrow r^\sigma = (r')^\tau)$$

  then $\mathfrak{R}$ is called *consistent*;
- if $\mathcal{R}$ is nonoverlapping and in addition the lefthand sides of no two distinct rules have overlap at the root, then $\mathfrak{R}$ is called *nonambiguous*;
- if for $\mathcal{R}$ it holds that

$$\forall \langle l, r \rangle, \langle l', r' \rangle \in \mathcal{R}\, \forall \sigma, \tau\ (l^\sigma = (l')^\tau \Longrightarrow \exists t \in T_{\Sigma,\mathcal{V}} (r^\sigma \twoheadrightarrow_{\mathcal{R}} t \twoheadleftarrow_{\mathcal{R}} (r')^\tau))$$

  then $\mathfrak{R}$ is said to have *confluent root overlaps*.

Observe that on the level of underlying ARSs the properties of consistency and nonambiguity coincide (the class of underlying ARSs corresponding to the class of consistent TRSs equals the class of underlying ARSs corresponding to the class of nonambiguous TRSs). Moreover, we have the following implications:

$$\text{nonambiguity} \Longrightarrow \text{consistency} \Longrightarrow \text{nonoverlapping \& confluent root overlaps}$$

Proposition 2.4 Every nonoverlapping TRS with confluent root overlaps is WCR.

Proof Let $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ be a nonoverlapping TRS with confluent root overlaps, and assume that $t_1 \xleftarrow{o_1}_{\mathcal{R}} s \xrightarrow{o_2}_{\mathcal{R}} t_2$ (with $s, t_1, t_2 \in T_{\Sigma,\mathcal{V}}$). We show that there is a $t \in T_{\Sigma,\mathcal{V}}$ such that $t_1 \twoheadrightarrow_{\mathcal{R}} t \twoheadleftarrow_{\mathcal{R}} t_2$.

We distinguish cases according to the relative positions of $o_1, o_2 \in O(s)$:

$o_1 = o_2$**:** If both contractions are applications of the same rule then we take $t = t_1 = t_2$. If not, then let $l_1 \to r_1$ be the rule applied in $s \xrightarrow{o_1}_{\mathcal{R}} t_1$ and $l_2 \to r_2$ the rule applied in $s \xrightarrow{o_2}_{\mathcal{R}} t_2$. Let $o = o_1 = o_2$. There are substitutions σ, τ such that $s/o = l_1^\sigma = l_2^\tau$, $t_1 = s[o \leftarrow r_1^\sigma]$ and $t_2 = s[o \leftarrow r_2^\sigma]$. Since $\mathfrak{R}$ has confluent root overlaps there is a $t' \in T_{\Sigma,\mathcal{V}}$ such that $r_1^\sigma \twoheadrightarrow_{\mathcal{R}} t' \twoheadleftarrow_{\mathcal{R}} r_2^\sigma$, and clearly, $t_1 \twoheadrightarrow_{\mathcal{R}} s[o \leftarrow t'] \twoheadleftarrow_{\mathcal{R}} t_2$.

$o_1 < o_2$ **or** $o_2 < o_1$**:** If $o_1 < o_2$, then a common reduct of t_1 and t_2 is reached by contracting in t_1 all descendants of o_2 and contracting o_1 in t_2. The case where $o_2 < o_1$ is similar.

o_1 **and** o_2 **disjoint:** Trivial. □

Corollary 2.5 Every nonambiguous (consistent) TRS is WCR.

If a TRS is both nonambiguous and left-linear, then it is called *orthogonal* and if it is both consistent and left-linear, then we call it *regular*.[4]

In term rewriting the classes of orthogonal and regular systems are of special importance, mainly because of the fact that descendants of redexes are still redexes. This leads to the following result:

Corollary 2.6 Every regular TRS is CR.

Proof See [BKM89] for a proof of confluence for orthogonal TRSs (there called regular). It is easy to verify that their proof generalizes to the class of what we call regular systems. □

[4] There is much confusion about this terminology in literature. One often encounters the name ‘regular’ for the class of nonambiguous, left-linear systems. The adherents of the name ‘orthogonal’ tend to call the class of consistent, left-linear systems *almost orthogonal* [Klo96]. Again we would like to point out that the difference between the two classes is minimal and not present on the level of the underlying ARSs.

3 Correctness Criteria

A criterion for correctness of some transformation of ARS $\mathfrak{A} = \langle A, \rightarrow_\alpha \rangle$ into $\mathfrak{B} = \langle B, \rightarrow_\beta \rangle$ essentially consists of two parts.

On the one hand, the elements of A must be in some way connected to those of B. In order to establish such a connection, we will use a pair of functions, one from A to B and one from B to A, and we will identify the transformation with this pair of functions. All possibilities we encountered in literature fit into this framework.

On the other hand, we need a connection between the reduction relations $\rightarrow_\alpha$ and $\rightarrow_\beta$. Especially this part is open to discussion. Until now there has not been much consensus about how to express this connection and we do not believe that such consensus is to be expected in the near future. We classify the various opinions we found roughly into two categories: the *direct approach* and the *indirect approach*.

By the direct approach we mean that the main interest lies in proving that certain properties are invariant under the transformation, such as the normal form relation, confluence, and termination. By its abstract formulation this approach hardly gives any direction for the prover, but it seems to be more powerful in the sense that more transformations can be proven "correct". However, much information about the exact relation that exists between the reduction relations of the original and its transform remains rather implicit.

The indirect approach has more to do with the structure of the reduction relations in terms of the reduction steps. Some connection between the reduction steps is established. Therefore, we could also call this approach *operational* or *behavioural*. It leads to notions that resemble the notions of simulation that occur in Automata Theory and Concurrency Theory (see for example [LV95]).

We will describe both approaches independently in the following sections. However, we feel that such a division does not have to be all that strict. To emphasize this, we will apply a mix of both approaches in our example transformations. The preservation of reduction graphs in combination with a specific mapping between the elements of the two systems has a number of pleasant consequences in terms of the preservation of properties. We will describe and prove these in section 4.

As we did with the properties CR, SN etc., we will define all our correctness notions for simple ARSs. They generalize to arbitrary ARSs with more than one reduction relation in the same way as in section 2.2.

3.1 Mappings

Generally speaking, to express a connection between the objects of the original system $\langle A, \rightarrow_\alpha \rangle$ and its transform $\langle B, \rightarrow_\beta \rangle$ a pair of mappings $\langle \phi, \psi \rangle$ is used. These mappings provide the there and back translations from the original to its transform (ϕ) and from the transform to its original (ψ).[5] We will demand that these translations are consistent with each other for all $a \in A$, in the sense that their composition $\phi \circ \psi$ is the identity function on A.

Definition 3.1 (inspired by **[FP96]**) Let $\mathfrak{A} = \langle A, \rightarrow_\alpha \rangle$ and $\mathfrak{B} = \langle B, \rightarrow_\beta \rangle$ be ARSs.

The pair $\langle \phi, \psi \rangle$, with $\phi : A \rightarrow B$ a total function and $\psi : B \rightarrow A$ a partial function, is called a *simulation mapping* between $\mathfrak{A}$ and $\mathfrak{B}$ if

1. $\mathsf{range}(\phi) \subseteq \mathsf{domain}(\psi)$; and

2. $\phi \circ \psi = \mathsf{id}(A)$. $\dashv$

[5] In [FP96] these mappings are also called `parse` and `print`, respectively.

Note that the definition above implies that ϕ is injective and that $\psi \upharpoonright \mathsf{range}(\phi) = \phi^{-1}$. Therefore, if $\mathsf{domain}(\psi) = \mathsf{range}(\phi)$, then the simulation mapping amounts to $\langle \phi, \phi^{-1} \rangle$, and ϕ acts as a bijection between A and a subset of B. This case will be of special importance in the rest of the paper. We will just write ϕ if we mean the simulation map $\langle \phi, \phi^{-1} \rangle$.

We also would like to mention another special case. It arises if $A \subseteq B$ and $\phi = \mathsf{id}(A)$ (see [KW96]). Such a simulation mapping is useful in transformations on TRSs where the transformed system contains some extended signature and the added symbols are related to symbols in the original system. Actually, ψ then becomes a mapping that translates a term by replacing some new symbols by their originals.

It is very well possible that the transformation introduces a lot of structure in the transform that is by no means related to anything in the original, which we will demonstrate in the following example:

Example 3.1 Let $\mathfrak{A} = \langle \{a, b\}, \{a \to b\} \rangle$ be an ARS.

Define $\mathfrak{A}_1$ by $\langle \{a, b, c\}, \{a \to_\alpha b,\ c \to_\alpha d,\ c \to_\alpha e\} \rangle$. It seems reasonable to consider $\mathfrak{A}_1$ a correct transformation of $\mathfrak{A}$ under the identity simulation map. In that case c, d and e do not play an important rôle in $\mathfrak{A}_1$ when used as a transform of $\mathfrak{A}$. Neither a nor b rewrites to any of these elements and c, d, e are not related to any element in $\mathfrak{A}$. However, c, d, and e do have the effect that $\mathfrak{A}_1$ is not CR.

It is not very useful to consider unreachable parts of some transform in the study of the preservation of properties. Although not relevant for the system as a transform, they do influence the validity of these properties. The next definition provides some notation for systems with unreachable parts removed.

Definition 3.2 (Reachability) Let $\mathfrak{A} = \langle A, \to \rangle$ be an ARS and $X \subseteq A$.

i. By $[A]^{\twoheadrightarrow}_X$ we denote the set A restricted to elements reachable from X by $\twoheadrightarrow$, i.e.

$$[A]^{\twoheadrightarrow}_X := \{a \in A \mid \exists x \in X (x \twoheadrightarrow a)\}$$

ii. Now we define $[\mathfrak{A}]_X$ by $[\mathfrak{A}]_X := \langle [A]^{\twoheadrightarrow}_X,\ \to \cap ([A]^{\twoheadrightarrow}_X \times [A]^{\twoheadrightarrow}_X) \rangle$.

iii. If ϕ is a function with $\mathsf{range}(\phi) \subseteq A$, then we will abbreviate $[\mathfrak{A}]_{\mathsf{range}(\phi)}$ as $[\mathfrak{A}]_\phi$. $\dashv$

3.2 Preservation of Properties

Most authors of papers on transformations and their correctness formulate a requirement regarding the preservation of the normal form relation. This is not surprising, because in automatic computation this relates input and result and in equational reasoning normal forms are used as the representatives of equivalence classes.

We adapt a definition of Fokkink and Van de Pol [FP96] for our purposes.

Definition 3.3 (Normal Form Preservation) Let $\mathfrak{A} = \langle A, \to \rangle$ be an ARS with transform $\mathfrak{B}$ and $\langle \phi, \psi \rangle$ a simulation mapping between $\mathfrak{A}$ and $\mathfrak{B}$.

$\langle \phi, \psi \rangle$ is said to *preserve the normal form relation* if for all $a \in A$:

$$\{\psi(b) \mid b \in \mathsf{nf}_{\mathfrak{B}}[\phi(a)]\} = \mathsf{nf}_{\mathfrak{A}}[a]$$

$\langle \phi, \psi \rangle$ is said to *weakly preserve the normal form relation* if for all $a \in A$:

$$\{\psi(b) \mid b \in \mathsf{nf}_{\mathfrak{B}}[\phi(a)]\} \subseteq \mathsf{nf}_{\mathfrak{A}}[a] \qquad \dashv$$

Such normal form preservation is usually not considered sufficient as a correctness criterion. Consider the following:

Example 3.2 Let $\mathfrak{A} = \langle\{a,b\},\{a \to b\}\rangle$ be an ARS. It is clear that $\mathfrak{A}$ is SN and CR.

The simulation mapping $\mathsf{id}(\{a,b\})$ from $\mathfrak{A}$ to the ARS $\mathfrak{A}_1 = \langle\{a,b,c\},\{a \to_\alpha c,\ c \to_\alpha c,\ c \to_\alpha b\}\rangle$ preserves the normal form relation. However, rewriting a in $\mathfrak{A}$ results in normal form b, while in $\mathfrak{A}_1$ this computation may never yield a result; it may loop in c. This behaviour is very undesirable if $\mathfrak{A}_1$ is used as an implementation of $\mathfrak{A}$ that is executed on a computer.

Furthermore, $\mathfrak{A}_1$ is still confluent, which cannot be said of $\mathfrak{A}_2 = \langle\{a,b,c\},\{a \to_\alpha c,\ c \to_\alpha c,\ a \to_\alpha b\}\rangle$, for which the same mapping is also normal form preserving.

To avoid situations as described in the above example Fokkink and Van de Pol add to their principal correctness criterion a requirement about the termination behaviour of the elements of $\mathsf{range}(\phi)$.

Definition 3.4 Let $\mathfrak{A} = \langle A, \to\rangle$ be an ARS with transform $\mathfrak{B}$ and $\langle\phi,\psi\rangle$ a simulation mapping between $\mathfrak{A}$ and $\mathfrak{B}$.

$\langle\phi,\psi\rangle$ is said to *preserve termination* if for all $a \in A$: $\mathfrak{A} \models \mathsf{SN}(a) \Longrightarrow \mathfrak{B} \models \mathsf{SN}(\phi(a))$.

$\langle\phi,\psi\rangle$ is said to *preserve weak termination* if for all $a \in A$: $\mathfrak{A} \models \mathsf{WN}(a) \Longrightarrow \mathfrak{B} \models \mathsf{WN}(\phi(a))$. ⊣

If the transformation preserves weak termination then the transform of a weakly normalizing ARS is not necessarily weakly normalizing. For instance, the simulation mapping $\mathsf{id}(\{a,b\})$ between $\mathfrak{A}$ and $\mathfrak{A}_2$ in Example 3.2 preserves weak termination, but $\mathfrak{A}_2$ is not WN while $\mathfrak{A}$ clearly is.

However, for termination preserving transformations we do have such a result:

Proposition 3.5 If $\mathfrak{A} = \langle A, \to_\alpha\rangle$ is an ARS with transform $\mathfrak{B} = \langle B, \to_\beta\rangle$ and $\langle\phi,\psi\rangle$ a termination preserving simulation mapping between $\mathfrak{A}$ and $\mathfrak{B}$, then $\mathfrak{A} \models \mathsf{SN} \Longrightarrow [\mathfrak{B}]_\phi \models \mathsf{SN}$.

Proof Left to the reader. □

If a simulation mapping preserves the normal form relation and termination it is a *correct transformation*, according to Fokkink and Van de Pol. They call a transformation *weakly correct* if the simulation mapping between original and transform weakly preserves the normal form relation and furthermore preserves both termination and weak termination.

Thatte [Tha85, Tha88] and Verma [Ver95b] base correctness of a transformation mainly on the preservation of confluence. They are not very explicit about conservation of termination (although this holds for the transformation they consider) and concentrate on proving that the transformation preserves confluence.

Definition 3.6 Let $\mathfrak{A} = \langle A, \to\rangle$ be an ARS with transform $\mathfrak{B}$ and $\langle\phi,\psi\rangle$ a simulation mapping between $\mathfrak{A}$ and $\mathfrak{B}$.

$\langle\phi,\psi\rangle$ is said to *preserve confluence* if for all $a \in A$: $\mathfrak{A} \models \mathsf{CR}(a) \Longrightarrow \mathfrak{B} \models \mathsf{CR}(\phi(a))$.

$\langle\phi,\psi\rangle$ is said to *preserve weak confluence* if for all $a \in A$: $\mathfrak{A} \models \mathsf{WCR}(a) \Longrightarrow \mathfrak{B} \models \mathsf{WCR}(\phi(a))$. ⊣

For confluence preserving transformations we have:

Proposition 3.7 If $\mathfrak{A} = \langle A, \to_\alpha\rangle$ is an ARS with transform $\mathfrak{B} = \langle B, \to_\beta\rangle$ and $\langle\phi,\psi\rangle$ a confluence preserving simulation mapping between $\mathfrak{A}$ and $\mathfrak{B}$, then $\mathfrak{A} \models \mathsf{CR} \Longrightarrow [\mathfrak{B}]_\phi \models \mathsf{CR}$.

Proof Contraposition. Suppose $[\mathfrak{B}]_\phi \models \mathsf{CR}$ does not hold. Then there is some reachable object b such that $[\mathfrak{B}]_\phi \models \mathsf{CR}(b)$ does not hold, and because b is reachable there is some $a \in A$, such that $\phi(a) \twoheadrightarrow_\beta b$. Clearly $[\mathfrak{B}]_\phi \models \mathsf{CR}(\phi(a))$ does not hold and thus $\mathfrak{A} \models \mathsf{CR}(a)$ does not hold. Therefore $\mathfrak{A} \models \mathsf{CR}$ does not hold. □

For a weak confluence preserving transformation it does not necessarily hold that the transform of a weakly confluent original is again weakly confluent:

Example 3.3 Consider the ARSs $\mathfrak{A}_1 = \langle\{a,b\},\{a \rightarrow_\alpha b\}\rangle$ and $\mathfrak{A}_2 = \langle\{a,b,c,d\},\{a \rightarrow_\beta c,\ c \rightarrow_\beta b,\ c \rightarrow_\beta d\}\rangle$ with simulation mapping $\mathsf{id}(\{a,b\})$ between them. Note that $\mathfrak{A}_1 \models \mathsf{WCR}$ and $\mathfrak{A}_2 \models \mathsf{WCR}(a)$, $\mathfrak{A}_2 \models \mathsf{WCR}(b)$, so $\mathsf{id}(\{a,b\})$ preserves weak confluence. Furthermore, the objects c, d are reachable from a in $\mathfrak{A}_2$, but $\mathfrak{A}_2 \models \mathsf{WCR}(c)$ does not hold, so $[\mathfrak{A}_2]_{\mathsf{id}(\{a,b\})} \models \mathsf{WCR}$ does not hold.

