Asymptotic results for injection of reactive solutes from a three-dimensional well

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In this paper we consider some asymptotic aspects related to the profile of a reactive solute, which is injected from a well (radius $\epsilon > 0$) into a three-dimensional porous medium. We present a convergence result for $\epsilon \downarrow 0$ as well as the large time behaviour. Regarding the latter we show that the solute profile evolves in a self-similar way towards a stationary distribution and we give an estimate for the rate of the convergence. This paper extends earlier work of van Duijn & Peletier [5], where the two-dimensional case was treated.

1. INTRODUCTION

Suppose a homogeneous and saturated porous medium occupies the region

$$\Omega_\epsilon = \{x \in \mathbb{R}^3 : |x| > \epsilon\}.$$

Here ϵ denotes the radius of an injection well, which induces a radially symmetric flow in Ω_ϵ. At a certain instance ($t = 0$), a reactive solute at tracer concentration is added to the fluid in the well and subsequently carried into the porous medium. Within the medium, the solute interacts with the porous matrix by means of equilibrium adsorption.

Following van Duijn & Knabner [3], where a detailed derivation was presented, we find for the scaled solute concentration $u \colon \Omega_\epsilon \times [0, \infty) \mapsto [0, \infty)$ the following nonlinear initial-boundary value problem:

$$(\mathrm{P}_\epsilon) \quad \begin{cases} \beta(u)_t + \mathrm{div}\bar{F} = 0 & \text{in} \quad \Omega_\epsilon, \quad t > 0 & (1) \\ \bar{F} \cdot \mathbf{e}_r = u_e \boldsymbol{q} \cdot \mathbf{e}_r & \text{on} \quad \partial\Omega_\epsilon, \quad t > 0 & (2) \\ u(\cdot, 0) = u_0(\cdot) & \text{in} \quad \Omega_\epsilon. & (3) \end{cases}$$

Here $\bar{F} = \boldsymbol{q}u - \nabla u$ denotes the solute flux, $\boldsymbol{q} = \frac{\Lambda}{|x|^2}\mathbf{e}_r$ the induced flow field, and $\Lambda > 0$ the Peclet number of the problem, which combines the effects of flow rate and dispersion. In (2), u_e denotes the solute concentration in the injection well and $\mathbf{e}_r$ is the unit vector in radial direction. The adsorption mechanism is accounted for by the nonlinear term $\beta = \beta(u)$. Generally it takes the form

$$\beta(u) = u + \psi(u), \tag{4}$$

where ψ is called the adsorption isotherm (see for instance van Duijn & Knabner [4]). Typical examples are

$$\psi(u) = \frac{k_1 u}{1 + k_2 u}, \quad k_1, k_2 > 0, \qquad \text{(Langmuir isotherm)}$$

or

$$\psi(u) = ku^p, \quad k > 0, \quad p \in (0,1) \qquad \text{(Freundlich isotherm)}.$$

In a two-dimensional setting, Problem P_ϵ was previously considered by van Duijn & Knabner [3] and van Duijn & Peletier [5]. In [3] the authors derived a radially symmetric self-similar solution of equation (1) of the form $u(r,t) = f(r/\sqrt{t})$. This solution is defined on all $\mathbb{R}^2$ but does not satisfy boundary condition (2). In [5] it was demonstrated that this solution describes the large-time behaviour for general two-dimensional radially symmetric solutions of (1–2) and rates of convergence were given.

The existence of self-similar solutions in two dimensions requires the well injection rate to be constant in time. In three spatial dimensions self-similar solutions still exist but require the injection rate and therefore Λ to grow as $\sqrt{t}$. From a practical point of view this is an unsatisfactory setup and the main goal of this paper is to investigate the large-time behaviour of solutions under a constant injection rate. We do this in the framework of a contamination event (see also [5]), i.e. assuming that far away from the well no solute (contaminant) is present.

Two natural questions arise form Problem P_ϵ: the behaviour as $\epsilon \downarrow 0$ and as $t \to \infty$. Since in [5] the authors were only concerned with radially symmetric solutions, their proofs of the limiting behaviour as $\epsilon \downarrow 0$ and as $t \to \infty$ follow essentially along the same lines. This is due to the scale invariance of the equation and the boundary condition. In this paper the proofs are quite different and are treated separately.

We first consider the behaviour as $\epsilon \downarrow 0$. Taking the formal limit in the combination (1–2) yields the equation

$$\beta(u)_t + \mathrm{div}(F) = u_e \delta_{x=0} \quad \text{in} \quad \mathbb{R}^3, \quad t > 0 \tag{5}$$

where $\delta_{x=0}$ denotes the Dirac distribution at the origin. Thus the boundary condition at the well appears as a source term in the equation. We refer to (5), together with the initial condition

$$u(\cdot,0) = u_0(\cdot) \qquad \text{in} \quad \mathbb{R}^3 \tag{6}$$

as Problem P or (P).

Regarding the initial conditions (3) and (6), we take (3) as the restriction of (6) to Ω_ϵ, and assume

$$(H_{u_0}) \quad u_0 \in L^\infty(\mathbb{R}^3);\ \ u_0 \geq 0 \text{ in } \mathbb{R}^3;\ \ \lim_{|x|\to\infty} u_0(x) = 0;\ \int\limits_{\mathbb{R}^3} \beta(u_0)\,dx < \infty.$$

Note that we allow non-radial initial data.

With respect to the nonlinear capacity term $\beta = \beta(u)$ we assume the regularity

$$(H_\beta 1) \qquad \beta \in C^\infty(0,\infty) \cap C([0,\infty)),$$

and the structural properties

$$(H_\beta 2) \qquad \beta(0) = 0,\ \beta'(s) > 0,\ \text{ and }\ \beta''(s) \leq 0\ \text{ for }\ s > 0.$$

Later, when we consider the large-time behaviour, we will add some additional hypotheses, essentially expressing that $\beta(u)$ behaves as u^p $(0 < p \leq 1)$ near $u = 0^+$.

Since equation (1) is scale invariant, we may set $\Lambda = 1$ after redefining $\epsilon := \epsilon/\Lambda$. By redefining $\beta(u) := \beta(u_e u)/u_e$ we may also set $u_e = 1$.

Our first theorem makes the stabilization as $\epsilon \downarrow 0$ precise.

THEOREM 1.A. *Let* (H_{u_0}) *and* $(H_\beta 1\text{–}2)$ *be satisfied. Further, let* u^ϵ *be the unique weak solution of* (P_ϵ). *Then*

$$u^\epsilon \to u \quad \textit{as} \quad \epsilon \to 0, \quad \textit{uniformly in compact subsets of } (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}^+,$$

where u *is a weak solution of Problem* P.

The definition of weak solutions as well as the proof of Theorem 1.A are given in Section 2.

Next we consider the large-time behaviour. We expect that different small well radii (ϵ) lead to the same large-time behaviour. This was shown rigorously [5] for the two-dimensional case. With this in mind we consider only the large-time behaviour for Problem P and for technical reasons we limit ourselves to radially symmetric solutions. Before we state the convergence result, we provide some motivation.

The radial form of equation (5) is:

$$\beta(u)_t + \frac{1-2r}{r^2}u_r - u_{rr} = 0 \quad \text{in} \quad 0 < r < \infty, \quad t > 0, \tag{7}$$

and, as shown in Proposition 2.4, its solutions satisfy the boundary condition

$$u(0,t) = 1 \quad \text{for all} \quad t > 0. \tag{8}$$

The initial condition takes the form

$$u(r,0) = u_0(r) \quad \text{for} \quad 0 < r < \infty. \tag{9}$$

Equation (7) admits a nontrivial stationary solution $w = w(r)$, satisfying $w(0) = 1$ and $w(\infty) = 0$. It is given by

$$w(r) = 1 - e^{-1/r}, \tag{10}$$

and under the conditions of Theorem 1.B below the solution u converges to this stationary state.

The appearance of (10) is quite different from the two-dimensional case. There the only bounded stationary solution satisfying $w(0) = 1$ is the constant state $w \equiv 1$. In [5] it was shown that the solution attains this state in a self similar way, namely

$$u(r,t) \sim f(r/\sqrt{t}) \quad \text{as} \quad t \to \infty$$

where $f(0) = 1$.

In this paper we assume an analogous behaviour with respect to (10), i.e.

$$\frac{u(r,t)}{w(r)} \sim f(r/t^\alpha) \quad \text{as} \quad t \to \infty \tag{11}$$

for some $\alpha > 0$, where $f(0) = 1$. To this end we set

$$\tilde{z}(r,t) := \frac{u(r.t)}{w(r)}$$

and introduce the coordinate transformation

$$\eta = r/t^\alpha, \qquad \tau = \log t.$$

Then $z(\eta,\tau) = \tilde{z}(r,t)$ satisfies:

$$e^{(2\alpha-1)\tau}[\beta(zw)_\tau - \alpha\eta\beta(zw)_\eta] + \frac{e^{-\alpha\tau} - 2\eta}{\eta^2}(zw)_\eta - (zw)_{\eta\eta} = 0. \quad (12)$$

To obtain the convergence (11), we study the large-τ behaviour of (12). In particular we need to select the exponent α so that the appropriated terms in (12) balance as $\tau \to \infty$. For this purpose we rewrite the equation as

$$e^{(2\alpha-1)\tau}\beta'(zw)z_\tau - \alpha e^{(2\alpha-1)\tau}\eta\beta'(zw)z_\eta - z_{\eta\eta} + \frac{1}{\eta}A(\frac{1}{\eta e^{\alpha\tau}})z_\eta = 0, \quad (13)$$

where $A(s) := \dfrac{2s}{e^s - 1} + s - 2$ with $\lim_{s\to 0} A(s) = 0$.

To find the appropriate balance, we observe that for fixed $\eta > 0$, $\tau \to \infty$ implies $r \to \infty$. Since $u(r,t) \to 0$ as $r \to \infty$, the behaviour of β near 0 is critical. Let us assume

$$\beta(s) \sim s^p \quad (0 < p \le 1) \quad \text{as } s \downarrow 0. \quad (14)$$

Using this and $w(r) \to 1/r$, as $r \to \infty$, we find that the second and third term in (13) balance if and only if $\alpha = 1/(3-p)$.