3.3 Preservation of Reduction Graphs

Kamperman and Walters advocate the use of a form of preservation of reduction graphs as correctness criterion. They call a simulation map $\langle\phi,\psi\rangle$ between $\mathfrak{A}$ and its transform $\mathfrak{B}$ *sound* if each reduction in $\mathfrak{B}$ simulates a reduction in $\mathfrak{A}$ and *complete* if each reduction in $\mathfrak{A}$ is actually simulated by a reduction in $\mathfrak{B}$. For their original definitions we refer to [KW96]. Their approach is interesting, because it turns out that if one wants correctness via the direct approach of the previous section one often needs to prove soundness and correctness as lemmas (see both Thatte [Tha85] and Verma [Ver95b]).

We will define a notion of *preservation of reduction graphs* that is based on the notions of Kamperman and Walters, but differs slightly. In the next section we will provide further justification for the claim that such reduction conservation in combination with a simulation mapping of the form $\langle\phi,\phi^{-1}\rangle$ is useful as a correctness criterion, by studying its consequences.

Definition 3.8 (Preservation of Reduction Graphs) Let $\mathfrak{A} = \langle A, \rightarrow_\alpha\rangle$ be an ARS with transform $\mathfrak{B} = \langle B, \rightarrow_\beta\rangle$ and $\langle\phi,\psi\rangle$ a simulation map between $\mathfrak{A}$ and $\mathfrak{B}$.

The transformation is said to *preserve reduction graphs* if the following two things hold:

1. if $a \rightarrow_\alpha b$ $(a,b \in A)$, then $\phi(a) \rightarrow_\beta^+ \phi(b)$;

2. if $a \rightarrow_\beta^+ b$ for $a \in \mathsf{domain}(\psi)$ and $b \in B$, then $\exists c \in \mathsf{domain}(\psi)(\psi(a) \rightarrow_\alpha^+ \psi(c) \wedge b \twoheadrightarrow_\beta c)$. ⊣

4 Simulation

In this section we will relate the direct and the indirect approach as described in the previous section. We will study what it means for a simulation mapping of the form $\langle \phi, \phi^{-1} \rangle$ to preserve reduction graphs. It will turn out that for such a simulation mapping we get the preservation of the normal form relation and confluence for free. An extra advantage of this notion is the internal resemblance between the original and its transform. First we introduce some extra terminology.

Definition 4.1 (ω-**simulation**[6]) Let $\mathfrak{A} = \langle A, \rightarrow_\alpha \rangle$ and $\mathfrak{B} = \langle B, \rightarrow_\beta \rangle$ be ARSs, and $\phi : A \rightarrow B$ a simulation mapping between $\mathfrak{A}$ and $\mathfrak{B}$.

We say that $\mathfrak{B}$ *ω-simulates $\mathfrak{A}$ via* ϕ (notation: $\mathfrak{A} \lesssim_{\omega}^{\phi} \mathfrak{B}$) if the following holds:

1. if $a \rightarrow_\alpha b$ $(a, b \in A)$, then $\phi(a) \rightarrow_\beta^+ \phi(b)$;

2. if $a \rightarrow_\beta^+ b$ for $a \in \mathsf{range}(\phi)$ and $b \in B$, then $\exists c \in \mathsf{range}(\phi)(\phi^{-1}(a) \rightarrow_\alpha^+ \phi^{-1}(c) \wedge b \twoheadrightarrow_\beta c)$.

If $\mathfrak{A} \lesssim_{\omega}^{\phi} \mathfrak{B}$ then ϕ is called an *ω-simulation.* Moreover, we will just write $\mathfrak{B}$ ω-simulates $\mathfrak{A}$ ($\mathfrak{A} \lesssim_{\omega} \mathfrak{B}$) if there is an ω-simulation between $\mathfrak{A}$ and $\mathfrak{B}$. $\dashv$

4.1 A Preorder

It is crucial that the composition of two correct transformations is again a correct transformation, i.e. the correctness criterion looked upon as a relation on the class of (simple) ARSs must be transitive. This may considerably alleviate the task of proving a transformation correct, because sometimes a complicated transformation can be divided into a number of independent transformations that are easier to prove correct and that are performed one after the other to yield the same result. This method is know as 'stepwise refinement' (see [LV95]).

It is trivial that all criteria defined in section 3.2 induce preorders (transitive reflexive relations) on the class of ARSs. It is moreover easy to see that ω-simulation is reflexive. Transitivity is a little less clear.

Lemma 4.2 Let $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ be ARSs and ϕ, ψ simulation mappings between $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{B}, \mathfrak{C}$ respectively.
If $\mathfrak{A} \lesssim_{\omega}^{\phi} \mathfrak{B} \lesssim_{\omega}^{\psi} \mathfrak{C}$ then $\mathfrak{A} \lesssim_{\omega}^{\phi \circ \psi} \mathfrak{C}$.

Proof We have to demonstrate that the conditions of Definition 4.1 hold for $\chi = \phi \circ \psi$.
Note that χ is an injection.
It is clear that the first condition holds between $\mathfrak{A}$ and $\mathfrak{C}$ under mapping χ. We verify the second condition. Assume $a \rightarrow_\gamma^+ b$ for $a \in \mathsf{range}(\chi)$ and $b \in C$. We need to derive the existence of a $c \in \mathsf{range}(\chi)$ such that $\chi^{-1}(a) \rightarrow_\alpha^+ \chi^{-1}(c)$ and $b \twoheadrightarrow_\gamma c$. Because $\mathfrak{B} \lesssim_{\omega}^{\psi} \mathfrak{C}$, there is a $c_2 \in \mathsf{range}(\psi)$ with $\psi^{-1}(a) \rightarrow_\beta^+ \psi^{-1}(c_2)$ and $b \twoheadrightarrow_\gamma c_2$, and because $\mathfrak{A} \lesssim_{\omega}^{\phi} \mathfrak{B}$, there is a $c_1 \in \mathsf{range}(\phi)$, such that $\psi^{-1}(c_2) \twoheadrightarrow_\beta c_1$ and $\phi^{-1}(\psi^{-1}(a)) \rightarrow_\alpha^+ \phi^{-1}(c_1)$. It now remains to show that $b \twoheadrightarrow_\gamma \psi(c_1)$. Note that from $\psi^{-1}(c_2) \twoheadrightarrow_\beta c_1$ and the first condition satisfied by $\mathfrak{B} \lesssim_{\omega}^{\psi} \mathfrak{C}$ we find that $c_2 \twoheadrightarrow_\gamma \psi(c_1)$. Combining this reduction with $b \twoheadrightarrow_\gamma c_2$, we find $b \twoheadrightarrow_\gamma \psi(c_1)$, so we may take $c = \psi(c_1)$. $\square$

Corollary 4.3 ω-simulation is a preorder on the class of ARSs. $\square$

[6]We use the name "ω-simulation" because of the familiarity of this notion with the simulation notions that occur in Concurrency Theory (see [LV95]). The ω emphasizes the fact that possibly one step in the original is simulated by infinitely many steps in the transform.

4.2 Preservation of Normal Forms

ω-simulation implies a very tight relationship between the normal form relations of the two systems.

Proposition 4.4 Let $\mathfrak{A} = \langle A, \to_\alpha \rangle$, $\mathfrak{B} = \langle B, \to_\beta \rangle$ be ARSs, with $\mathfrak{A} \overset{\omega}{\rightrightarrows}^{\phi} \mathfrak{B}$.
Then for $a, b \in A$ we have:

i. $n \in \mathsf{NF}(\mathfrak{A})$ iff $\phi(n) \in \mathsf{NF}(\mathfrak{B})$.

ii. If $a \twoheadrightarrow_\beta n$, with $n \in \mathsf{NF}(\mathfrak{B})$ and $a \in \mathsf{range}(\phi)$, then $n \in \mathsf{range}(\phi)$ and $\phi^{-1}(a) \twoheadrightarrow_\alpha \phi^{-1}(n)$.

Proof

i. If $\phi(n) \notin \mathsf{NF}(\mathfrak{B})$, then $\phi(n) \to_\beta b$ for some $b \in B$, and using the second condition of Definition 4.1 we find that there must be a $c \in \mathsf{range}(\phi)$, such that $n \to_\alpha^+ \phi^{-1}(c)$, so $n \notin \mathsf{NF}(\mathfrak{A})$.
On the other hand, if $n \notin \mathsf{NF}(\mathfrak{A})$ then there exists $b \in A$ such that $n \to_\alpha b$. By the first condition of Definition 4.1 we find $\phi(n) \to^+ \phi(b)$, so $\phi(n) \notin \mathsf{NF}(\mathfrak{B})$.

ii. Assume $a \twoheadrightarrow_\beta n$, with $n \in \mathsf{NF}(\mathfrak{B})$ and $a \in \mathsf{range}(\phi)$.
If $a = n$, then $\phi^{-1}(a) \twoheadrightarrow_\alpha \phi^{-1}(n)$ and if $a \to_\beta^+ n$, then by the second condition of Definition 4.1 $\exists c \in \mathsf{range}(\phi)(\phi^{-1}(a) \to_\alpha^+ \phi^{-1}(c) \wedge n \twoheadrightarrow_\beta c)$, and $n = c$ because $n \in \mathsf{NF}(\mathfrak{B})$. □

Corollary 4.5 Let $\mathfrak{A}$ and $\mathfrak{B}$ be ARSs such that $\mathfrak{A} \overset{\omega}{\rightrightarrows}^{\phi} \mathfrak{B}$. Then ϕ preserves the normal form relation.

Proof Straightforward by the previous proposition. □

4.3 Preservation of Confluence

An arbitrary simulation mapping $\langle \phi, \psi \rangle$ that preserves reduction graphs does not necessarily preserve confluence:

Example 4.1 Consider ARSs $\langle \{a, b\}, \{a \to_\alpha b\} \rangle$ and $\langle \{c, d_1, d_2\}, \{c \to_\beta d_1,\ c \to_\beta d_2\} \rangle$ and define ϕ and ψ by:

$$\begin{array}{rclcrcl} \phi(a) & = & c & \quad & \psi(c) & = & a \\ \phi(b) & = & d_1 & & \psi(d_1) & = & b \\ & & & & \psi(d_2) & = & b \end{array}$$

It is clear that $\langle \phi, \psi \rangle$ preserves reduction graphs as defined in Definition 3.8. Furthermore, it is clear that $\mathfrak{B} = [\mathfrak{B}]_\phi$. However, $\mathfrak{A} \models \mathsf{CR}(a)$ while it is not the case that $\mathfrak{B} \models \mathsf{CR}(\phi(a))$.

The problem in the above counterexample lies in the fact that $\psi \neq \phi^{-1}$. If we assume a mapping of the form $\langle \phi, \phi^{-1} \rangle$ then we exclude the above example and w.r.t. confluence the situation takes a pleasant turn.

Theorem 4.6 (Confluence) Let $\mathfrak{A} = \langle A, \to_\alpha \rangle$, $\mathfrak{B} = \langle B, \to_\beta \rangle$ be ARSs, with $\mathfrak{A} \overset{\omega}{\rightrightarrows}^{\phi} \mathfrak{B}$.
Then we have for all $a \in A$: $\mathfrak{A} \models \mathsf{CR}(a) \Longrightarrow \mathfrak{B} \models \mathsf{CR}(\phi(a))$.

Proof Fix an $a \in A$ and assume $\mathsf{CR}(a)$ and $c \twoheadleftarrow_\beta \phi(a) \twoheadrightarrow_\beta b$, with $b, c \in [B]^{\twoheadrightarrow_\beta}_{\mathsf{range}(\phi)}$.
If $\phi(a) \to_\beta^0 b$, or $\phi(a) \to_\beta^0 c$ then there is nothing to prove, so assume $\phi(a) \to_\beta^+ b$ and $\phi(a) \to_\beta^+ c$. Then by the second condition of Definition 4.1 there are $d_1, d_2 \in A$ such that $a \to_\alpha^+ d_1$, $a \to_\alpha^+ d_2$, $b \twoheadrightarrow_\beta \phi(d_1)$ and $b \twoheadrightarrow_\beta \phi(d_2)$. Because $\mathfrak{A} \models \mathsf{CR}(a)$, d_1 and d_2 must have a common reduct d, i.e. $d_1 \twoheadrightarrow_\alpha d \twoheadleftarrow_\alpha d_2$, which, by the first

condition of Definition 4.1 and a simple induction argument, implies that we also have $\phi(d_1) \twoheadrightarrow_\beta \phi(d) \twoheadleftarrow_\beta \phi(d_2)$. In other words, b and c have a common reduct in $\mathfrak{B}$, so $\mathfrak{B} \models \mathsf{CR}(\phi(a))$. $\square$

The reverse does not hold:

Example 4.2 Consider the following ARSs (see also Figure 1 below):

$$\begin{array}{rcll}
\mathfrak{A} & = & \langle & \{a\} \cup \{b_i, c_i \mid i \in \omega\}, \\
 & & & \{a \rightarrow_\alpha b_0,\ a \rightarrow_\alpha c_0\} \cup \{b_i \rightarrow_\alpha b_{i+1},\ c_i \rightarrow_\alpha c_{i+1} \mid i \in \omega\} \quad \rangle \text{ and} \\
\mathfrak{A}' & = & \langle & \{a\} \cup \{b_i, c_i, d_i \mid i \in \omega\}, \\
 & & & \{a \rightarrow_\beta b_0,\ a \rightarrow_\beta c_0\} \cup \{b_i \rightarrow_\beta b_{i+1},\ c_i \rightarrow_\beta c_{i+1} \mid i \in \omega\} \cup \\
 & & & \{b_i \rightarrow_\beta d_i, c_i \rightarrow_\beta d_i, d_i \rightarrow_\beta b_{i+2}, d_i \rightarrow_\beta c_{i+2} \mid i \in \omega\} \quad \rangle
\end{array}$$

Clearly, all objects of $\mathfrak{A}'$ are reachable. We will denote the object set of $\mathfrak{A}$ by A and we show that $\mathfrak{A} \overset{\omega}{\rightsquigarrow}^{\mathsf{id}(A)} \mathfrak{A}'$.

Note that each reduction in $\mathfrak{A}$ is also a reduction in $\mathfrak{A}'$ (this verifies the first condition) and that each d_i has a reduction to all b_j, c_j with $j > i + 1$. This means that:

1. each $a \rightarrow_\beta^+ d_i$ can be followed by $d_i \rightarrow_\beta b_{i+2}$ to simulate $a \rightarrow_\alpha^+ b_{i+2}$;
2. each $b_i \rightarrow_\beta^+ d_j$ can be followed by $d_j \rightarrow_\beta b_{j+2}$ to simulate $b_i \rightarrow_\alpha^+ b_{j+2}$;
3. each $c_i \rightarrow_\beta^+ d_j$ can be followed by $d_j \rightarrow_\beta c_{j+2}$ to simulate $c_i \rightarrow_\alpha^+ c_{j+2}$.

Moreover it is clear that $\mathsf{CR}(a)$ does not hold in $\mathfrak{A}$, while in $\mathfrak{A}'$ each pair of objects has a common reduct.

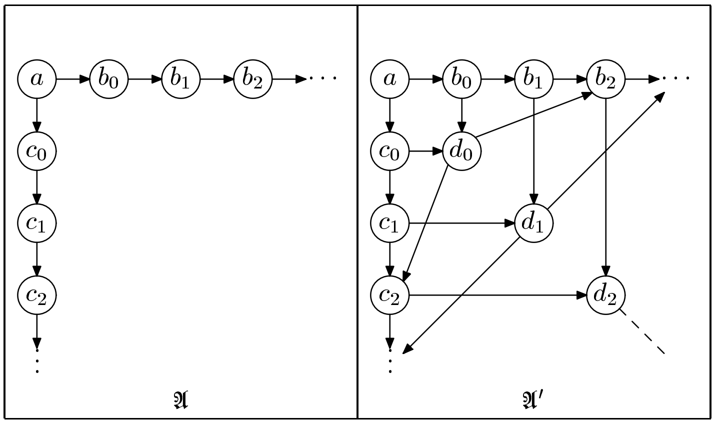


Figure 1: Counterexample: ω-simulation does not preserve confluence in the reverse direction.

ω-simulation does not preserve weak confluence in either direction. Observe that by Theorem 4.6 we already know that to show this, we need a system with at least one element that is WCR, but not CR. We use the famous counterexample by R. Hindley that shows that WCR does not imply CR.