The resulting equation is

$$\alpha\eta^{2-p}(f^p)_\eta + f_{\eta\eta} = 0 \quad \text{or} \quad \alpha\eta^{3-p}(f^p)_\eta + (\eta f_\eta - f)_\eta = 0 \text{ for } 0 < \eta < \infty, \quad (15)$$

where $f(\eta) := \lim_{\tau\to\infty} z(\eta,\tau)$. Note the resemblance between (15) and the limiting equation obtained in [3].

Before we state the main convergence theorem, we specify some additional hypotheses on β. Related to (14) we assume that there exists $0 < p \le 1$ such that

$$(H_\beta 3) \qquad \frac{\beta'(s)}{ps^{p-1}} = \ell + O(s^\gamma) \qquad \text{as} \quad s \downarrow 0,$$

for some $\ell > 0$ and $\gamma \in (0, 3-p)$. Furthermore we assume the lower bound

$$(H_\beta 4) \qquad \inf_{s\in[0,1]} \frac{\beta'(s)}{ps^{p-1}} = m > 0.$$

Let $\beta_p(s) := \ell s^p$ and $\varphi(s) := \dfrac{\beta'(s) - \beta_p'(s)}{ps^{p-1+\gamma}}$.

Remark 1. 1. The simplest function β that satisfies (H_β3–4) is

$$\beta(s) = ks^p \quad p \in (0,1],$$

with $\ell = m = k$, $\varphi \equiv 0$, and for any $\gamma \in (0, 3-p)$. Hypotheses $(H_\beta 3$–$4)$ are also fulfilled by the examples given at the beginning of the introduction. In the case of the Freundlich isotherm,

$$\beta(s) = s + ks^p \quad p \in (0,1),$$

we have $\ell = m = k$, and $\gamma = 1 - p$. Note that this choice implies $\varphi(s) = 1/p > 0$. In the Langmuir isotherm case,

$$\beta(s) = s + \frac{k_1 s}{k_2 s + 1} \quad k_1, k_2 > 0,$$

we have $p = 1$, $\ell = k_1 + 1$, $m = 1 + \frac{k_1}{(k_2+1)^2}$, $\gamma = 1$, and $\varphi(s) = -k_1 k_2 \frac{k_2 s + 2}{(k_2 s+1)^2} \leq 0$.

Below we use the notation $[\cdot]_+ := \max\{\cdot, 0\}$, $\varphi_+ := [\varphi]_+$, and $\varphi_- := [-\varphi]_+$.

THEOREM 1.B. *Let hypotheses $(H_\beta 1$–$4)$ and (H_{u_0}) be satisfied, and let u be a weak solution of Problem* P. *Then we have the following estimates:*

$$0 \leq e^{p\alpha\tau} \int_0^\infty [u^p - f^p w^p]_+ \eta^2 \, d\eta \leq L_1 e^{-\alpha\tau} + L\|\varphi_-\|_{L^\infty} e^{-\alpha\gamma\tau} \tag{16}$$

for all $\tau \in \mathbb{R}$, and

$$0 \leq e^{p\alpha\tau} \int_0^\infty [f^p w^p - u^p]_+ \eta^2 \, d\eta \leq L_2 e^{-\alpha\tau} + L\|\varphi_+\|_{L^\infty} e^{-\alpha\gamma\tau} \tag{17}$$

for all $\tau \in \mathbb{R}$. Here L_1, L_2, and L are positive constants and $\alpha = 1/(3-p)$.

The function f is the unique solution of

$$\text{(S)} \qquad \begin{cases} \alpha\eta^{2-p}\beta_p(f)_\eta + f_{\eta\eta} = 0 & \textit{for } 0 < \eta < \infty, \\ f(0) = 1, \quad f(\infty) = 0. \end{cases}$$

Figure 1 shows the limit function $r \mapsto w(r) f(r/\sqrt{t})$ for different t, in the case $p = 1$.

Remark 1. 2. Note that the constants in the estimates of Theorem 1.B depend on p. For instance, it follows from the proof that if $p = 1$ then

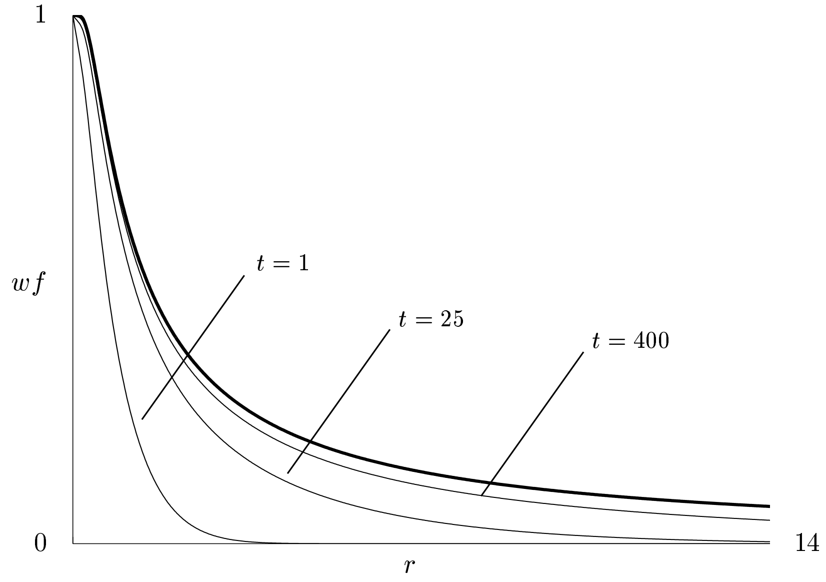

FIG. 1. The function $r \mapsto w(r)f(r/\sqrt{t}); t = 1, 25, 400$.

$L_2 = 0$. An immediate consequence of this fact concerns functions β of the form

$$\beta(s) = s + \frac{k_1 s}{k_2 s + 1} \quad k_1, k_2 > 0.$$

Here $p = 1$ and $\varphi \leq 0$ (Remark 1.1), so that $fw \leq u$.

Remark 1. 3. The mass of the system increases linearly in time. The scaling used in (16) (and (17)) is chosen to normalize the increase of mass:

$$\frac{1}{t}\int_0^\infty [u^p - f^p w^p]_+ r^2 \, d\eta = e^{p\alpha\tau}\int_0^\infty [u^p - f^p w^p]_+ \eta^2 \, d\eta.$$

In this scaled metric the solutions u and fw converge. In the unscaled (original) metric the distance increases without bound.

2. CONVERGENCE AS $\epsilon \to 0$

2.1. Weak solutions of (P_ϵ)

Let T be a fixed positive number which eventually tends to infinity and let $E_T^\epsilon = \Omega_\epsilon \times (0,T]$. Note that we have rescaled the problem such that $u_e = \Lambda = 1$.

DEFINITION 2.1. A weak solution of Problem P_ϵ is a non-negative function u such that

(i) $u \in C(E_T^\epsilon)$ and $\nabla u \in L^2(E_T^\epsilon)$,
(ii) For every test function $\phi \in L^2(0,T;H^1(\Omega_\epsilon)) \cap H^1(0,T,L^2(\Omega_\epsilon))$ that vanishes for large $|x|$ and at $t = T$,

$$\int_{E_T^\epsilon} \{\beta(u)\phi_t + (\boldsymbol{q}\, u - \nabla u)\nabla\phi\}\, dxdt + \int_{\Omega_\epsilon} \beta(u_0)\phi(0)\, dx + \\ + \frac{1}{\epsilon^2}\int_0^T \int_{\partial\Omega_\epsilon} \phi\, dSdt = 0. \tag{18}$$

If u satisfies (i) and (ii) with the equality replaced by $\geq$ ($\leq$) and with $\phi \geq 0$ in E_T^ϵ then we call u a sub(super)solution. Here and in the sequel, we use the obvious notation $\phi(0) = \phi(t = 0)$.

THEOREM 2.1 (Existence for (P_ϵ)). *Let* (H_{u_0}) *and* (H_β*1–2*) *be satisfied. Then there exists a unique weak solution of* (P_ϵ).

The proof of existence will be given in Section 2.3, the uniqueness follows from Proposition 2.2 below.

PROPOSITION 2.1. *Let u be the weak solution of Problem* P_ϵ. *For each $t > 0$,*

$$\int_{\Omega_\epsilon} \beta(u(t))\, dx = \int_{\Omega_\epsilon} \beta(u_0)\, dx + 4\pi t.$$

The proof of Proposition 2.1 follows along the same lines as in [7].

2.2. Uniqueness of (P_ϵ)

Throughout this section we denote $\Omega_\epsilon^m = \{x \in \mathbb{R}^3 \mid \epsilon < |x| < m\}$ and similarly $E_T^{\epsilon,m} = \Omega_\epsilon^m \times (0,T]$.

In order to prove the comparison result for Problem P_ϵ, we introduce as in [2] an equivalent definition of solution, which we call *generalized solution:*

DEFINITION 2.2. A generalized solution of Problem P_ϵ is a function u satisfying:

(i) u is bounded, nonnegative, and continuous on E_T^ϵ;
(ii) for any $t \in (0,T]$ and any bounded domain $\Omega'_\epsilon \subset \Omega_\epsilon$ with smooth boundary $\partial\Omega'_\epsilon := \Gamma_\epsilon \cup \Gamma$, such that $\Gamma_\epsilon \subset \partial B_\epsilon$ and $\Gamma \cap \partial B_\epsilon = \emptyset$,

$$\int_{\Omega'_\epsilon} \beta(u(t))\phi(t)\,dx - \int_{\Omega'_\epsilon}\int_0^t \{\beta(u)\partial_t\phi + u\boldsymbol{q}\nabla\phi + u\Delta\phi\}\,dxdt +$$

$$+\frac{1}{\epsilon^2}\int_0^t\int_{\Gamma_\epsilon} \phi\,dSdt + \int_0^t\int_\Gamma u\partial_\nu\phi\,dSdt = \int_{\Omega'_\epsilon}\beta(u_0)\phi(0)\,dx \tag{19}$$

for all $\phi \in C^{2,1}(\Omega'_\epsilon \times (0,t])$, $\phi \geq 0$ with $\partial\phi/\partial\nu = 0$ on $\Gamma_\epsilon \times (0,t)$ and $\phi = 0$ on $\Gamma \times (0,t]$.

We define a subsolution (supersolution) by (i) and (ii) with the equality replaced by $\leq$ ($\geq$).

For the proof of equivalence between generalized and weak solutions we refer to [2].