Example 4.3 Consider the following ARSs (see also Figure 2):

$$\begin{array}{rcl}
\mathfrak{A} & = & \langle \{a, b, c, d\}, \{b \rightarrow_\alpha a,\ b \rightarrow_\alpha c,\ c \rightarrow_\alpha b,\ c \rightarrow_\alpha d\} \rangle \\
\mathfrak{A}' & = & \langle \{a, b, c, d\}, \{b \rightarrow_\beta a,\ b \rightarrow_\beta c,\ c \rightarrow_\beta b,\ c \rightarrow_\beta d,\ b \rightarrow_\beta d\} \rangle
\end{array}$$

It is easy to see that $\mathfrak{A} \lessapprox^{\phi} \mathfrak{A}'$, as well as $\mathfrak{A}' \lessapprox^{\phi} \mathfrak{A}$, where $\phi = \mathsf{id}(\{a,b,c,d\})$. Note that operationally, $b \to_\beta d$ adds nothing to the original system. It is just a shortcut for the reduction $b \to_\alpha c \to_\alpha d$.

Neither of the systems is CR, but obviously $\mathfrak{A} \models \mathsf{WCR}$, while $\mathfrak{A}' \models \mathsf{WCR}(b)$ does not hold; we have $a \leftarrow_\beta b \to_\beta d$, but a and d have no common reduct.

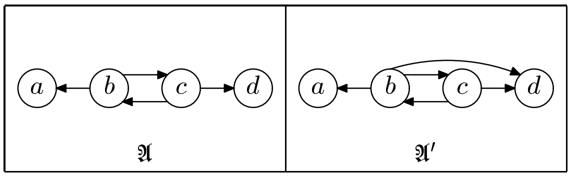


Figure 2: Counterexample: ω-simulation does not preserve WCR in either direction.

4.4 Preservation of Termination

Unfortunately, the existence of an ω-simulation does not imply that the simulation map preserves termination. We obtain the following counterexample:

Example 4.4 Let $\mathfrak{A} = \langle \{a,c\}, \{a \to_\alpha c\}\rangle$ and $\mathfrak{A}' = \langle \{a,b,c\}, \{a \to_\beta b,\ b \to_\beta b,\ b \to_\beta c\}\rangle$ be ARSs. Then $\mathfrak{A} \lessapprox^{\phi} \mathfrak{A}'$, with $\phi = \mathsf{id}(\{a,c\})$, because the only reduction possible in $\mathfrak{A}$ is $a \to_\alpha c$ and in $\mathfrak{A}'$ we have $a \to_\beta^+ c$ by $a \to_\beta b \to_\beta c$ and that $b \twoheadrightarrow_\beta c$, so all reductions $a \to_\beta^+ b$, i.e.

$$\begin{array}{l} a \to_\beta b \\ a \to_\beta b \to_\beta b \\ a \to_\beta b \to_\beta b \to_\beta b \\ \qquad\quad \vdots \end{array}$$

can be followed by $b \twoheadrightarrow_\beta c$. It is also easy to see that $[\mathfrak{A}']_\phi = \mathfrak{A}'$, but $\mathfrak{A} \models \mathsf{SN}(a)$, while $\mathfrak{A}'$ has the infinite reduction $b \to_\beta b \to_\beta b \to_\beta \cdots$.

However, ω-simulation does preserve weak termination in both directions:

Theorem 4.7 (Weak Termination) Let $\mathfrak{A} = \langle A, \to_\alpha\rangle$, $\mathfrak{B} = \langle B, \to_\beta\rangle$ be ARSs, with $\mathfrak{A} \lessapprox^{\phi} \mathfrak{B}$. Then we have for all $a \in A$: $\mathfrak{A} \models \mathsf{WN}(a) \iff \mathfrak{B} \models \mathsf{WN}(\phi(a))$.

Proof

- ($\Rightarrow$) Suppose that $\mathfrak{A} \models \mathsf{WN}(a)$ for some $a \in A$. Then $a \twoheadrightarrow_\alpha n$, where n is some normal form, and thus $\phi(a) \twoheadrightarrow_\beta \phi(n)$ (by the first condition of Definition 4.1 and induction) and $\phi(n) \in \mathsf{NF}(\mathfrak{B})$ (by Proposition 4.4(i)). Thus, $\mathfrak{B} \models \mathsf{WN}(\phi(a))$.
- ($\Leftarrow$) If $\mathfrak{B} \models \mathsf{WN}(\phi(a))$, then $\phi(a) \twoheadrightarrow_\beta n$ for some $n \in \mathsf{NF}(\mathfrak{B})$. By Proposition 4.4(i–ii) $a \twoheadrightarrow_\alpha \phi^{-1}(n)$ and $\phi^{-1}(n) \in \mathsf{NF}(\mathfrak{A})$, so $\mathfrak{A} \models \mathsf{WN}(a)$. □

Moreover, note that if $\mathfrak{A} \lessapprox^{\phi} \mathfrak{B}$ then by the second condition of Definition 4.1 every reachable object reduces to an object in the range of ϕ, so we get:

Corollary 4.8 Let $\mathfrak{A} = \langle A, \to_\alpha\rangle$, $\mathfrak{B} = \langle B, \to_\beta\rangle$ be ARSs, with $\mathfrak{A} \lessapprox^{\phi} \mathfrak{B}$.
Then we have: $\mathfrak{A} \models \mathsf{WN} \iff \mathfrak{B} \models \mathsf{WN}$. □

4.5 Correctness of Transformations

Recall that Fokkink and Van de Pol [FP96] call a transformation *correct* if it preserves both the normal form relation and termination. They called a transformation *weakly correct* if it weakly preserved the normal form relation and preserved both weak termination and termination.

Note that by Theorem 4.7 ω-simulation preserves weak termination. However, as Example 4.4 demonstrates, it does not preserve termination, so according to Fokkink and Van de Pol, ω-simulation is not a sufficient correctness criterion. ω-simulation does however preserve the normal form relation (Corollary 4.5).

Theorem 4.9 Assume a transformation of $\mathfrak{A}$ into $\mathfrak{B}$, where $\mathfrak{A}, \mathfrak{B}$ are ARSs and ϕ a simulation map between $\mathfrak{A}$ and $\mathfrak{B}$. Then:

$$(\mathfrak{A} \rightrightarrows_{\omega}^{\phi} \mathfrak{B} + \phi \text{ preserves termination}) \Longrightarrow \text{the transformation is correct} \qquad \square$$

Note furthermore that ω-simulation in combination with preservation of termination is essentially stronger then the transformation correctness of [FP96]. This follows from the fact that correctness of transformation does not imply preservation of confluence (see Example 4.1), while ω-simulation does (Theorem 4.6). The following example is to emphasize that this not only comes from the fact that in Example 4.1 a simulation map with $\psi \neq \phi^{-1}$ is used:

Example 4.5 Define $\mathfrak{A} = \langle\{a, b\}, \{a \to_\alpha a,\ a \to_\alpha b\}\rangle$ and $\mathfrak{B} = \langle\{a, b, c\}, \{a \to_\alpha b,\ a \to_\alpha c,\ c \to_\alpha c\}\rangle$. Note that $\phi = \mathsf{id}(\{a, b\})$ preserves termination and the normal form relation, but there is no ω-simulation between these two systems.

5 Thatte's Transformation for Left-Linear Systems

Thatte's transformation (named after its inventor Satish R. Thatte) is a transformation on TRSs that has been the subject of some discussion in literature. In 1985, Thatte presented a transformation on the restricted class of regular TRSs turning an arbitrary regular system into a constructor-based system [Tha85]. He proved two simple lemmas stating that each rewrite step in the original system could still be performed in the transform (possibly in more steps), and that (up to some translation on terms) every reduction in the transform corresponds with a reduction in the original system. These two lemmas were to support Thatte's conclusion that every regular system could be 'simulated' by a constructor-based regular system.

In this section we will discuss Thatte's transformation on the class of left-linear, nonoverlapping systems (a superclass of the class of regular systems). We will prove that the transformation preserves termination, and that the transform ω-simulates its original provided that the latter is nonoverlapping and left-linear.

5.1 The Transformation

The aim of Thatte's transformation is to obtain systems that are constructor-based. The signature of these systems is partitionable into disjoint sets of *defined symbols* (appearing only as outermost function symbols of lefthand sides of rules) and *constructor symbols* (not appearing as outermost symbols).

This is not only interesting for reasons of clarity. Regular constructor-based systems are important in the context of strong sequentiality, a property that implies the existence of a normalizing reduction strategy. This property is decidable for the class of all regular systems, but the decision procedure is complicated. Thatte in [Tha87] introduces an alternative in the form of 'left sequentiality' for which the decision procedure is much simpler. He proves that on the class of regular constructor-based systems this property coincides with strong sequentiality. Klop and Middeldorp conjecture that this is not the case for the general class of regular systems [KM91].

Definition 5.1 Let $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ be a TRS.
We define the set of *defined symbols* of $\mathfrak{R}$ by

$$\mathcal{F}^{\mathcal{R}} = \{f \in \Sigma \mid \exists t_1, \ldots, t_n \in T_{\Sigma,\mathcal{V}}(f(t_1, \ldots, t_n) \in \mathcal{L}^{\mathcal{R}})\}$$

and the set of *argument symbols* of $\mathfrak{R}$ by

$$\mathcal{A}^{\mathcal{R}} = \{l(o) \mid l \in \mathcal{L}^{\mathcal{R}} \wedge o \in O^{\mathcal{F}}(l) \wedge o \neq \lambda\}$$

Denote by $\mathcal{C}^{\mathcal{R}}$ the set of *constructor symbols* (*constructors*) of $\mathcal{R}$, that is, $\mathcal{C}^{\mathcal{R}} = \Sigma - \mathcal{F}^{\mathcal{R}}$.
$\mathfrak{R}$ is called *constructor-based* if $\mathcal{A}^{\mathcal{R}} \subseteq \mathcal{C}^{\mathcal{R}}$ (or equivalently: $\mathcal{F}^{\mathcal{R}} \cap \mathcal{A}^{\mathcal{R}} = \emptyset$).
If no confusion can arise, then we will drop the superscript $\mathcal{R}$. ⊣

Systems that are not constructor-based have a symbol that is both defined and appears in some argument of a lefthand side, i.e. such a symbol has a dual rôle. The idea is to remove this duality by adding a fresh symbol c_f for each $f \in \mathcal{F} \cap \mathcal{A}$, that will act as the constructor variant of f (take over the rôle of f as a constructor).

Assume a TRS $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ that is not constructor-based. Thatte's transform $\mathfrak{R}^\sharp = \langle \Sigma^\sharp, \mathcal{R}^\sharp \rangle$ of $\mathfrak{R}$ is a system constructed from $\mathfrak{R}$ as follows.

We first extend Σ by adding a fresh symbol c_f of arity n (a constructor variant) for each $f \in \mathcal{F} \cap \mathcal{A}$ of arity n:

$$\Sigma^\sharp = \Sigma \cup \{c_f \mid f \in \mathcal{F} \cap \mathcal{A}\}$$

We need translations from Σ-terms to $\Sigma^\sharp$-terms and vice versa. Therefore, we define functions $h : T_{\Sigma^\sharp,\mathcal{V}} \to T_{\Sigma,\mathcal{V}}$ and $c : T_{\Sigma,\mathcal{V}} \to T_{\Sigma^\sharp,\mathcal{V}}$. h replaces each occurrence of a constructor variant c_f by its corresponding $f \in \mathcal{F} \cap \mathcal{A}$, while c does the inverse: it replaces each $f \in \mathcal{F} \cap \mathcal{A}$ by its constructor variant c_f. In addition, we define a mapping $ic : T_{\Sigma,\mathcal{V}} \to T_{\Sigma^\sharp,\mathcal{V}}$ that replaces each *inner occurrence* of a symbol $f \in \mathcal{F} \cap \mathcal{A}$ by its constructor variant f, i.e. $ic(f(t_1,\ldots,t_n)) = f(c(t_1),\ldots,c(t_n))$. For convenience we write t^h, t^c and t^{ic} for $h(t)$, $c(t)$ and $ic(t)$, respectively.

Now, we define two sets of rules $\mathcal{R}^\sharp{}_1$ and $\mathcal{R}^\sharp{}_2$. $\mathcal{R}^\sharp{}_1$ will consist of transformed versions of rules in $\mathcal{R}$:

$$\mathcal{R}^\sharp{}_1 = \{\langle l^{ic}, r\rangle \mid \langle l, r\rangle \in \mathcal{R}\}$$

In $\mathcal{R}^\sharp{}_2$ we put rules to transform a symbol $f \in \mathcal{F}^\mathcal{R} \cap \mathcal{A}^\mathcal{R}$ into its constructor variant. That is, $\mathcal{R}^\sharp{}_2$ is the smallest set satisfying:

> If u is a subtemplate of a rule in $\mathcal{R}$, with $u = f(t_1,\ldots,t_n)$ and $f \in \mathcal{F}^\mathcal{R} \cap \mathcal{A}^\mathcal{R}$, then $\langle u^{ic}, u^c\rangle \in \mathcal{R}^\sharp{}_2$.

We define $\mathcal{R}^\sharp = \mathcal{R}^\sharp{}_1 \cup \mathcal{R}^\sharp{}_2$. In the sequel we will write $\langle \Sigma^\sharp, \mathcal{R}^\sharp{}_1 \cup \mathcal{R}^\sharp{}_2\rangle$ if we are interested in the TRS with underlying ARS $\langle T_{\Sigma^\sharp,\mathcal{V}}, (\to_\alpha)_{\alpha\in\{\mathcal{R}^\sharp{}_1,\mathcal{R}^\sharp{}_2\}}\rangle$ and $\langle \Sigma^\sharp, \mathcal{R}^\sharp\rangle$ if we mean the TRS with underlying simple ARS $\langle T_{\Sigma^\sharp,\mathcal{V}}, \to_{\mathcal{R}^\sharp}\rangle$.

We illustrate the transformation as described above by applying it to to Combinatory Logic ($\mathfrak{CL}$), a system that is regular, but not constructor-based.

Example 5.1 $\mathfrak{CL}$ is the TRS with $\Sigma = \{Ap, S, K, I\}$ and the following set of rules

$$\mathcal{CL} = \left\{\begin{array}{lcl} Ap(Ap(Ap(S,x),y),z) & \to & Ap(Ap(x,z),Ap(y,z)), \\ Ap(Ap(K,x),y) & \to & x, \\ Ap(I,x) & \to & x \end{array}\right\}$$

Since $\mathcal{F}^{\mathcal{CL}} \cap \mathcal{A}^{\mathcal{CL}} = \{Ap\}$, $\mathfrak{CL}$ is not constructor-based. Therefore, we extend Σ with a fresh symbol c_{Ap}, i.e. $\Sigma^\sharp = \Sigma \cup \{c_{Ap}\}$ and define:

$$\mathcal{CL}^\sharp{}_1 = \left\{\begin{array}{lcl} Ap(c_{Ap}(c_{Ap}(S,x),y),z) & \to & Ap(Ap(x,z),Ap(y,z)), \\ Ap(c_{Ap}(K,x),y) & \to & x, \\ Ap(I,x) & \to & x \end{array}\right\}$$

and

$$\mathcal{CL}^\sharp{}_2 = \left\{\begin{array}{lcl} Ap(c_{Ap}(S,x),y) & \to & c_{Ap}(c_{Ap}(S,x),y), \\ Ap(S,x) & \to & c_{Ap}(S,x), \\ Ap(K,x) & \to & c_{Ap}(K,x) \end{array}\right\}$$

Now $\mathfrak{CL}^\sharp = \langle \Sigma^\sharp, \mathcal{CL}^\sharp{}_1 \cup \mathcal{CL}^\sharp{}_2\rangle$ is constructor-based.

Proposition 5.2 Let $\mathfrak{R}$ be a TRS, with $\mathfrak{R}^\sharp$ its transform (as defined above). $\mathfrak{R}^\sharp$ is a constructor-based TRSs.

Proof $\mathfrak{R}^\sharp$ does not contain rules with a lefthand side of which the outermost symbol is a c_f-symbol or a symbol in $\mathcal{C}$ and the map ic produces terms in which all innermost symbols are either a c_f or a symbol in $\mathcal{C}$. □

Proposition 5.3 Let $\langle \Sigma, \mathcal{R}\rangle$ be a TRS and $\Sigma^\sharp$, $\mathcal{R}^\sharp{}_2$ obtained from Σ and $\mathcal{R}$ according to the above described transformation. The TRS $\langle \Sigma^\sharp, \mathcal{R}^\sharp{}_2\rangle$ is SN & CR.