PROPOSITION 2.2. *Let u^1 and u^2 be generalized sub- and supersolutions with initial data u_0^1 and u_0^2 respectively. Then for any $t \in [0,T]$, we have*

$$\int_{\Omega_\epsilon}[\beta(u^1(t)) - \beta(u^2(t))]_+\,dx \leq \int_{\Omega_\epsilon}[\beta(u_0^1) - \beta(u_0^2)]_+\,dx.$$

Proof. Let $\bar{u} = u^1 - u^2$ and $\bar{\beta} = \beta(u^1) - \beta(u^2)$. Subtracting equations (19) we find

$$\int_{\Omega'_\epsilon}\bar{\beta}(t)\phi(t)\,dx - \int_{\Omega'_\epsilon}\bar{\beta}(0)\phi(0)\,dx \leq \int_0^t\int_{\Omega'_\epsilon}\{\bar{\beta}(t)\partial_t\phi + \bar{u}(\boldsymbol{q}\nabla\phi + \Delta\phi)\}\,dxdt$$

$$-\int_0^t\int_\Gamma \bar{u}\partial_\nu\phi\,dSdt. \tag{20}$$

Following [1] we define a family of weight functions $\omega_\lambda\colon \mathbb{R}^3 \to \mathbb{R}^+$, for each $\lambda > 0$, by

$$\omega_\lambda(x) = \begin{cases} 1 & \text{if } |x| \in (\epsilon, 1), \\ e^{-\sqrt{\lambda}(|x|-1)} & \text{if } |x| \in (1,\infty). \end{cases}$$

Hypothesis $(H_\beta 2)$ implies that there exists $b_0 > 0$ such that $\beta'(s) \geq b_0$ for all $s \in \mathbb{R}$. We define $A\colon \Omega'_\epsilon \times \mathbb{R} \to \mathbb{R}$ by:

$$A(x,t) = \begin{cases} \frac{\beta(u^1)-\beta(u^2)}{u^1-u^2} & \text{if } u^1 \neq u^2, \\ b_0 & \text{if } u^1 = u^2. \end{cases}$$

We choose $\xi \in C_c^\infty(\Omega_\epsilon)$ such that $0 \leq \xi \leq 1$, with $\partial\xi/\partial\nu = 0$ in Γ_ϵ. In addition let $\Omega'_\epsilon = \Omega_\epsilon^m$ where $m > 0$ is such that $\operatorname{supp}\xi \subset B_m$. We introduce smooth functions $A_m\colon \Omega_\epsilon^m \times (0,T) \mapsto \mathbb{R}$, satisfying

$$0 < b_0 \leq A_m \leq \|A\|_{L^\infty(E_T^\epsilon)} + \frac{1}{m}, \qquad \|\frac{A_m - A}{\sqrt{A_m}}\|_{L^2(E_T^{\epsilon,m})} \to 0. \tag{21}$$

Consider for each A_m the problem

$$(\mathrm{PA_m})\begin{cases} A_m\partial_\tau\phi + \boldsymbol{q}\nabla\phi + \Delta\phi = \lambda\phi & \text{in} \quad \Omega_\epsilon^m \times [0,t] \\ \partial_\nu\phi = 0 & \text{on} \quad \partial B_\epsilon \times [0,t] \\ \phi = 0 & \text{on} \quad \partial B_m \times [0,t] \\ \phi(x,t) = \xi(x)\omega_\lambda(x) & \text{in} \quad \Omega_\epsilon^m. \end{cases}$$

This equation has a unique solution $\phi_m \in C^{2,1}(\overline{\Omega_\epsilon^m} \times [0,t])$, $\phi_m \geq 0$. Using ϕ_m as a test function, we find

$$\int_{\Omega_\epsilon^m} \bar\beta(t)\xi(x)\omega_\lambda(x)\,dx - \int_{\Omega_\epsilon^m} \bar\beta(0)\phi_m(x,0)\,dx \leq \int_{E_t^\epsilon} \bar u(A - A_m)\partial_t\phi_m\,dxdt$$
$$+\lambda\int_{E_t^\epsilon} \bar u\phi_m\,dxdt - \int_0^t\int_{\partial\Gamma_m} \bar u\partial_\nu\phi_m\,dSdt. \tag{22}$$

LEMMA 2.1. *The functions ϕ_m satisfy the following properties:*

(i)$0 \leq \phi_m \leq \omega_\lambda$ *in* E_t^ϵ

(ii) $\displaystyle\int_{E_t^{\epsilon,m}} A_m|\partial_\tau\phi_m|^2 dxdt \leq C;$

(iii) $\displaystyle\sup_{0\leq\tau\leq t}\int_{\Omega_\epsilon^m} |\nabla\phi_m(\tau)|^2 dx \leq C;$

(iv)$0 \leq -\phi_{m\nu} \leq Ce^{-\sqrt{\lambda}m}$ *on* $\partial B_m \times [0,t]$.