Proof The only effect of applying a rule in $\mathcal{R}^\sharp{}_2$ to a $t \in T_{\Sigma^\sharp,\mathcal{V}}$ is that some $f \in \mathcal{F}$ is replaced by its constructor variant c_f. Therefore, we are able to define a complexity-measure on the terms t in $T_{\Sigma^\sharp,\mathcal{V}}$, counting the number of occurences of symbols in $\mathcal{F} \cap \mathcal{A}$,

i.e. $||t|| = |\{o \in O^{\mathcal{F}}(t) \mid t(o) \in \mathcal{F} \cap \mathcal{A}\}|$ (clearly, because terms are finite, $||t|| < \omega$). For $t, t' \in T_{\Sigma^\sharp,\mathcal{V}}$ it holds that if $t \to_{\mathcal{R}^\sharp{}_2} t'$ then $||t'|| < ||t||$. SN then follows, because $<$ is a well-founded ordering on ω.
$\mathcal{L}_2^{\mathcal{R}^\sharp}$ is nonoverlapping ($\mathcal{F} \cap \mathcal{A} = \emptyset$). Therefore, the only overlap possible between lefthand sides of rules is root overlap, and if the two lefthand sides of rules $l_1 \to r_1$ and $l_2 \to r_2$ have root overlap, then for each substitution σ such that $l_1^\sigma = l_2^\sigma$, we get $r_1^\sigma = r_2^\sigma$ (just by definition of $\mathcal{R}^\sharp{}_2$). We conclude that $\mathcal{R}^\sharp{}_2$ is regular and therefore we get confluence by Corollary 2.6. □

Thatte proved the following two 'simulation lemmas':

Lemma 5.4 Let $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ be a TRS, with transform $\mathfrak{R}^\sharp$. Then:

i. Given $t_1, t_2 \in T_{\Sigma^\sharp,\mathcal{V}}$: $t_1 \to_{\mathcal{R}^\sharp} t_2 \implies t_1^h \twoheadrightarrow_{\mathcal{R}} t_2^h$.

ii. Given $t_1, t_2 \in T_{\Sigma,\mathcal{V}}$: $t_1 \to_{\mathcal{R}} t_2 \implies t_1 \to^+_{\mathcal{R}^\sharp} t_2$.

Proof Lemmas 1 and 2 of [Tha85], respectively. □

The connection between $\mathfrak{R}$ and $\mathfrak{R}^\sharp$ thus expressed is not so strong. In our opinion still some important questions remain unanswered: Is it possible that some t^h is a normal form of $\mathfrak{R}$, while t is not a normal form of $\mathfrak{R}^\sharp$? Can it be that there is an infinite reduction sequence $t \to_{\mathfrak{R}^\sharp} t_1 \to_{\mathfrak{R}^\sharp} \cdots \to_{\mathfrak{R}^\sharp} t_n \to_{\mathfrak{R}^\sharp} \cdots$, with $t^h = t_i^h$ for all $i \in \omega$?

In the next section we will analyze the connection between $\mathfrak{R}$ and $\mathfrak{R}^\sharp$ using correctness criteria we defined earlier and answer the above questions. More specifically, we will establish an ω-simulation between $\mathfrak{R}$ and $\mathfrak{R}^\sharp$ (provided that $\mathfrak{R}$ is nonoverlapping and left-linear) and we will show that termination is preserved. Together these two facts imply that Thatte's transformation is correct according to Fokkink and Van de Pol.

5.2 Correctness for Nonoverlapping Left-Linear TRSs

In the sequel we assume a TRS $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ with Thatte transform $\mathfrak{R}^\sharp = \langle \Sigma^\sharp, \mathcal{R}^\sharp \rangle$. Unless otherwise stated, in this section it will be assumed that $\mathfrak{R}$ is a nonoverlapping left-linear TRS. Note that in this case $\mathfrak{R}^\sharp$ is nonoverlapping and left-linear as well. More specifically, by the proof of Proposition 5.3 no lefthand side of an $\mathcal{R}^\sharp{}_2$-rule overlaps (either root or nonroot) a lefthand side of an $\mathcal{R}^\sharp{}_1$-rule (notice that this would imply that $\mathcal{R}$ has overlaps as well).

To show that Thatte's transformation is correct we show that $\mathfrak{R}^\sharp$ ω-simulates $\mathfrak{R}$ via $\mathsf{nf}_{\mathfrak{R}_2^\sharp} \restriction T_{\Sigma,\mathcal{V}}$, the normal form function generated by $\mathfrak{R}_2^\sharp$ restricted to Σ-terms, and that this simulation map is termination preserving. For the rest of this section and the next, let $\phi = \mathsf{nf}_{\mathfrak{R}_2^\sharp} \restriction T_{\Sigma,\mathcal{V}}$.

Lemma 5.5 ϕ is an injection.

Proof First note that $\mathsf{nf}_{\mathfrak{R}_2^\sharp}$ is a total function by Proposition 5.3. Clearly the restriction of a total function is again a total function, so ϕ is total.
It remains to verify that $\forall s, t \in T_{\Sigma,\mathcal{V}} (\phi(s) = \phi(t) \implies s = t)$. Suppose $s \neq t$. We show that then also the $\mathcal{R}^\sharp{}_2$ normal forms of s and t differ, and the result follows by contraposition.
First note that $\mathcal{R}^\sharp{}_2$-rules do not alter the structure of a term. They only replace some symbols by other symbols. So if $O(s) \neq O(t)$, then also the $\mathcal{R}^\sharp{}_2$ normal forms of these terms differ in structure and thus are different. The remaining case, where $O(s) = O(t)$ now follows by structural induction w.r.t. s and t.
If $s = a \neq b = t$ or $s = x \neq y = t$ (a, b constants, $x, y \in \mathcal{V}$) then s and t are already in $\mathcal{R}^\sharp{}_2$-normal form (note that if a constant would be an $\mathcal{R}^\sharp{}_2$-redex, then in $\mathcal{R}$ it would

appear both as a lefthand side and as a proper subterm of a lefthand side, so there would be overlap).

Suppose that for $s_1, \ldots, s_n$ and $t_1, \ldots, t_n$ it holds that if $s_i \neq t_i$ $(1 \leq i \leq n)$ then their respective $\mathcal{R}^\sharp{}_2$-normal forms differ. If $f(s_1, \ldots, s_n) = s \neq t = f'(t_1, \ldots, t_n)$ because $s_i \neq t_i$ for some $1 \leq i \leq n$, then clearly also the normal forms of s and t differ (by induction hypothesis). If $s \neq t$ because $f \neq f'$, and either $f \notin \mathcal{F} \cap \mathcal{A}$ or $f' \notin \mathcal{F} \cap \mathcal{A}$ or both, then the result follows by the definition of $\mathcal{R}^\sharp{}_2$. If $f, f' \in \mathcal{F} \cap \mathcal{A}$, then also $c_f \neq c_{f'}$. □

The following lemma expresses that every $\Sigma^\sharp$-term t that is reachable from an element in the range of ϕ is reachable from t^h by an $\mathcal{R}^\sharp{}_2$-reduction.

Lemma 5.6 If $s \in \mathsf{range}(\phi)$, $t \in T_{\Sigma^\sharp,\mathcal{V}}$, then $s \twoheadrightarrow_{\mathcal{R}^\sharp} t \Longrightarrow t^h \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} t$.

Proof Induction on the length of the reduction $s \twoheadrightarrow_{\mathcal{R}^\sharp} t$. If $s \rightarrow^0_{\mathcal{R}^\sharp} t$, then $t = \mathsf{nf}_{\mathcal{R}^\sharp{}_2}(t^h)$.
Suppose the lemma holds for reductions $s \twoheadrightarrow_{\mathcal{R}^\sharp} t'$ of length $n \in \omega$ and $s \twoheadrightarrow^n_{\mathcal{R}^\sharp} t' \rightarrow_{\mathcal{R}^\sharp} t$. If $t' \rightarrow_{\mathcal{R}^\sharp{}_2} t$, then we are done because by induction hypothesis $(t')^h \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} t'$ and clearly $t^h = (t')^h$. If $t' \rightarrow_{\mathcal{R}^\sharp{}_1} t$, say by application of an $\mathcal{R}^\sharp{}_1$-rule $l \rightarrow r$ in a context $C[\,]$ using a substitution σ (i.e. $t' = C[l^\sigma] \rightarrow_{\mathcal{R}^\sharp{}_1} C[r^\sigma] = t$), then $C[\,]^h \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} C[\,]$ (using that the $\mathcal{R}^\sharp{}_2$-rules are left-linear and have no overlap with $\mathcal{R}^\sharp{}_1$-rules). Moreover, if $l^\sigma = l(x_1, \ldots, x_n)^\sigma = l(t_1, \ldots, t_n)$, then $t_i^h \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} t_i$ for all $1 \leq i \leq n$, since $(t')^h \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} t'$. Finally, note that the righthand sides of the $\mathcal{R}^\sharp{}_1$-rules are Σ-terms. □

Lemma 5.7 ($\mathcal{R}^\sharp{}_1$-$\mathcal{R}^\sharp{}_2$ **Strip Lemma**) Let $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ be a nonoverlapping left-linear TRS and $\mathfrak{R}^\sharp = \langle \Sigma^\sharp, \mathcal{R}^\sharp{}_1 \cup \mathcal{R}^\sharp{}_2 \rangle$ its transform.
For all $s, s', t \in T_{\Sigma^\sharp,\mathcal{V}}$: $s' \twoheadleftarrow_{\mathcal{R}^\sharp{}_2} s \rightarrow_{\mathcal{R}^\sharp{}_1} t$ implies that there is a $t' \in T_{\Sigma^\sharp,\mathcal{V}}$ such that $s' \rightarrow_{\mathcal{R}^\sharp{}_1} t' \twoheadleftarrow_{\mathcal{R}^\sharp{}_2} t$.

Proof First we show that $s' \xleftarrow{o_2}_{\mathcal{R}^\sharp{}_2} s \xrightarrow{o_1}_{\mathcal{R}^\sharp{}_1} t$ implies that there is a $t' \in T_{\Sigma^\sharp,\mathcal{V}}$ such that $s' \rightarrow_{\mathcal{R}^\sharp{}_1} t' \twoheadleftarrow_{\mathcal{R}^\sharp{}_2} t$.
We distinguish cases according to the relative positions of o_1 and o_2 in s:

$o_1 = o_2$**:** This is impossible. If $o_1 = o_2$, then $(s/o_1)^h = (s/o_2)^h$ would be both an instance of a proper subterm of a lefthand side of a rule in $\mathcal{R}$ as well an instance of some lefthand side of a rule, so $\mathcal{L}^{\mathcal{R}}$ would contain nonroot overlap, which would contradict the assumption that $\mathfrak{R}$ is nonoverlapping.

$o_1 < o_2$**:** If $l \rightarrow r$ is the $\mathcal{R}^\sharp{}_1$-rule applied at s/o_1 and $o_2 = o_1.p$. Because $\mathfrak{R}^\sharp$ is nonoverlapping, it must be the case that $p \notin O^{\mathcal{F}}(l)$. Thus, in t we can contract all descendants of o_2 using the same rule as the one applied in $s \rightarrow_{\mathcal{R}^\sharp{}_2} s'$, yielding a term u and $t \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} u$. On the other hand, again because $\mathfrak{R}^\sharp$ is nonoverlapping and left-linear, in s' it is still possible to reduce s'/o_1, which clearly results in the same u.

$o_2 < o_1$**:** Similar (but note that o_2 has only one descendant, viz. o_2, and that by construction and by left-linearity of $\mathcal{R}$ a righthand side of an $\mathcal{R}^\sharp{}_2$-rule cannot contain the same variable more than once).

o_1 **and** o_2 **disjoint:** Obvious.

Now the lemma follows by induction w.r.t. the length of the reduction $s \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} s'$, using the above. □

Theorem 5.8 If $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ is a nonoverlapping left-linear TRS with transform $\mathfrak{R}^\sharp = \langle \Sigma^\sharp, \mathcal{R}^\sharp{}_1 \cup \mathcal{R}^\sharp{}_2 \rangle$, then $\mathfrak{R}^\sharp$ ω-simulates $\mathfrak{R}$ via ϕ ($\mathfrak{R} \lesssim^\phi \mathfrak{R}^\sharp$).

Proof We prove that the two conditions of Definition 4.1 hold.
First, assume $s \rightarrow_{\mathcal{R}} t$ for $s, t \in T_{\Sigma,\mathcal{V}}$. We have to show that $\phi(s) \rightarrow^+_{\mathcal{R}^\sharp} \phi(t)$.
If $s \rightarrow_{\mathcal{R}} t$ then there are an $o \in O(s)$, a substitution σ, and an $\mathcal{R}$-rule $l \rightarrow r$, such that $s/o = l^\sigma$ and $t = s[o \leftarrow r^\sigma]$. Furthermore, $\mathfrak{R}^\sharp$ is constructed in such a way that

$s/o = l^\sigma \twoheadrightarrow_{\mathcal{R}^\sharp_2} (l^{ic})^\sigma$, and because $\langle l^{ic}, r\rangle \in \mathcal{R}^\sharp{}_1$ we find $s \twoheadrightarrow_{\mathcal{R}^\sharp_2} s[o \leftarrow (l^{ic})^\sigma] \rightarrow_{\mathcal{R}^\sharp_1} s[o \leftarrow r^\sigma] = t$.
Now, observe that by Proposition 5.3 $s[o \leftarrow (l^{ic})^\sigma] \twoheadrightarrow_{\mathcal{R}^\sharp_2} \phi(s)$, so by Lemma 5.7 there is some $t' \in T_{\Sigma^\sharp,\mathcal{V}}$ such that $\phi(s) \rightarrow_{\mathcal{R}^\sharp_1} t'$ and $t \twoheadrightarrow_{\mathcal{R}^\sharp_2} t'$. Clearly, both $t \twoheadrightarrow_{\mathcal{R}^\sharp_2} \phi(t)$ and $t' \twoheadrightarrow_{\mathcal{R}^\sharp_2} \phi(t)$ by another application of Proposition 5.3, so $\phi(s) \rightarrow^+_{\mathcal{R}^\sharp} \phi(t)$. The following diagram illustrates the procedure just described:

$$\begin{tikzcd}
s \arrow[r, two heads, "\mathcal{R}^\sharp{}_2"] \arrow[d, two heads, "\mathcal{R}^\sharp{}_2"'] \arrow[dr, phantom, "\text{Prop. 5.3}"] & s[o \leftarrow (l^{ic})^\sigma] \arrow[r, "\mathcal{R}^\sharp{}_1"] \arrow[d, two heads, "\mathcal{R}^\sharp{}_2"] \arrow[dr, phantom, "\text{Lem. 5.7}"] & t \arrow[r, equal] \arrow[d, two heads, "\mathcal{R}^\sharp{}_2"'] \arrow[dr, phantom, "\text{Prop. 5.3}"] & t \arrow[d, two heads, "\mathcal{R}^\sharp{}_2"] \\
\phi(s) \arrow[r, equal] & \phi(s) \arrow[r, "\mathcal{R}^\sharp{}_1"'] & t' \arrow[r, two heads, "\mathcal{R}^\sharp{}_2"'] & \phi(t)
\end{tikzcd}$$

Secondly, assume $s \rightarrow^+_{\mathcal{R}^\sharp} t$ for some $s \in \mathsf{range}(\phi)$ and $t \in T_{\Sigma^\sharp,\mathcal{V}}$. According to Definition 4.1 we have to show the existence of a term $t' \in \mathsf{range}(\phi)$ such that $\phi^{-1}(s) \rightarrow^+_{\mathcal{R}} \phi^{-1}(t')$ and $t \twoheadrightarrow_{\mathcal{R}^\sharp} t'$.
By Lemma 5.6 $t^h \twoheadrightarrow_{\mathcal{R}^\sharp_2} t$. Moreover, because $\mathcal{R}^\sharp{}_2$ is confluent and terminating (Proposition 5.3), and because $\phi(t^h)$ is obtained from t^h by an $\mathcal{R}^\sharp{}_2$-reduction we find $t \twoheadrightarrow_{\mathcal{R}^\sharp_2} \phi(t^h)$. So we take $t' = \phi(t^h)$ and it remains to show that $\phi^{-1}(s) = s^h \rightarrow^+_{\mathcal{R}} t^h = \phi^{-1}(\phi(t^h))$
Since s is an $\mathcal{R}^\sharp{}_2$-normal form, the first step of the reduction to t must be an application of a $\mathcal{R}^\sharp{}_1$-rule, so we have $s \rightarrow_{\mathcal{R}^\sharp_1} s' \twoheadrightarrow_{\mathcal{R}^\sharp} t$. Furthermore, $s \rightarrow_{\mathcal{R}^\sharp_1} s'$ implies $s^h \rightarrow_{\mathcal{R}} (s')^h$. So by Lemma 5.4(i) and induction w.r.t. the length of the reduction $s' \twoheadrightarrow_{\mathcal{R}^\sharp} t$, we get $(s')^h \twoheadrightarrow_{\mathcal{R}} t^h$. Therefore $s^h \rightarrow_{\mathcal{R}} (s')^h \twoheadrightarrow_{\mathcal{R}} t^h$, i.e. $\phi^{-1}(s) \rightarrow^+_{\mathcal{R}} t^h$. □

Theorem 5.9 Let $\mathfrak{R} = \langle \Sigma, \mathcal{R}\rangle$ be a TRS (not necessarily nonoverlapping and left-linear!) with transform $\mathfrak{R}^\sharp = \langle \Sigma^\sharp, \mathcal{R}^\sharp\rangle$:

i. $\langle \mathsf{id}(T_{\Sigma,\mathcal{V}}), h\rangle$ preserves termination; and

ii. ϕ preserves termination.

Proof

i. First observe that (1) $t \rightarrow_{\mathcal{R}^\sharp_1} t'$ implies $t^h \rightarrow_{\mathcal{R}} (t')^h$; and (2) $t \rightarrow_{\mathcal{R}^\sharp_2} t'$ implies $t^h = (t')^h$.
Now, assume $s \in T_{\Sigma,\mathcal{V}}$ is terminating in $\mathfrak{R}$, while s has an infinite reduction in $\mathfrak{R}^\sharp$. Because $\langle \Sigma^\sharp, \mathcal{R}^\sharp{}_2\rangle$ is SN (Proposition 5.3) and $T_{\Sigma^\sharp,\mathcal{V}}$ contains only finite terms, there must be infinitely many $\mathcal{R}^\sharp{}_1$-steps in this reduction, so this reduction can be written as follows:
$$s \twoheadrightarrow_{\mathcal{R}^\sharp_2} s_0 \rightarrow_{\mathcal{R}^\sharp_1} s_1 \twoheadrightarrow_{\mathcal{R}^\sharp_2} {s_1}' \rightarrow_{\mathcal{R}^\sharp_1} \cdots \twoheadrightarrow_{\mathcal{R}^\sharp_2} {s_{n-1}}' \rightarrow_{\mathcal{R}^\sharp_1} s_n \rightarrow_{\mathcal{R}^\sharp_2} \cdots$$
However, because of (1) and (2) together, this reduction corresponds to
$$s = s_0^h \rightarrow_{\mathcal{R}} s_1^h \rightarrow_{\mathcal{R}} \cdots \rightarrow_{\mathcal{R}} s_n^h \rightarrow^+_{\mathcal{R}} \cdots$$
which contradicts the assumption that s was terminating in $\mathfrak{R}$.

ii. This follows from the first part of this theorem and the trivial fact that $s \twoheadrightarrow_{\mathcal{R}^\sharp_2} \phi(s)$ for all $s \in T_{\Sigma,\mathcal{V}}$ (if $\phi(s)$ would not be terminating, then s would not be terminating either). □

Now, combining Theorems 5.8 and 5.9(ii) we get by Theorem 4.9 the following:

Corollary 5.10 Thatte's transformation is correct for the class of nonoverlapping left-linear TRSs (according to the definition of [FP96]).

6 Enlarging the Domain of Thatte's Transformation

In 1988, Thatte tried to extend his result to a larger class of nonoverlapping confluent systems (without the restriction of left-linearity) [Tha88]. He based his correctness argument on the presumed fact that for the given class of rewrite systems his transformation resulted in a system that was confluent for reachable terms.

Unfortunately, Thatte's proof for the class of nonoverlapping confluent systems does not work, and a counterexample of Rakesh M. Verma [Ver95b] shows that his theorem does not even hold. Inspired by his counterexample, Verma somewhat restricted the domain of the transformation and stated the theorem that for this domain the transformation preserved confluence.

Verma's proof however was also incorrect, as we will demonstrate in this section. As of yet, we have not been able to either give a sound proof of Verma's theorem or a counterexample, so this problem remains open. We will however prove preservation of confluence for all the relevant subclasses of the class of weakly persistent systems that Verma mentions. In addition we will prove some of the numerous unproved statements in [Ver95b], some of which are by no means trivial.

6.1 Weak Persistence

In [Tha88] Thatte stated the following as a theorem:

Claim 6.1 ([Tha88]) If $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ is nonoverlapping and confluent, then $[\mathfrak{R}^\sharp]_{T_{\Sigma,\mathcal{V}}}$ is confluent.

This is clearly the same as stating that the simulation mapping $\langle \mathsf{id}(T_{\Sigma,\mathcal{V}}), h \rangle$ (h as defined in the previous section) preserves confluence. Verma refutes this claim. We will repeat his counterexample and show that it also works for the mapping ϕ that we used in the previous section. We will then verify that his counterexample is indeed valid, by giving the nontrivial proof that the original TRS is confluent, while its transform is not.

From his counterexample, Verma tried to derive an extra restriction that was sufficient to obtain a confluence preserving simulation map. We will define this restriction (Verma called it 'weak persistence') at the end of this section.

Example 6.1 ([Ver95b]) Consider the TRS $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ and its transform $\mathfrak{R}^\sharp = \langle \Sigma^\sharp, \mathcal{R}^\sharp \rangle$, where $\Sigma = \{f, g, h, a, c\}$ and $\Sigma^\sharp = \Sigma \cup \{c_f\}$. $\mathcal{R}$ and $\mathcal{R}^\sharp$ respectively consist of the rules:

$$\mathcal{R} = \left\{ \begin{array}{lcl} h(f(x,x)) & \to & h(a), \\ f(x,g(x)) & \to & a, \\ c & \to & g(c) \end{array} \right\}, \quad \mathcal{R}^\sharp = \left\{ \begin{array}{lcl} h(c_f(x,x)) & \to & h(a), \\ f(x,g(x)) & \to & a, \\ c & \to & g(c), \\ f(x,x) & \to & c_f(x,x) \end{array} \right\}$$

In Lemma 6.2 we show that $\mathfrak{R}$ is indeed confluent. However, $\mathfrak{R}^\sharp$ is not confluent for reachable terms. Consider the Σ-term $f(g(c),c)$.[7] Note that it is a ${\mathcal{R}^\sharp}_2$-normal form, so under $\langle \mathsf{id}(T_{\Sigma,\mathcal{V}}), h \rangle$ as well as under ϕ this term is simulated in $\mathfrak{R}^\sharp$ by $f(g(c),c)$. Furthermore, $f(g(c),c) \to_{{\mathcal{R}^\sharp}_1} f(g(c),g(c))$ so $f(g(c),g(c))$ is a reachable term under both simulation maps. However, $\mathfrak{R}^\sharp \models \mathsf{CR}(f(g(c),g(c)))$ does not hold (see the diagram below).

[7] Verma uses the term $f(c,c)$. This term works for the mapping $\langle \mathsf{id}(T_{\Sigma,\mathcal{V}}), h \rangle$, but not for our mapping ϕ.

$$\begin{tikzcd}
f(g(c),g(c)) \arrow[r, "\mathcal{R}^\sharp{}_1"'] \arrow[d, "\mathcal{R}^\sharp{}_2"] & f(g(c),g(g(c))) \arrow[r, "\mathcal{R}^\sharp{}_1"'] & a \\
c_f(g(c),g(c)) \arrow[d, dashed, two heads, "\mathcal{R}^\sharp{}_1"] & & \\
{} & &
\end{tikzcd}$$

It is clear that $c_f(g(c),g(c))$ never reduces to a, because there is no rule that removes the c_f-symbol from this term.

Lemma 6.2 The TRS $\mathfrak{R} = \langle \Sigma, \mathcal{R}^c \cup \mathcal{R}^f \rangle$, where $\Sigma = \{a, c, f, g, h\}$, $\mathcal{R}^c = \{c \to g(c)\}$ and $\mathcal{R}^f = \{h(f(x,x)) \to h(a),\ f(x,g(x)) \to a\}$ is confluent.

Proof Note that if $s \to_{\mathcal{R}^c} t$, then $O^{\mathcal{F}}(s) \subset O^{\mathcal{F}}(t)$. Now, we will first prove three simple facts:

1. $\langle \Sigma, \mathcal{R}^c \rangle \models \mathsf{CR}$.

   Because it is clearly regular, this is an immediate consequence of Corollary 2.6.

2. $\forall s, t_1, t_2 \in T_{\Sigma,\mathcal{V}}(t_2 \twoheadleftarrow_{\mathcal{R}^c} s \xrightarrow{o}_{\mathcal{R}^f} t_1 \Longrightarrow \exists t, t' \in T_{\Sigma,\mathcal{V}}(t_2 \twoheadrightarrow_{\mathcal{R}^c} t' \xrightarrow{o}_{\mathcal{R}^f} t \twoheadleftarrow_{\mathcal{R}^c} t_1)$, and for all positions $o_1, \ldots, o_n$ of redexes contracted in the reduction $t_2 \twoheadrightarrow_{\mathcal{R}^c} t'$, it holds that $o_i > o$ $(1 \leq i \leq n)$.

   Either $s/o = f(u, g(u))$ or $s/o = h(f(u,u))$. Note that $o \in O^{\mathcal{F}}(t_2)$ and that $t_2/o = f(u_1, g(u_2))$ or $t_2/o = h(f(u_1,u_2))$. If $u_1 = u_2$, then the same rule as applied in $s \to_{\mathcal{R}^f} t_1$ is still applicable. In this case we take $t' = t_2$. If $u_1 \neq u_2$, then they have a common reduct u' by the previous fact. We take $t' = f(u', g(u'))$ or $t' = h(f(u',u'))$ respectively, and $t_2 \twoheadrightarrow_{\mathcal{R}^c} t'$. The redexes contracted in this reduction clearly all lie below o, and the $\mathcal{R}^f$-rule can be applied to t'/o, so $t' \to_{\mathcal{R}^f} t$. Finally, contracting all descendants over $s \to_{\mathcal{R}^f} t_1$ of redexes contracted in the reduction $s \twoheadrightarrow_{\mathcal{R}^c} t_2 \twoheadrightarrow_{\mathcal{R}^c} t'$, results in $t_1 \twoheadrightarrow_{\mathcal{R}^c} t$.

3. For all $s, t_1, t_1', t_2, t_2' \in T_{\Sigma,\mathcal{V}}$: $t_1 \xleftarrow{o_1}_{\mathcal{R}^f} t_1' \twoheadleftarrow_{\mathcal{R}^c} s \twoheadrightarrow_{\mathcal{R}^c} t_2' \xrightarrow{o_2}_{\mathcal{R}^f} t_2$, with for all redex positions $p_1, \ldots, p_m$ and $q_1, \ldots, q_n$ contracted in $s \twoheadrightarrow_{\mathcal{R}^c} t_1'$ and $s \twoheadrightarrow_{\mathcal{R}^c} t_2'$, respectively, $p_i > o_1$ and $q_j > o_2$ $(1 \leq i \leq m,\ 1 \leq j \leq n)$, implies that there is a $t \in T_{\Sigma,\mathcal{V}}$, such that $t_1 \twoheadrightarrow_{\mathcal{R}} t \twoheadleftarrow_{\mathcal{R}} t_2$.

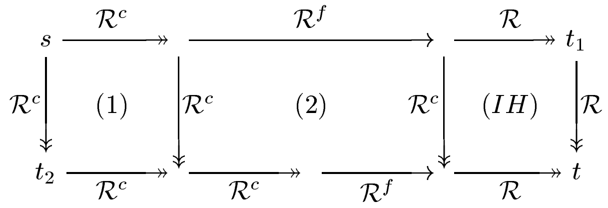
   The cases where $o_1 \not< o_2$ and $o_2 \not< o_1$ (so either $o_1 = o_2$, or they are disjunct) are trivial. Moreover, if $o_1 < o_2$, or $o_2 < o_1$, and $s/o_1 = h(s/o_2)$, or $s/o_2 = h(s/o_1)$, then we find that $t_1/o_1 = h(a) = t_2/o_1$, or $t_1/o_2 = h(a) = t_2/o_2$, respectively, and thus $t_1 = t_2$.

   So assume $o_1 < o_2$ and not $s/o_1 = h(s/o_2)$. Either $s/o_1 = f(u, g(u))$ or $s/o_1 = h(f(u,u))$, and $t_2/o_1 = f(u_1, g(u_2))$, or $t_2/o_1 = h(f(u_1,u_2))$, with $u = u_1$, or $u = u_2$. Clearly, we can reduce t_2/o_1 to $f(u', g(u'))$, or to $h(f(u',u'))$, using the same rules as applied in $s \twoheadrightarrow_{\mathcal{R}^c} t_2' \xrightarrow{o_2}_{\mathcal{R}^f} t_2$ (either $u' = u_1$, or $u' = u_2$) to obtain a term t', and then apply the same rule as applied in $t_1' \xrightarrow{o_1}_{\mathcal{R}^f} t_1$. This gives us the reduction $t_2 \twoheadrightarrow_{\mathcal{R}} t' \xrightarrow{o_1}_{\mathcal{R}^f} t_1$.

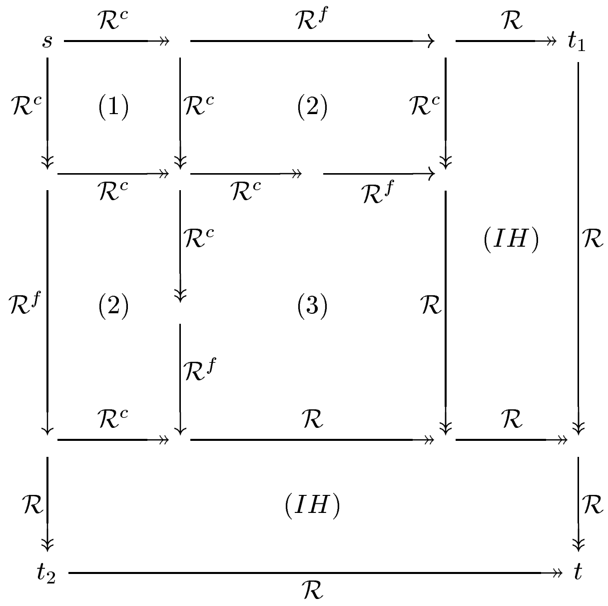
   The case where $o_2 < o_1$ and not $s/o_2 = h(s/o_1)$ goes similarly.

We define on $s \in T_{\Sigma,\mathcal{V}}$ a complexity measure $||s|| = |\{o \in O^{\mathcal{F}}(s) \mid s(o) = f\}|$. Clearly $||s|| < \omega$ for all $s \in T_{\Sigma,\mathcal{V}}$ and if $s \to_{\mathcal{R}^f} s'$, then $||s|| < ||s'||$. With the above facts at hand, we will derive confluence for all terms $\forall s \in T_{\Sigma,\mathcal{V}}(\mathfrak{R} \models \mathsf{CR}(s))$, using induction w.r.t. $||s||$.

The base case follows immediately from the first fact, for if $t_1 \twoheadleftarrow_{\mathcal{R}} s \twoheadrightarrow_{\mathcal{R}} t_2$ and $||s|| = 0$, then $s \twoheadrightarrow_{\mathcal{R}^c} t_1$ and $s \twoheadrightarrow_{\mathcal{R}^c} t_2$.
Assume that $t_1 \twoheadleftarrow_{\mathcal{R}} s \twoheadrightarrow_{\mathcal{R}} t_2$, with $||s|| = n + 1$ and $\forall t \in T_{\Sigma,\mathcal{V}}(\mathfrak{R} \models \mathsf{CR}(t))$ with $||t|| = n$ (IH).
Note that if both reductions only contain applications of $\mathcal{R}^c$-rules, then we get a common reduct using the first fact. If only one of the reductions contains applications of $\mathcal{R}^f$-rules, then we have the following diagram:



If both reductions contain applications of $\mathcal{R}^f$-rules then we can tile the following diagram:



□

The problem in the above counterexample lies in the fact that $\mathfrak{R}$ (although it is a nonoverlapping TRS) contains some ambiguity. For instance, for the Σ-term $h(f(g(c), g(c)))$, we have two different kinds of reductions to $h(a)$:

1. the reductions using $h(f(x, x)) \to h(a)$, e.g. $h(f(g(c), g(c)) \to_{\mathcal{R}} h(a)$; and
2. the reductions using $f(x, g(x)) \to a$, e.g. $h(f(g(c), g(c)) \to_{\mathcal{R}} h(f(g(c), g(g(c)))) \to_{\mathcal{R}} h(a)$.

Note that in the reductions of the first kind f acts as a constructor symbol, while in the reductions of the second kind it acts as a defined symbol. Intuitively, the first step in the second reduction, that is, the application of the rule $c \to g(c)$ to the second argument of f, switches the rôle of f from constructor to defined symbol. Note that this switch is not rigid, because if we apply $c \to g(c)$ to the first argument of f in $h(f(g(c), g(g(c)))$ the rule $h(f(x, x)) \to h(a)$ is applicable again.

In $\mathfrak{R}^\sharp$ things are a bit different. We still have this switching capability, but now we also have the possibility of making the switch rigid. In the situation where f appears in its rôle of constructor (say in terms $f(t,t)$) we can now make that situation explicit by applying the rule $f(x,x) \to c_f(x,x)$, and then there is no return.

To avoid this situation, Verma introduces his 'weak persistence'. Informally, this notion makes sure that instances of terms in which the outermost symbol acts as a constructor symbol (instances of proper subterms of lefthand sides of rules) do not reduce to instances of lefthand sides of rules.

Definition 6.3 (Weak Persistence) Let $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ be a TRS.
We say that $\mathfrak{R}$ is *weakly persistent* if there is no subtemplate s of $\mathcal{R}$ and s-pseudo-redex $t \in T_{\Sigma,\mathcal{V}}$ such that $\exists t_1, t_2 \in T_{\Sigma,\mathcal{V}} (t \xrightarrow[s]{nt}{}^{*}_{\mathcal{R}} t_1 \wedge t_1 \xrightarrow{r}_{\mathcal{R}} t_2)$. $\dashv$

We refer to section 2.3.2 for definitions of 'subtemplate', 'pseudo-redex', $\xrightarrow{nt}_{\mathcal{R}}$ and $\xrightarrow{r}_{\mathcal{R}}$.

The above formulation of weak persistence differs from that of Verma. See Appendix A.1 for a proof that our definition is equivalent with that of Verma.

Note that $\mathfrak{R}$ of Example 6.1 is not weakly persistent, because the switching reductions are the nontemplate reductions of the above definition.

Proposition 6.4 i. Every weakly persistent TRS is nonoverlapping.

ii. Every left-linear nonoverlapping TRS is weakly persistent.