Proof. Part (i) is a consequence of the maximum principle. Parts (ii–iii) are standard estimates. To prove (iv), we follow the ideas of [1]. We fix

$m_0 < m$ such that $\operatorname{supp}\xi \subset B_{m_0}$ and define $\widetilde{\omega}_\lambda \colon B_m \to [0,1]$ separately on the two subsets B_{m_0} and $\Omega^m_{m_0}$. In B_{m_0} we set $\widetilde{\omega}_\lambda = \omega_\lambda$, and in $\Omega^m_{m_0}$ we define $\widetilde{\omega}_\lambda$ as the solution of

$$\begin{aligned} \boldsymbol{q}\nabla\widetilde{\omega}_\lambda + \Delta\widetilde{\omega}_\lambda - \lambda\widetilde{\omega}_\lambda &= 0 \quad \text{in} \quad \Omega^m_{m_0} \\ \widetilde{\omega}_\lambda &= \omega_\lambda \quad \text{on} \quad \partial B_{m_0} \\ \widetilde{\omega}_\lambda &= 0 \quad \text{on} \quad \partial B_m. \end{aligned} \tag{23}$$

By (i) we have $0 \le \phi_m \le \widetilde{\omega}_\lambda$ on $B_{m_0} \times (0,t]$; by an application of the comparison principle on $\Omega^m_{m_0} \times (0,t]$ it follows that $0 \le \phi_m \le \widetilde{\omega}_\lambda$ on $\Omega^m_\epsilon \times (0,t]$. Therefore $0 \le -\phi_{m\nu} \le -\widetilde{\omega}_{\lambda\nu}$ on ∂B_m. To estimate $\widetilde{\omega}_{\lambda\nu}$ we introduce another auxiliary function $\bar{\omega}_\lambda$, defined by $\bar{\omega}_\lambda = \omega_\lambda$ in B_{m_0} and the solution of

$$\begin{aligned} \Delta\bar{\omega}_\lambda - \lambda\bar{\omega}_\lambda &= 0 \quad \text{in} \quad \Omega^m_{m_0} \\ \bar{\omega}_\lambda &= \omega_\lambda \quad \text{on} \quad \partial B_{m_0} \\ \bar{\omega}_\lambda &= 0 \quad \text{on} \quad \partial B_m, \end{aligned} \tag{24}$$

in $\Omega^m_{m_0}$. By a standard argument we have $\nabla\widetilde{\omega}_\lambda \cdot \mathbf{e}_r < 0$ in $\Omega^m_{m_0}$. The function $\widetilde{\omega}_\lambda$ is therefore subsolution for (24). Then

$$0 \le -\phi_{m\nu} \le -\widetilde{\omega}_{\lambda\nu} \le -\bar{\omega}_{\lambda\nu} \quad \text{on} \quad \partial B_m$$

which proves (iv), because $\bar{\omega}_{\lambda\nu} \le c(\lambda, m_0)e^{-\sqrt{\lambda}m}$ on ∂B_m. ∎

We continue the proof of Theorem 2.2. Using (21) and Lemma 2.1 the inequality (22) yields

$$\int_{\Omega^m_\epsilon} (\beta(u^1(t)) - \beta(u^2(t)))\xi\omega_\lambda\, dx \le \int_{\Omega^m_\epsilon} [\beta(u^1_0) - \beta(u^2_0)]_+\omega_\lambda\, dx$$
$$+ \int_{E^{\epsilon,m}_t} \bar{u}(A - A_m)\partial_\tau\phi_m\, dxdt + \int_{E^{\epsilon,m}_t} \lambda(u^1 - u^2)\omega_\lambda\, dxdt + Cm^2 e^{-\sqrt{\lambda}m}.$$

With the estimate

$$\left\|\bar{u}(A - A_m)\partial_\tau\phi_m\right\|_{L^1(E^{\epsilon,m}_t)} \le C\left\|\frac{A - A_m}{\sqrt{A_m}}\right\|_{L^2(E^{\epsilon,m}_t)}\left\|\sqrt{A_m}\partial_t\phi_m\right\|_{L^2(E^{\epsilon,m}_t)},$$

we find in the limit $m \to \infty$,

$$\int_{\Omega_\epsilon} (\beta(u^1(t)) - \beta(u^2(t)))\xi\omega_\lambda \, dx \le \int_{\Omega_\epsilon} [\beta(u_0^1) - \beta(u_0^2)]_+ \omega_\lambda \, dx + \int_{E_t^\epsilon} \lambda(u^1 - u^2)\omega_\lambda \, dxdt. \qquad (25)$$

In (25), we take a sequence $\{\xi_n\}$ that converges pointwise to $sgn(\bar\beta_+)$. We then let $\lambda \to 0$ to obtain the result; the convergence of the term $\int_{E_t^{\epsilon,m}} \lambda(u^1 - u^2)\omega_\lambda \, dxdt$ follows from the L^1-bound (Prop. 2.1) and $(H_\beta 2)$. ∎

2.3. Existence for (P_ϵ)

Now we use solutions of a regularized problem to prove the existence of solutions for (P_ϵ). Let $\delta_n := 1/n$ and introduce the approximations $\{u_{0n}\}$ and $\{u_{ne}\}$,

$$\begin{array}{l}
u_{0n} \in C^\infty(\mathbb{R}^3), \quad \text{with} \quad \|u_{0n}\|_{L^\infty} \le \|u_0\|_{L^\infty} + \delta_n; \\
u_{0n} \downarrow u_0 \quad \text{uniformly on compact subsets of } \Omega_\epsilon; \\
u_{0n}(x) = \delta_n \quad \text{for} \quad n-1 \le |x| \le n; \\
\nabla u_{0n}(x) \cdot \mathbf{e}_r = 0 \quad \text{at} \quad |x| = \epsilon
\end{array}$$

and

$$u_{ne}(x,t) := 1 - (1 - u_{0n}(x))e^{-nt} \quad \text{for} \quad |x| = \epsilon \quad \text{and} \quad 0 \le t \le T.$$