Proof

i. Let $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ be a weakly persistent TRS, and assume that $l, l' \in \mathcal{L}^{\mathcal{R}}$, $o \in O(l)$, $o \neq \lambda$, and σ is a substitution such that $\sigma(l/o) = \sigma(l')$. Obviously, l/o is a subtemplate of $\mathcal{R}$, so $\sigma(l/o)$ is both a l/o-pseudo-redex and a l'-redex. This contradicts weak persistence.
ii. Suppose $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ is not weakly persistent. Then there is a substitution and some proper subterm s (say at position $o \in O(l)$) of lefthand side l of some rule in $\mathcal{R}$, such that $s^\sigma \xrightarrow[s]{nt}{}^{*}_{\mathcal{R}} s'$ and $s' \xrightarrow{r}_{\mathcal{R}} t$. But clearly, because s is a linear term, there must be some substitution τ, such that $s' = s^\tau$. Now $s^\tau \xrightarrow{r}_{\mathcal{R}} t$ implies that there is some rule $(l', r) \in \mathcal{R}$ and a substitution ρ, such that $(l')^\rho = s^\tau = (l/o)^\tau$. Thus, there is overlap and $\mathfrak{R}$ is not nonoverlapping. $\Box$

Remark 6.5 Observe that a weakly persistent TRS not necessarily has confluent root overlaps (is consistent/nonambiguous). This is demonstrated by the trivial counterexample $\langle \{f, a, b\}, \{f(x) \to a,\ f(x) \to b\} \rangle$, which is clearly weakly persistent and not consistent.

6.2 Is Weak Persistence Enough?

The question is whether weak persistence suffices to obtain a confluence preserving transformation. Verma tried to prove this, but we discovered some fundamental errors in Verma's arguments. In the proof of his Lemma 13 [Ver95b] Verma claimes that if $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ is a weakly persistent system with transform $\mathfrak{R}^\sharp = \langle \Sigma^\sharp, \mathcal{R}^\sharp{}_1 \cup \mathcal{R}^\sharp{}_2 \rangle$ and $s \in [T_{\Sigma^\sharp,\mathcal{V}}]_{T_{\Sigma,\mathcal{V}}}$, then $s^h \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} s$ in $\mathfrak{R}^\sharp$. The following example demonstrates that this is not true.

Example 6.2 Consider the TRS $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$, with transform $\mathfrak{R}^\sharp = \langle \Sigma^\sharp, \mathcal{R}^\sharp \rangle$. Here $\Sigma = \{a, b, f, g, h\}$ and $\Sigma^\sharp = \Sigma \cup \{c_f\}$. $\mathcal{R}$ and $\mathcal{R}^\sharp$ respectively consist of the rules:

$$\mathcal{R} = \left\{ \begin{array}{lll} g(f(x,x)) & \to & a, \\ f(a,b) & \to & a, \\ h(x) & \to & b \end{array} \right\}, \quad \mathcal{R}^\sharp = \left\{ \begin{array}{lll} g(c_f(x,x)) & \to & a, \\ f(a,b) & \to & a, \\ h(x) & \to & b, \\ f(x,x) & \to & c_f(x,x) \end{array} \right\}$$

Observe that $\mathfrak{R} \models \mathsf{SN}$[8] and $\mathfrak{R}$ is nonambiguous. Applying Corollary 2.5 and Lemma 2.3(ii) we find $\mathfrak{R} \models \mathsf{CR}$, so by Lemma 2.3(i) $\mathfrak{R}$ has the unique normal form property. Now, if $\mathfrak{R}$ would not be weakly persistent, then there would be some term $t \in T_{\Sigma,\mathcal{V}}$ such that $f(t,t) \twoheadrightarrow_{\mathcal{R}} f(a,b)$. However, a and b are both normal forms in $\mathfrak{R}$, so that $a \twoheadleftarrow_{\mathcal{R}} t \twoheadrightarrow_{\mathcal{R}} b$, which contradicts $\mathfrak{R} \models \mathsf{UN}$. Therefore, $\mathfrak{R}$ is weakly persistent.

In $\mathfrak{R}^\sharp$ we have $f(h(b),h(b)) \to_{\mathcal{R}^\sharp{}_2} c_f(h(b),h(b)) \to_{\mathcal{R}^\sharp{}_1} c_f(h(b),b)$. This reduction demonstrates that the term $c_f(h(b),b)$ is reachable, under both simulation mappings $\langle \mathsf{id}(T_{\Sigma,\mathcal{V}}), h\rangle$ and ϕ (for the latter note that $\phi(f(h(b),h(b))) = c_f(h(b),h(b)))$. But $(c_f(h(b),b))^h = f(h(b),b)$ is in $\mathcal{R}^\sharp{}_2$-normal form.

In fact, the above example also demonstrates that weak persistence is not sufficient to establish $\mathfrak{R} \lesssim^\phi \mathfrak{R}^\sharp$. In $\mathfrak{R}$ we have $f(h(b),h(b)) \to_{\mathcal{R}} f(h(b),b)$, but $\phi(f(h(b),b)) = f(h(b),b)$ and in $\mathfrak{R}^\sharp$ we do not have $c_f(h(b),h(b)) \to^+_{\mathcal{R}^\sharp} f(h(b),b)$. This is a violation of the first condition of Definition 4.1. Recall that $\mathfrak{R} \lesssim^\phi \mathfrak{R}^\sharp$ would have implied confluence right away (Theorem 4.6).

By introducing weak persistence Verma dealt with the "switching problem" as described in the previous section; in a transform of a weakly persistent system the $\mathcal{R}^\sharp{}_2$-rules are never responsible for a rigid switch. The above counterexample reveals another (and very complex) problem in nonlinear systems. Verma believed that only *unwanted introductions* of constructor variants were problematic. He forgot that also the *impossibility of introducing* constructor variants imposes a problem.

We now adapt the above counterexample a little to obtain a counterexample to Verma's Lemma 12:

Claim 6.6 (Lemma 12 in [Ver95b]) Let $\mathfrak{R} = \langle \Sigma, \mathcal{R}\rangle$ be a weakly persistent TRS with transform $\mathfrak{R}^\sharp = \langle \Sigma^\sharp, \mathcal{R}^\sharp\rangle$.

If $s \in [T_{\Sigma^\sharp,\mathcal{V}}]_{T_{\Sigma,\mathcal{V}}}$, $t \in T_{\Sigma,\mathcal{V}}$ and $s^h \to_{\mathcal{R}} t$, then there is a $t' \in T_{\Sigma^\sharp,\mathcal{V}}$ such that $s \twoheadrightarrow_{\mathcal{R}^\sharp} t'$ and $(t')^h = t$.

Example 6.3 Consider the TRS $\mathfrak{R} = \langle \Sigma, \mathcal{R}\rangle$, with transform $\mathfrak{R}^\sharp = \langle \Sigma^\sharp, \mathcal{R}^\sharp\rangle$. Here $\Sigma = \{a,b,f,g,h\}$ and $\Sigma^\sharp = \Sigma \cup \{c_f\}$. $\mathcal{R}$ and $\mathcal{R}^\sharp$ respectively consist of the rules:

$$\mathcal{R} = \left\{ \begin{array}{lcl} g(f(x,x)) & \to & a, \\ f(a,b) & \to & a, \\ h(x,x) & \to & x \end{array} \right\}, \quad \mathcal{R}^\sharp = \left\{ \begin{array}{lcl} g(c_f(x,x)) & \to & a, \\ f(a,b) & \to & a, \\ h(x,x) & \to & x, \\ f(x,x) & \to & c_f(x,x) \end{array} \right\}$$

It is easy to adapt the argument of Example 6.2 to obtain $\mathfrak{R} \models \mathsf{SN}$, $\mathfrak{R} \models \mathsf{CR}$, $\mathfrak{R} \models \mathsf{UN}$ and weak persistence.

The term $c_f(h(b,b),b)$ is reachable, as witnessed by the reduction:

$$f(h(b,b),h(b,b)) \to_{\mathcal{R}^\sharp{}_2} c_f(h(b,b),h(b,b)) \to_{\mathcal{R}^\sharp{}_1} c_f(h(b,b),b)$$

Furthermore, in $\mathfrak{R}$ we have $h(f(h(b,b),b), f(h(b,b),b)) \to_{\mathcal{R}} f(h(b,b),b)$. Now, take $s = h(f(h(b,b),b), c_f(h(b,b),b))$; it is clear that $s^h = h(f(h(b,b),b), f(h(b,b),b))$, so Verma's Lemma 12 suggests the existence of a $t' \in T_{\Sigma^\sharp,\mathcal{V}}$ such that $s \twoheadrightarrow_{\mathcal{R}^\sharp} t'$ and $(t')^h = f(h(b,b),b)$.

However, there is no such t'. We argue as follows. First note that s is not a redex of $\mathcal{R}^\sharp$ (while s^h is a redex of $\mathcal{R}$). The only redexes in s are the $h(b,b)$ at occurrences 1.1 and 2.1. Moreover, contraction of either one of these two redexes does not yield any new redexes and neither of the resulting terms (say s') satisfy $(s')^h = f(h(b,b),b)$. So in our search for such a term we will have to contract them both. That is, we get the reduction

[8] If we take $||s|| = |O^{\mathcal{F}}(s)|$ as a complexity measure, then $s \to_{\mathcal{R}} t$ in $\mathcal{R}$ implies $|t| < |s|$.

$h(f(h(b,b),b),c_f(h(b,b),b)) \twoheadrightarrow_{\mathcal{R}^\sharp_1} h(f(b,b),c_f(b,b))$. Still, we have not found our t', because $h(f(b,b),c_f(b,b))^h \neq f(h(b,b),b)$, but we created one new redex: $f(b,b)$. Thus, the only possibility we have is to contract this redex: $h(f(b,b),c_f(b,b)) \to_{\mathcal{R}^\sharp_2} h(c_f(b,b),c_f(b,b))$. Again, this reduction does not yield the t' we were looking for, so we contract this whole term (note that this is the only redex): $h(c_f(b,b),c_f(b,b)) \to_{\mathcal{R}^\sharp_1} c_f(b,b)$. Since, $c_f(b,b)$ is a normal form, it is our last option for t', but $c_f(b,b)^h = f(b,b) \neq f(h(b,b),b)$. This refutes Verma's claim.

6.3 Positive Results

Verma remarks that the class of weakly persistent systems contains the following three subclasses:

1. the subclass of left-linear nonoverlapping systems with confluent root overlaps;
2. the subclass of nonoverlapping, strongly normalizing systems with confluent root overlaps; and
3. the subclass consisting of systems $\langle \Sigma \cup \{D_i, E\}, \mathcal{R} \cup \mathcal{D}_i \rangle$ ($i = k$ or $i = s$), where it is assumed that $D_k, D_s, E \notin \Sigma$, $\langle \Sigma, \mathcal{R} \rangle$ is a regular TRS, $\mathcal{D}_s = \{D_s(x,x) \to E\}$, and $\mathcal{D}_k = \{D_k(x,x) \to E(x)\}$ (these systems appear in [Klo80]).

In section 5.2 we already proved the correctness of Thatte's transformation (including the preservation of confluence) for the first subclass.

In Appendix A.2 we show that all systems in the second and third subclass are weakly persistent. In this section we will show that by a slight generalization of the proof given in section 5.2 we also get an ω-simulation for a superclass of the third and preservation of confluence for the second subclass. Note that the TRS of Example 6.2 is nonoverlapping, strongly normalizing and has confluent root overlaps, so we will not obtain an ω-simulation via ϕ for this subclass.

Definition 6.7 Let $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ be a TRS.

We say that $\mathfrak{R}$ has *linear constructor variants* if all subtemplates s with outermost symbol from $\mathcal{F} \cap \mathcal{A}$ (i.e. $s = f(t_1, \ldots, t_n)$ where $f \in \mathcal{F} \cap \mathcal{A}$, $t_1, \ldots, t_n \in T_{\Sigma,\mathcal{V}}$) are linear. $\dashv$

Observe that if $\mathfrak{R}$ is a nonoverlapping TRS with linear constructor variants, then $\langle \Sigma^\sharp, \mathcal{R}^\sharp{}_2 \rangle$ is still left-linear. Therefore, Lemma 5.6 still holds for this class of systems (the proof of this lemma only uses the left-linearity of the $\mathcal{R}^\sharp{}_2$-rules). Unfortunately, Lemma 5.7 does not hold for this class. Below, we will prove a weaker version, which will turn out to be enough. In the proof of this weaker version we need the following proposition.

Proposition 6.8 Let $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ be a nonoverlapping TRS with linear constructor variants and $\mathfrak{R}^\sharp = \langle \Sigma^\sharp, \mathcal{R}^\sharp{}_1 \cup \mathcal{R}^\sharp{}_2 \rangle$ its transform.
For all $s, s', t \in T_{\Sigma^\sharp,\mathcal{V}}$: $s' \leftarrow_{\mathcal{R}^\sharp_2} s \twoheadrightarrow_{\mathcal{R}^\sharp_2} t$ implies that there is a $t' \in T_{\Sigma^\sharp,\mathcal{V}}$ such that $s' \twoheadrightarrow_{\mathcal{R}^\sharp_2} t'$ and $t \to^i_{\mathcal{R}^\sharp_2} t'$, where $i \in \{0,1\}$.

Proof Induction w.r.t. the length of the reduction $s \twoheadrightarrow_{\mathcal{R}^\sharp_2} t$.

The case where $s \to^0_{\mathcal{R}^\sharp_2} t$ is trivial: take $t' = s'$. If $s \to^n_{\mathcal{R}^\sharp_2} u \to_{\mathcal{R}^\sharp_2} t$ and the lemma holds for reductions of length n, then the induction hypothesis yields a $u' \in T_{\Sigma^\sharp,\mathcal{V}}$ such that $s' \twoheadrightarrow_{\mathcal{R}^\sharp_2} u'$ and $u \to^i_{\mathcal{R}^\sharp_2} u'$ ($i \in \{0,1\}$). If $i = 0$, then $u = u'$ and thus $s \twoheadrightarrow_{\mathcal{R}^\sharp_2} u \to_{\mathcal{R}^\sharp_2} t$, so we are done.

If $i = 1$ then assume that $u \xrightarrow{o_1}_{\mathcal{R}^\sharp_2} t$ and $u \xrightarrow{o_2}_{\mathcal{R}^\sharp_2} u'$. If $o_1 = o_2$, then we take $t' = u' = t$, and if o_1 and o_2 are disjoint, then in t, u' we contract the single descendant of o_1, o_2 respectively yielding $u' \to_{\mathcal{R}^\sharp_2} t' \leftarrow_{\mathcal{R}^\sharp_2} t$. If $o_1 < o_2$, or $o_2 < o_1$, then since all rules in $\mathcal{R}^\sharp{}_2$ are left-linear by assumption, so t/o_2 and u'/o_1 are again $\mathcal{R}^\sharp{}_2$-redexes. Contraction of these redexes yields $u' \to_{\mathcal{R}^\sharp_2} t' \leftarrow_{\mathcal{R}^\sharp_2} t$. $\square$

Lemma 6.9 (Weak $\mathcal{R}^\sharp{}_1$-$\mathcal{R}^\sharp{}_2$ Strip Lemma) Let $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ be a nonoverlapping TRS with linear constructor variants and $\mathfrak{R}^\sharp = \langle \Sigma^\sharp, \mathcal{R}^\sharp{}_1 \cup \mathcal{R}^\sharp{}_2 \rangle$ its transform.
For all $s, s', t \in T_{\Sigma^\sharp, \mathcal{V}}$: $s' \twoheadleftarrow_{\mathcal{R}^\sharp{}_2} s \rightarrow_{\mathcal{R}^\sharp{}_1} t$ implies that there are $t', t'' \in T_{\Sigma^\sharp, \mathcal{V}}$ such that $s' \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} t' \rightarrow_{\mathcal{R}^\sharp{}_1} t'' \twoheadleftarrow_{\mathcal{R}^\sharp{}_2} t$.

Proof We first show that $s' \xleftarrow{o_2}_{\mathcal{R}^\sharp{}_2} s \xrightarrow{o_1}_{\mathcal{R}^\sharp{}_1} t$ implies that there are $t', t'' \in T_{\Sigma^\sharp, \mathcal{V}}$ such that $s' \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} t' \rightarrow_{\mathcal{R}^\sharp{}_1} t'' \twoheadleftarrow_{\mathcal{R}^\sharp{}_2} t$ by distinguishing cases according to the relative positions of o_1 and o_2 in s. The case where $o_1 = o_2$ is impossible as in the proof of Lemma 5.7, and for the cases where o_1 and o_2 are disjoint the proofs are trivial. We prove the remaining cases:

$o_1 < o_2$: Suppose $s \to t$ was an application of a (possibly nonlinear) $\mathcal{R}^\sharp{}_1$-rule $l \to r$ and $o_2 = o_1.p.q$, where $p \in O^{\mathcal{V}}(l)$. If l/p is a variable that occurs only once in l, then nothing is the matter; we reason exactly as in the proof of Lemma 5.7. If, however, the variable l/p occurs more than once in l then the reduction $s \xrightarrow{o_2}_{\mathcal{R}^\sharp{}_2} s'$ disturbs the redex s/o_1. Now observe that the redex can be restored by applying the same $\mathcal{R}^\sharp{}_2$-rule to other occurrences of s/o_2, so there is a term $s'' \in T_{\Sigma^\sharp, \mathcal{V}}$ such that $s' \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} s''$ and s''/o_1 is again an l-redex. Contraction of this redex yields a term u that is reachable from t by contracting all descendants of the disjoint redexes contracted in the reduction $s \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} s''$.[9]

$o_2 < o_1$: Since $\mathfrak{R}$ has linear constructor variants, all rules in $\mathcal{R}^\sharp{}_2$ are left-linear. Therefore, in t/o_2 is still an $\mathcal{R}^\sharp{}_2$-redex. For the remainder of this case the proof is trivial.

Now the lemma follows by induction w.r.t. the length of the reduction $s \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} s'$.
The case where $s \twoheadrightarrow^0_{\mathcal{R}^\sharp{}_2} s'$ is trivial. If $s \rightarrow^n_{\mathcal{R}^\sharp{}_2} u \rightarrow_{\mathcal{R}^\sharp{}_2} s'$ and the lemma holds for reductions of length n, then the induction hypothesis yields terms $u', v \in T_{\Sigma^\sharp, \mathcal{V}}$ such that $u \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} u' \rightarrow_{\mathcal{R}^\sharp{}_1} v$ and $t \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} v$. By Proposition 6.8 there exists $v' \in T_{\Sigma^\sharp, \mathcal{V}}$ such that $s' \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} v'$ and $u' \rightarrow^i_{\mathcal{R}^\sharp{}_2} v'$ with $i \in \{0, 1\}$. If $i = 0$ then $u' = v'$, so we are done. If $i = 1$ then by the above there are $v'', w \in T_{\Sigma^\sharp, \mathcal{V}}$ such that $v' \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} v'' \rightarrow_{\mathcal{R}^\sharp{}_1} w$ and $v \twoheadrightarrow_{\mathcal{R}^\sharp{}_2} w$. The proof for this latter case is depicted in the diagram below.

$$\begin{tikzcd}
s \arrow[rrr, "\mathcal{R}^\sharp{}_1"] \arrow[d, two heads, "\mathcal{R}^\sharp{}_2"'] \arrow[drrr, phantom, "\text{(IH)}"] & & & t \arrow[d, two heads, "\mathcal{R}^\sharp{}_2"] \\
u \arrow[r, two heads, "\mathcal{R}^\sharp{}_2"] \arrow[d, "\mathcal{R}^\sharp{}_2"'] \arrow[dr, phantom, "\text{Prop. 6.8}"] & u' \arrow[rr, "\mathcal{R}^\sharp{}_1"] \arrow[d, "\mathcal{R}^\sharp{}_2"] & & v \arrow[d, two heads, "\mathcal{R}^\sharp{}_2"] \\
s' \arrow[r, two heads, "\mathcal{R}^\sharp{}_2"'] & v' \arrow[r, two heads, "\mathcal{R}^\sharp{}_2"'] & v'' \arrow[r, "\mathcal{R}^\sharp{}_1"'] & w
\end{tikzcd}$$

□

Theorem 6.10 If $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ is a nonoverlapping TRS with linear constructor variants, $\mathfrak{R}^\sharp = \langle \Sigma^\sharp, \mathcal{R}^\sharp \rangle$ its transform, then $\mathfrak{R}^\sharp$ ω-simulates $\mathfrak{R}$ via ϕ ($\mathfrak{R} \lesssim^{\phi} \mathfrak{R}^\sharp$).

Proof We need to prove that the two conditions of Definition 4.1 hold.
For the first condition we argue as in the proof of Theorem 5.8, using Lemma 6.9 instead of Lemma 5.7.
For the second condition we can apply the same argument as in the proof of Theorem 5.8, literally. □

[9]Note that, because of the way we defined this concept, the sets of descendants of these redexes over the reduction $s \rightarrow_{\mathcal{R}^\sharp{}_1} t$ are equal.

It is an immediate consequence of the above theorem that Thatte's transformation preserves confluence for the third subclass mentioned above (Theorem 4.6). Another consequence, using Theorem 5.9(ii), is the following:

Corollary 6.11 Thatte's transformation is correct for the class of nonoverlapping TRSs with linear constructor variants (according to the definition of [FP96]).

We now proceed with a proof that Thatte's transformation preserves confluence for the class of weakly persistent, strongly normalizing systems. Verma proved confluence preservation for the class of strongly normalizing, consistent systems [Ver95b, Theorem 7]. Note that our result is stronger. By Corollary A.5 every strongly normalizing consistent system is weakly persistent. On the other hand, the TRS $\langle\{a,b,c,d\},\{a\to b,\ a\to c,\ b\to d,\ c\to d\}\rangle$ is trivially weakly persistent, strongly normalizing and confluent, but clearly not consistent.

For the first time we will really need weak persistence. More specifically, we need the following consequence of weak persistence:

Proposition 6.12 Let $\mathfrak{R} = \langle\Sigma,\mathcal{R}\rangle$ be a weakly persistent TRS and $\mathfrak{R}^\sharp = \langle\Sigma^\sharp,\mathcal{R}^\sharp\rangle$ its transform.
If $s \in [T_{\Sigma^\sharp,\mathcal{V}}]_{T_{\Sigma,\mathcal{V}}}$ and there is a $t \in T_{\Sigma,\mathcal{V}}$ such that $s^h \xrightarrow{r}_{\mathcal{R}} t$, then $s(\lambda) = s^h(\lambda)$ (i.e. if s is reachable and s^h a redex, then the outermost symbol of s is not a constructor variant).

Proof Assume $s \in [T_{\Sigma^\sharp,\mathcal{V}}]_{T_{\Sigma,\mathcal{V}}}$, $s = c_f(s_1,\ldots,s_n)$, $s^h = f(s_1^h,\ldots,s_n^h)$ (c_f the constructor variant of f) and $s^h \xrightarrow{r}_{\mathcal{R}} t$ for some $t \in T_{\Sigma,\mathcal{V}}$.
Since s is reachable, there is a $u \in T_{\Sigma,\mathcal{V}}$ such that $u \twoheadrightarrow_{\mathcal{R}^\sharp} s$ and at some point in this reduction an $\mathcal{R}^\sharp{}_2$-rule $l \to r$ must be applied to obtain the c_f-symbol. Note that without loss of generality we may assume that this application is at the root, so there are $u_1,\ldots,u_n \in T_{\Sigma^\sharp,\mathcal{V}}$ such that $u \twoheadrightarrow_{\mathcal{R}^\sharp} l(u_1,\ldots,u_n) \to_{\mathcal{R}^\sharp{}_2} r(u_1,\ldots,u_n) \twoheadrightarrow_{\mathcal{R}^\sharp} s$. By construction of $\mathcal{R}^\sharp$, r consists entirely of constructor symbols and variables, so $r(u_1,\ldots,u_n) \xrightarrow[r]{nt}{}^*_{\mathcal{R}^\sharp} s$.
It is easy to see that then also $(r(u_1,\ldots,u_n))^h \xrightarrow[h(r)]{nt}{}^*_{\mathcal{R}} s^h$ (see Lemma 5.4(i); Thatte's proof of this lemma is applicable here). Moreover, by construction of $\mathcal{R}^\sharp$, r^h is a subtemplate of $\mathcal{R}$. However, then we have a contradiction with the assumption that $\mathfrak{R}$ is weakly persistent, because $s^h \xrightarrow{r}_{\mathcal{R}} t$. □

Definition 6.13 Let $\mathfrak{R} = \langle\Sigma,\mathcal{R}\rangle$ be a TRS and $\mathfrak{R}^\sharp = \langle\Sigma^\sharp,\mathcal{R}^\sharp\rangle$ its transform.
We call $s \in T_{\Sigma^\sharp,\mathcal{V}}$ *balanced*, if $(u')^h = (u'')^h \Longrightarrow u' = u''$ for all $u', u'' \subseteq s$. ⊣

Proposition 6.14 Let $\mathfrak{R} = \langle\Sigma,\mathcal{R}\rangle$ be a weakly persistent TRS and $\mathfrak{R}^\sharp = \langle\Sigma^\sharp,\mathcal{R}^\sharp\rangle$ its transform.
If $s \in \mathsf{NF}([\mathfrak{R}^\sharp]_{T_{\Sigma,\mathcal{V}}})$ and s is balanced, then $s^h \in \mathsf{NF}(\mathfrak{R})$.

Proof Assume $s \in \mathsf{NF}([\mathfrak{R}^\sharp]_{T_{\Sigma,\mathcal{V}}})$ a balanced term, but $s^h \notin \mathsf{NF}(\mathfrak{R})$; we derive a contradiction.
There must be an occurrence $o \in O(s^h)$, a substitution σ and a rule $l \to r$ in $\mathcal{R}$ such that $s^h/o = l^\sigma$, i.e. there are $t_1,\ldots,t_n \in T_{\Sigma,\mathcal{V}}$ such that $s^h/o = l(t_1,\ldots,t_n)$.
Observe that if l contains some variable more than once, say at occurrences $o_1,\ldots,o_m \in O(l)$ $(m \geq 2)$, then $s/o_1 = s/o_2 = \cdots = s/o_m$, because s is balanced. Therefore, there is some $l' \in T_\Sigma$ and a substitution τ such that $(l')^h = l$ and $s = (l')^\tau$. Moreover, since s is balanced, $s(o) = s^h(o)$ (using Proposition 6.12), so $l'(\lambda) = l(\lambda)$.
By assumption $s \in \mathsf{NF}([\mathfrak{R}^\sharp]_{T_{\Sigma,\mathcal{V}}})$, so $(l') \neq l^{ic}$. Therefore, there must be an occurrence $o' \in O(l')$ such that $l'/o' \neq l^{ic}/o'$, while $o' < p \Longrightarrow l'/p = l^{ic}/p$ for all $p \in O(l')$. However, by construction of $\mathcal{R}^\sharp$, this l'/o' is an $\mathcal{R}^\sharp{}_2$-redex, which contradicts $s \in \mathsf{NF}([\mathfrak{R}^\sharp]_{T_{\Sigma,\mathcal{V}}})$. □

In the proof of the following lemma we will speak of 'linearized versions' of terms. We will informally introduce this notion. Let s be a term, possibly containing some variables more than once. s' is a *linearized version* of s if s' is a linear term that can be obtained from s by renaming occurrences of variables such that $o_1 \neq o_2 \Longrightarrow s/o_1 \neq s/o_2$ for all $o_1, o_2 \in O^{\mathcal{V}}(s)$. For instance, if $s = f(x, x, y, y)$ then $f(x_1, x_2, y_1, y_2)$ would be a linearized version of s. Of course, every term has infinitely many distinct linear versions. This will not impose a problem, because we are only interested in the instances of such linearized versions and all linearized versions of a term are instances of each other.

Lemma 6.15 Let $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ be a confluent, weakly persistent, and strongly normalizing TRS with transform $\mathfrak{R}^\sharp = \langle \Sigma^\sharp, \mathcal{R}^\sharp \rangle$.
For all $s, t \in \mathsf{NF}([\mathfrak{R}^\sharp]_{T_{\Sigma,\mathcal{V}}})$: $s^h = t^h \Longrightarrow s = t$.

Proof By Theorem 5.9 $\langle \mathsf{id}(T_{\Sigma,\mathcal{V}}), h \rangle$ preserves termination, so $[\mathfrak{R}^\sharp]_{T_{\Sigma,\mathcal{V}}} \models \mathsf{SN}$ (Proposition 3.5).
Fix $s, t \in \mathsf{NF}([\mathfrak{R}^\sharp]_{T_{\Sigma,\mathcal{V}}})$ and assume that $s^h = t^h$, but $s \neq t$; we derive a contradiction.
Since both s and t are in normal form all their subterms are in normal form. Therefore, without loss of generality, we may assume that s and t are balanced terms (if s is not balanced, then there are $u', u'' \subseteq s$ such that $u' \neq u''$, while $(u')^h = (u'')^h$, so we take $s = u'$ and $t = u''$; if t is not balanced then we apply a similar argument; repeating this will eventually yield balanced terms for s and t, because we only consider finite terms).
Moreover, because trivially all subterms of a balanced term are balanced, we may assume without loss of generality that s and t are minimal in the sense that s and t only differ at the root, i.e. $o \neq \lambda \Longrightarrow s/o = t/o$ for all $o \in O(s), O(t)$ (again we can find such minimal terms by inspection of s and t: if s, t are not minimal, then for some $o \neq \lambda$ we have $s/o \neq t/o$, so we take $s = s/o$ and $t = t/o$; this will eventually yield that s, t are minimal, by finiteness of terms).
If s and t are minimal, then $s = c_f(s_1, \ldots, s_n)$ and $t = f(t_1, \ldots, t_n)$ with $n = \mathsf{arity}(f)$ and $s_i = t_i$ for all $1 \leq i \leq n$ (or vice versa, but in that case we just interchange s and t).
Since s is reachable, there is a $u \in T_{\Sigma,\mathcal{V}}$ such that $u \twoheadrightarrow_{\mathcal{R}^\sharp} s$ and at some point in this reduction some $\mathcal{R}^\sharp{}_2$-rule $l \to r$ is applied to obtain the c_f-symbol. Without loss of generality we may assume that this application is at the root, so there are $u_1, \ldots, u_n \in T_{\Sigma^\sharp,\mathcal{V}}$ such that $u \twoheadrightarrow_{\mathcal{R}^\sharp} l(u_1, \ldots, u_m) \to_{\mathcal{R}^\sharp{}_2} r(u_1, \ldots, u_m) \twoheadrightarrow_{\mathcal{R}^\sharp} s$.
Moreover, by construction of $\mathcal{R}^\sharp{}_2$, all symbols at nonvariable positions in r are constructor symbols, so $r(u_1, \ldots, u_m) \xrightarrow[r]{nt}{}^*_{\mathcal{R}^\sharp} s$. Therefore, s is an instance of a linearized version of r and t is an instance of a linearized version of l (using minimality and the construction of $\mathcal{R}^\sharp{}_2$). Since by assumption t was in normal form, it cannot be an instance of l, so l contains at least one variable more than once, say at occurrences $o_1, o_2 \in O^{\mathcal{V}}(l)$, and $t/o_1 \neq t/o_2$. This implies $s/o_1 \neq s/o_2$.
However, $l(u_1, \ldots, u_m)/o_1 = l(u_1, \ldots, u_m)/o_2 = u_i$ for some $1 \leq i \leq m$, so, because $l(u_1, \ldots, u_m) \to_{\mathcal{R}^\sharp{}_2} r(u_1, \ldots, u_m) \xrightarrow[r]{nt}{}^*_{\mathcal{R}^\sharp} s$, we find $s/o_2 \twoheadleftarrow_{\mathcal{R}^\sharp} u_i \twoheadrightarrow_{\mathcal{R}^\sharp} s/o_1$. From this $(s/o_2)^h \twoheadleftarrow_{\mathcal{R}} u_i^h \twoheadrightarrow_{\mathcal{R}} (s/o_1)^h$ follows (using Lemma 5.4(i)).
Now observe that by minimality $(s/o_1)^h \neq (s/o_2)^h$. Moreover, since $s/o_1, s/o_2 \in \mathsf{NF}([\mathfrak{R}^\sharp]_{T_{\Sigma,\mathcal{V}}})$ are balanced terms, we find $(s/o_1)^h, (s/o_2)^h \in \mathsf{NF}(\mathfrak{R})$. Now we have reached a contradiction, because by assumption $\mathfrak{R}$ was confluent, and thus uniquely normalizing by Lemma 2.3(i). □

Corollary 6.16 Let $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ be a confluent, weakly persistent, and strongly normalizing TRS with transform $\mathfrak{R}^\sharp = \langle \Sigma^\sharp, \mathcal{R}^\sharp \rangle$.
If $s \in \mathsf{NF}([\mathfrak{R}^\sharp]_{T_{\Sigma,\mathcal{V}}})$, then s is balanced.

Proof Contraposition. Suppose s is not balanced, because there are $u', u'' \subseteq s$ such that $(u')^h = (u'')^h$ and not $u' = u''$. u' and u'' cannot be both in normal form, for this would contradict Lemma 6.15. However, in that case s is not in normal form either. □

Theorem 6.17 Let $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ be a weakly persistent and strongly normalizing TRS with transform $\mathfrak{R}^\sharp = \langle \Sigma^\sharp, \mathcal{R}^\sharp \rangle$.

i. $\langle \mathsf{id}(T_{\Sigma,\mathcal{V}}), h \rangle$ preserves confluence; and

ii. ϕ preserves confluence.