Then consider the regularized version of (P_ϵ),

$$(P_{n\epsilon}) \quad \begin{cases}
\beta(u)_t + \operatorname{div}(\bar F) = 0 & \text{in} \quad E_T^{\epsilon,n}, \\
\bar F \cdot \mathbf{e}_r = u_{ne}\boldsymbol{q} \cdot \mathbf{e}_r & \text{at} \quad |x| = \epsilon, \quad t > 0, \\
u = \delta_n & \text{at} \quad |x| = n, \quad t > 0, \\
u(x,0) = u_{0n}(x) & \text{in} \quad \Omega_\epsilon^n.
\end{cases}$$

Let $u_n^\epsilon \in C^\infty(E_T^{\epsilon,n}) \cap C^{2+\alpha,1+\alpha/2}(\overline{E_T^{\epsilon,n}})$, be the unique solution of $(P_{n\epsilon})$ (see [8], Theorem 7.4), which satisfies

$$\delta_n \le u_n^\epsilon(x,t) \le \max\{\|u_0\|_{L^\infty}, 1\} + \delta_n,$$

and

$$\int\limits_{E_T^{\epsilon,n}} |\nabla u_n^\epsilon|^2 \, dxdt \leq M, \tag{26}$$

where M is independent of n and ϵ (see [11], Theorem 4).

With the above estimates, we are ready to prove the existence for (P_ϵ).

Proof (Proof of Theorem 2.1). For this proof we fix $\epsilon > 0$. Using Bernšteĭn estimates as in [10], we find

$$\|\nabla u_n^\epsilon(x,t)\|_{L^\infty(\Omega^m_{\epsilon+1/m}\times[\frac{1}{m},T])} \leq C(m) \quad \text{for all} \quad n \geq m. \tag{27}$$

Using Gilding [6], we find that, for $n \geq m$,

$$|u_n^\epsilon(x,t_2) - u_n^\epsilon(x,t_1)| \leq C(m)|t_2 - t_1|^{\frac{1}{2}} \tag{28}$$

for all $1/m \leq t_1 \leq t_2 \leq T$ and $x \in \Omega^m_{\epsilon+1/m}$. By a standard argument we combine estimates (26), (27), and (28), to conclude the existence of a solution of (P_ϵ). ∎

2.4. Weak solutions of Problem P and proof of Theorem 1.A

We now turn to Problem P. Let $E_T = \mathbb{R}^3 \times (0,T)$.

Definition 2.3. A weak solution of Problem P is a non-negative function u such that

(i) $u \in C(E_T)$ and $\nabla u \in L^2(E_T)$.

(ii) For every test function $\phi \in H^1(E_T)$ with $\int\limits_{\mathbb{R}^3} |\boldsymbol{q}||\nabla\phi|^2 \, dx < \infty$, that vanishes for large $|x|$ and at $t = T$,

$$\int\limits_{E_T} [\beta(u)\phi_t + \{\boldsymbol{q}\, u - \nabla u\}\nabla\phi] \, dxdt + \int\limits_{\mathbb{R}^3} \beta(u_0)\phi(0) \, dx + 4\pi \int\limits_0^T \phi(0,t) \, dt = 0. \tag{29}$$

If u satisfies (18) with the equality replaced by $\geq$ $(\leq)$ and with $\phi \geq 0$ in E_T then we call u sub(super)solution.

Remark 2. 1. Since $|\boldsymbol{q}| \in L^1_{loc}(\mathbb{R}^3)$, the integrals in (29) are well-defined,

$$\left|\int_{\mathbb{R}^3} \boldsymbol{q} u \nabla\phi\, dx\right|^2 \le \left(\int_{\operatorname{supp}\phi} |\boldsymbol{q}| u^2\, dx\right)\left(\int_{\operatorname{supp}\phi} |\boldsymbol{q}||\nabla\phi|^2\, dx\right) < \infty.$$

The existence of a weak solution of (P) is a consequence of Theorem 1.A. Uniqueness holds in the class of solutions of (P) that are obtained as limits of solutions of (P_ϵ), since the comparison principle (Proposition 2.2) carries over to the limit. However, due to the singularity of $\boldsymbol{q}$ at the origin, uniqueness in the class of all solutions of (P) remains an open question.

We have the following properties of the weak solution of (P).