Proof

i. Fix $s \in T_{\Sigma,\mathcal{V}}$ and suppose $\mathfrak{R} \models \mathsf{CR}(s)$. In order to prove that $\mathfrak{R}^\sharp \models \mathsf{CR}(s)$ we show that if $t_1 \twoheadleftarrow_{\mathcal{R}^\sharp} s \twoheadrightarrow_{\mathcal{R}^\sharp} t_2$, then t_1 and t_2 have a common reduct.
Because $[\mathfrak{R}^\sharp]_{T_{\Sigma,\mathcal{V}}} \models \mathsf{SN}$ (Theorem 5.9 and Proposition 3.5) there are normal forms $n_1, n_2 \in [T_{\Sigma^\sharp,\mathcal{V}}]_{T_{\Sigma,\mathcal{V}}}$ such that $t_1 \twoheadrightarrow_{\mathcal{R}^\sharp} n_1$ and $t_2 \twoheadrightarrow_{\mathcal{R}^\sharp} n_2$. By Corollary 6.16 n_1 and n_2 are both balanced. Using Lemma 5.4(i) we find that $s \twoheadrightarrow_{\mathcal{R}} n_1^h$ and $s \twoheadrightarrow_{\mathcal{R}} n_2^h$, and by Proposition 6.14 $n_1^h, n_2^h \in \mathsf{NF}(\mathfrak{R})$. Now, since $\mathfrak{R}$ is confluent, $n_1^h = n_2^h$ and thus, by Lemma 6.15, $n_1 = n_2$, so $\mathfrak{R}^\sharp \models \mathsf{CR}(s)$.

ii. Immediate by the first part of this theorem and the fact that $s \in [T_{\Sigma^\sharp,\mathcal{V}}]_\phi \Longrightarrow s \in [T_{\Sigma^\sharp,\mathcal{V}}]_{T_{\Sigma,\mathcal{V}}}$. □

Corollary 6.18 Thatte's transformation preserves confluence for the subclass of nonoverlapping, strongly normalizing systems with confluent root overlaps.

Proof By Corollary A.5 systems that are nonoverlapping, strongly normalizing and have confluent root overlaps are weakly persistent. Preservation of confluence then follows from Theorem 6.17. □

7 Concluding Remarks

In this paper we investigated notions of correctness for transformations of Reduction Systems. We defined a notion of ω-simulation and studied its consequences in terms of the preservation of certain general properties defined for Reduction Systems. It turned out that our notion preserves the normal form relation, confluence and weak termination. Moreover, together with the condition that the transformation preserves termination, it constitutes correctness as defined by Fokkink and Van de Pol.

Furthermore, we considered the behaviour of Thatte's transformation on the various classes of Reduction Systems. We gave a counterexample to a key lemma in Verma's proof that this transformation preserves confluence for the class of weakly persistent systems. It is remarkable that the reason why Verma's proof fails is just the reason why there is no ω-simulation for this domain. We consider this an indication that our notion has its use in demonstrating the correctness of transformations for Reduction Systems.

Unfortunately, we did not succeed in our attempt to confirm or refute Verma's claim. However, we were able to give confluence preservation proofs for all the important subclasses of the class of weakly persistent systems that Verma mentions in his paper. Only in one case (Theorem 6.17) we really needed the weak persistence condition. We think that should there be a proof of Verma's claim, then the proof of this theorem might be very useful in finding it.

To our regret, this paper must end with the statement of a complex open problem:

> Does Thatte's transformation preserve confluence for the class of weakly persistent systems?

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A Weak Persistence

To apply Thatte's transformation to a larger domain containing nonlinear sytems, Verma introduces a concept that he calls 'weak persistence' [Ver95b], and as a kind of partial characterization Verma tries to prove a theorem stating that each subclass of TRSs that meets some requirements contains only weakly persistent systems. Unfortunately, the proof of this theorem in [Ver95b] is incomplete.

In this section we will give a simpler definition of weak persistence and show its equivalence with Verma's definition. Using our simpler definition we were able to give a complete proof of the above described theorem. As corollaries we obtain that (1) strongly normalizing systems with confluent root overlaps, and (2) some nonlinear systems with no overlaps as described in [Klo80] are weakly persistent.[10]

A.1 Alternative Definition

First we will introduce the concept of 'weak persistence' as it was formulated in [Ver95b] and we will repeat our own Definition 6.3. Then we show that the definitions are equivalent.

Definition A.1 (Definition 8 of [Ver95b]) A TRS $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ is *weakly persistent according to Verma* iff for every subtemplate s and s-pseudo-redex t such that:

1. $t \xrightarrow{nr}{}^{*}_{\mathcal{R}} t_1$, and

2. $t_1 \to_{\mathcal{R}} t_2$ by application of some rule $l \to r$ in $\mathcal{R}$ applied at $o \in O(t)$

there is an occurrence $o' \in O^{\mathcal{V}}(s)$, with $o' \leq o$. ⊣

Definition 6.3 (Weak Persistence) Let $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ be a TRS.
We say that $\mathfrak{R}$ is *weakly persistent* if there is no subtemplate s of $\mathcal{R}$ and s-pseudo-redex $t \in T_{\Sigma,\mathcal{V}}$ such that $\exists t_1, t_2 \in T_{\Sigma,\mathcal{V}} (t \xrightarrow[s]{nt}{}^{*}_{\mathcal{R}} t_1 \wedge t_1 \xrightarrow{r}_{\mathcal{R}} t_2)$. ⊣

Lemma A.2 Let $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ be a TRS. The following statements are equivalent:

1. $\mathfrak{R}$ is weakly persistent according to Verma.

2. $\mathfrak{R}$ is weakly persistent as defined in Definition 6.3.

Proof

(1 ⇒ 2) Suppose $\mathfrak{R}$ is weakly persistent according to Verma; we show that the conditions of Definition 6.3 are met. Let s be a subtemplate of $\mathcal{R}$ and t an s-pseudo-redex. Moreover, let $t \xrightarrow[s]{nt}{}^{*}_{\mathcal{R}} t_1$ for some $t_1 \in T_{\Sigma,\mathcal{V}}$. Clearly, the root of t is untouched in this reduction (i.e. $t \xrightarrow{nr}{}^{*}_{\mathcal{R}} t_1$) and thus it cannot be the case that $t_1 \xrightarrow{r}_{\mathcal{R}} t_2$ for some $t_2 \in T_{\Sigma,\mathcal{V}}$, for this would contradict weak persistence according to Verma.

(2 ⇒ 1) Contraposition. Suppose s is a subtemplate and t is an s-pseudo-redex contradicting the weak persistence according to Verma of $\mathfrak{R}$, i.e. there are a $t_1 \in T_\Sigma$ such that $t \xrightarrow{nr}{}^{*}_{\mathcal{R}} t_1$ and an $o \in O(t_1) \cap O^{\mathcal{F}}(s)$ with t_1/o a redex. We show that t/o is an s/o-pseudo-redex that contradicts weak persistence as in Definition 6.3 of $\mathfrak{R}$.

First observe that without loss of generality we may assume that $t \xrightarrow[s]{nt}{}^{*}_{\mathcal{R}} t_1$. If $t = t_1$ this is clear and if $t = t'_0 \xrightarrow{nr}_{\mathcal{R}} t'_1 \xrightarrow{nr}_{\mathcal{R}} \cdots \xrightarrow{nr}_{\mathcal{R}} t'_n = t_1$ $(n \geq 1)$, then the the first t'_{i+1} $(0 < i < n)$ such that $t'_i \xrightarrow[s]{nt}_{\mathcal{R}} t'_{i+1}$ does not hold, would already contradict

[10] As opposed to what Verma states, this does not include "certain kind of systems (more specifically in which the nonlinear steps can be postponed) with no overlaps in which no function symbol appearing in the lefthand side of a nonlinear rule appears in the righthand side of *any* rule" [Klo96].

weak persistence according to Verma, so we could have taken $t_1 = t'_i$ and $o = o'$, for t'_i/o' the contracted redex of $t'_i \xrightarrow{nr}_{\mathcal{R}} t'_{i+1}$.
Furthermore, observe that s/o is a subtemplate of $\mathcal{R}$, so t/o is an s/o-pseudo-redex, $t/o \xrightarrow[s/o]{nt}{}^*_R t_1/o$ and there is some rule of $\mathcal{R}$ applicable at the root of t_1/o.
Thus, we conclude that the existence of a pseudo-redex contradicting weak persistence according to Verma implies the existence of a a pseudo-redex contradicting weak persistence as in Definition 6.3. □

A.2 Implications

We will now give a complete proof of Theorem 9 in [Ver95b]. This theorem establishes a partial characterization of the class of weakly persistent systems. Verma's aim was to obtain as corollaries that three special subclasses of TRS contain only weakly persistent systems: (1) the subclass of left-linear nonoverlapping systems with confluent root overlaps; (2) the subclass of nonoverlapping, strongly normalizing systems with confluent root overlaps; and (3) the subclass of systems $\langle \Sigma \cup \{D_i, E\}, \mathcal{R} \cup \mathcal{D}_i \rangle$ ($i = k$ or $i = s$), where it is assumed that $D_k, D_s, E \notin \Sigma$, $\langle \Sigma, \mathcal{R} \rangle$ is a regular TRS, $\mathcal{D}_s = \{D_s(x,x) \to E\}$, and $\mathcal{D}_k = \{D_k(x,x) \to E(x)\}$ (these systems appear in [Klo80]).

For the subclass of left-linear nonoverlapping systems with confluent root overlaps we note that this is an immediate consequence of Proposition 6.4(ii). We will prove weak persistence for the other two subclasses below.[11]

In the sequel we assume that if Σ is a signature then $\underline{\Sigma}$ denotes the signature $\underline{\Sigma} = \{\underline{f} \mid f \in \Sigma\}$. We extend this underlining to terms, that is, if $s \in T_{\Sigma,\mathcal{V}}$ then $\underline{s}$ denotes the $\underline{\Sigma}$-term obtained by replacing all symbols at occurrences in $O^{\mathcal{F}}(s)$ by its corresponding underlined symbol and leaving variables unchanged. Moreover, if $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$, then $\underline{\mathfrak{R}}$ will denote the TRS $\langle \underline{\Sigma}, \underline{\mathcal{R}} \rangle$, where $\underline{\mathcal{R}} = \{\langle \underline{l}, \underline{r} \rangle \mid \langle l, r \rangle \in \mathcal{R}\}$. Finally we extend this notation to substitutions. If σ is a substitution that substitutes Σ-term t_i for variable x_i, then $\underline{\sigma}$ substitutes $\underline{\Sigma}$-term $\underline{t_i}$ for x_i.

Lemma A.3 (Overlap lemma, Lemma 10 of [Ver95b]) Suppose $s, t \in T_{\Sigma,\mathcal{V}}$ and that some function symbols in s and t are replaced by their underlined versions from $\underline{\Sigma}$ to get s' and t'. If s' (root or nonroot) overlaps t' then s (root or nonroot, respectively) overlaps t.

Proof Trivial. □

Theorem A.4 (Theorem 9 of [Ver95b]) Let $\mathcal{C}$ be a class of nonoverlapping, uniquely normalizing TRSs $\langle \Sigma, \mathcal{R} \rangle$ closed under name permutations and extensions $\mathcal{R} := \mathcal{R} \cup \{l \to a\}$ ($l \in T_{\Sigma,\mathcal{V}}$, a a new constant symbol, i.e. $a \notin \Sigma$) such that:

- $\{l\}$ is nonoverlapping;
- for all nonvariable subterms t_1 of a lhs m of a rule in $\mathcal{R}$ and t_2 of l:

$$\neg\, \exists \sigma, \tau\ (\sigma(t_1) = \tau(l) \vee \sigma(t_2) = \tau(m))$$

Then $\mathcal{C}$ only contains weakly persistent systems.

[11] Verma claims the weak persistence of this subclass as a consequence of Theorem A.4. In order to apply this theorem one needs to show that all systems in the subclass are uniquely normalizing. We did not find a proof of this particular fact and it is certainly not trivial. Verma introduced this subclass as a generalization of the concept 'regular'. We would like to point out that, for instance, the confluence proof in [BKM89] cannot be generalized to this subclass. It uses a notion of underlining and the fact that if some $\mathfrak{R}$ is regular, then its underlined version is WCR. That is not the case for arbitrary nonoverlapping systems with confluent root overlaps!

Proof Assume that $\mathcal{C}$ is as above, $\mathfrak{R} \in \mathcal{C}$ and $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ is *not* weakly persistent.

There must be a subtemplate s of some lhs l of a rule in $\mathcal{R}$ and an s-pseudo-redex t, such that $t \xrightarrow[s]{nt}_{\mathcal{R}} t_1$, with t_1 a redex for a rule $l_1 \to r_1$ in $\mathcal{R}$ ($t = \tau(l_1)$, for substitution τ).

Assume that $q, d, c \notin \Sigma \cup \underline{\Sigma}$ and define $\mathfrak{G} = \langle \Gamma, \mathcal{G} \rangle$ by

$$\mathfrak{G} := \langle \Sigma \cup \underline{\Sigma} \cup \{q, d, e\}, \underline{\mathcal{R}} \cup \{q(s') \to d, l_1' \to e\} \rangle$$

in which s' and l' are obtained by underlining function symbols at all occurrences except the root.

It is easy to verify that the extension is such that it implies that $\mathfrak{G} \in \mathcal{C}$ (note that, by Lemma A.3, if l_1' overlaps s', then l_1 overlaps s, quod non).

Now, to arrive at a contradiction, we prove that $\mathfrak{G}$ is not uniquely normalizing. Let $t = \sigma(s)$; $u_1 = q(\underline{\sigma}(s'))$. By our choice of u_1 and construction of $\mathfrak{G}$, there is a term u_2 corresponding to t_1, such that $u_1 \xrightarrow[q(s')]{nt}_{\underline{\mathcal{R}}} q(u_2)$ and u_2 is an instance of l_1'. Now $u_1 \xrightarrow{r}_{\mathcal{G}} d$ via $q(s') \to d$ and $q(u_2) \to_{\mathcal{G}} q(e)$. Clearly, both d and $q(e)$ are normal forms, so $\mathfrak{G}$ is not uniquely normalizing. We conclude that there cannot be a $\mathcal{C}$ satisfying all conditions mentioned that contains a system that is not weakly persistent. □

Remark 1 Note that the requirement that classes as meant in the above theorem should be closed under extensions $\{l \to a\}$ in which a is a constant, leads to an essentially stronger theorem, than if the requirement on a had been that is is just an arbitrary normal form.

Furthermore, Verma included in his formulation of the theorem the requirement that l should be linear if all $\mathfrak{R} \in \mathcal{C}$ are left-linear. This requirement does not play a rôle in the proof and is only to make the theorem also applicable to subclasses of the class of left-linear, nonoverlapping systems (these classes trivially are not closed under extension with a nonlinear rule). However, this is not very useful, as our Proposition 6.4(ii) already states that the whole class of left-linear, nonoverlapping systems only contains weakly persistent systems.

With the above theorem at hand we now show that two of the three subclasses of TRSs mentioned above only contain weakly persistent systems.

Corollary A.5 The subclass of nonoverlapping, strongly normalizing TRSs with confluent root overlaps contains only weakly persistent TRSs.

Proof We show that this subclass meets the conditions of Theorem A.4.

Note that if $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ is a nonoverlapping TRS with confluent root overlaps, then $\mathfrak{R} \models \mathsf{WCR}$ by Proposition 2.4. In addition it is $\mathfrak{R} \models \mathsf{SN}$ by assumption, so by an application of Newman's Lemma (Lemma 2.3(ii)) $\mathfrak{R} \models \mathsf{CR}$. Thus, using Lemma 2.3(i), we get that $\mathfrak{R} \models \mathsf{UN}$.

Moreover, if a rule $l \to a$ is added ($\mathcal{R}^+ = \mathcal{R} \cup \{l \to a\}$), such that l has no overlap with a rule in $\mathcal{R}$, then (trivially) the resulting system is nonoverlapping and has confluent root overlaps.

Finally, it is not difficult to see that the resulting system is again strongly normalizing, for if not, then we derive a contradiction as follows. Suppose that the resulting system has an infinite reduction

$$s_0 \to_{\mathcal{R}^+} s_1 \to_{\mathcal{R}^+} \cdots \to_{\mathcal{R}^+} s_n \to_{\mathcal{R}^+} \cdots$$

for $s_i \in T_{\Sigma \cup \{a\}, \mathcal{V}}$, $i \in \omega$. Observe that because all rules in $\mathcal{R}^+$ are left-linear and the lefthand sides of these rules do not contain the new constant symbol a, the number of redexes in a term strictly decreases by an application of $l \to a$ to it. Therefore, there must be infinitely many applications of $\mathcal{R}$-rules in the above reduction. More specifically, if $s \to_{\mathcal{R}^+} t \to_{\mathcal{R}^+} u$ then $s \to_{\mathcal{R}^+} u$, so there is some infinite reduction

$$s_{i_0} \to_{\mathcal{R}} s_{i_1} \to_{\mathcal{R}} \cdots \to_{\mathcal{R}} s_{i_n} \to_{\mathcal{R}} \cdots$$

with $i_0 = 0$ and $i_{j+1} > i_j$ for all $j \in \omega$. Replacing all a-symbols in the s_{i_j} by (for instance) l yields an infinite reduction in $\mathfrak{R}$, which contradicts the assumption that $\mathfrak{R}$ is strongly normalizing. □

Corollary A.6 The subclass of systems $\langle \Sigma \cup \{D_i, E\}, \mathcal{R} \cup \mathcal{D}_i \rangle$ ($i = k$ or $i = s$), where it is assumed that $D_k, D_s, E \notin \Sigma$, $\langle \Sigma, \mathcal{R} \rangle$ is a regular TRS, $\mathcal{D}_s = \{D_s(x,x) \to E\}$, and $\mathcal{D}_k = \{D_k(x,x) \to E(x)\}$ contains only weakly persistent TRSs.

Proof Note that a regular system extended in the way Theorem A.4 describes is still regular.

Furthermore, Klop proves in [Klo80] that the mentioned systems are confluent, so they are uniquely normalizing as well. □