Proposition 2.3. *Let u be a weak solution of Problem P. Then*

$$\int_{\mathbb{R}^3} \beta(u(t))\, dx = \int_{\mathbb{R}^3} \beta(u_0)\, dx + 4\pi t \quad \textit{for all} \quad t \ge 0.$$

The proof of this proposition is similar to the proof of Proposition 2.1. The singularity of $\boldsymbol{q}$ at the origin creates a "pseudo-boundary condition":

Proposition 2.4. *For any weak solution u of Problem P we have*

$$u(0,t) = 1 \quad \textit{for} \quad 0 < t \le T.$$

Proof. Consider a fixed function $\rho \in C_c^\infty(0,T)$, and the functions $\eta_n : \mathbb{R}^3 \mapsto \mathbb{R}$ given by

$$\eta_n(r) = \begin{cases} 1 - nr & \text{if } 0 \le r \le \frac{1}{n}, \\ 0 & \text{if } \frac{1}{n} < r. \end{cases}$$

Let $\phi_n(x,t) := \rho(t)\eta_n(|x|)$. We estimate $\int_{E_T} \boldsymbol{q} u \nabla\phi_n\, dx$ by

$$\int_0^T \inf_{x\in B_{\frac{1}{n}}} \{u(x,t)\}\rho(t)\, dt \le -\frac{1}{4\pi}\int_{E_T} \boldsymbol{q} u \nabla\phi_n\, dxdt \le \int_0^T \sup_{x\in B_{\frac{1}{n}}} \{u(x,t)\}\rho(t)\, dt,$$

therefore in the limit, $n \to \infty$, we find

$$\lim_{n\to\infty} \int_{E_T} \boldsymbol{q} u \nabla\phi_n\, dxdt = -4\pi \int_0^T u(0,t)\rho(t)\, dt.$$

Using the boundedness of $\int\limits_{E_T} |\nabla u|^2\,dxdt$,

$$\left|\int\limits_{E_T} \nabla u \nabla \phi_n\,dxdt\right| \le \left(\int\limits_{E_T} |\nabla u|^2\,dxdt\right)^{\frac{1}{2}} \left(\int\limits_{E_T} |\nabla \eta_n|^2\,dxdt\right)^{\frac{1}{2}} \to 0$$

as $n \to \infty$. As u is bounded near the origin $0 \le |\int\limits_{E_T} \beta(u)(\phi_n)_t\,dxdt| \le C\int\limits_{\mathbb{R}^3} \eta_n\,dx \to 0$ as $n \to \infty$, and with a similar argument $\int\limits_{\mathbb{R}^3} \beta(u_0)\phi_n(0)\,dx \to 0$ as $n \to \infty$.

Using the above estimates and taking the limit in (29) as $n \to \infty$ we have

$$\int\limits_0^T (u(0,t)-1)\rho(t)\,dt = 0 \quad \text{for all} \quad \rho \in C_c^\infty(0,T),$$

which proves the lemma. ▌

Finally we are ready to prove Theorem 1.A.

Proof (Proof of Theorem 1.A). Using estimates (27) and (28), we have

$$\|u^\epsilon\|_{C^{0+1,0+\frac{1}{2}}(\Omega^m_{1/m}\times[\frac{1}{m},T])} \le C(m) \quad \text{for all} \quad \epsilon < 1/m.$$

Extending u^ϵ by zero on B_ϵ, we extract a subsequence of u^ϵ that converges *a.e.* in E_T to a limit u.

Fix $\phi \in C_c^\infty([0,T) \times \mathbb{R}^3)$. Since u^ϵ is uniformly bounded, and $|\boldsymbol{q}| \in L^1_{loc}(\mathbb{R}^3)$, the pointwise convergence of u^ϵ implies

$$\lim_{\epsilon\to 0} \int\limits_{E_T} \boldsymbol{q} u^\epsilon \nabla\phi\,dxdt = \int\limits_{E_T} \boldsymbol{q} u \nabla\phi\,dxdt.$$

Using the bound $\int\limits_{E_T^\epsilon} |\nabla u^\epsilon|^2\,dxdt \le M$, we have (after extracting a subsequence),

$$\int\limits_{E_T^\epsilon} \nabla u^\epsilon \nabla\phi\,dxdt \to \int\limits_{E_T} \nabla u \nabla\phi\,dxdt \quad \text{as} \quad \epsilon \to 0.$$

Therefore

$$\lim_{\epsilon\to 0} \int\limits_{E_T^\epsilon} \beta(u^\epsilon)\phi_t + (\boldsymbol{q}u^\epsilon - \nabla u^\epsilon)\nabla\phi\,dxdt = \int\limits_{E_T} \beta(u)\phi_t + (\boldsymbol{q}u - \nabla u)\nabla\phi\,dxdt.$$

Furthermore by the continuity of ϕ we have

$$\int_{\Omega_\epsilon} \beta(u_0)\phi(x,0)\,dx + \int_0^T \int_{\partial\Omega_\epsilon} \frac{\phi}{\epsilon^2}\,dSdt \to \int_{\mathbb{R}^3} \beta(u_0)\phi(x,0)\,dx + 4\pi \int_0^T \phi(0,t)dt.$$

as $\epsilon \to 0$. Combining these results we conclude that u satisfies equation (29) for all $\phi \in C_c^\infty([0,T) \times \mathbb{R}^3)$. To extend this equation to all ϕ as mentioned in the definition we note that the set $C_c^\infty([0,T) \times \mathbb{R}^3)$ is dense in the set of all such ϕ with respect to the norm

$$\|\phi\|^2_{L^2(E_T)} + \|\phi_t\|^2_{L^2(E_T)} + \|(\sqrt{|\boldsymbol{q}|} + 1)\nabla\phi\|^2_{L^2(E_T)}.$$

∎

3. ASYMPTOTIC BEHAVIOUR FOR A SOLUTION OF (P)

3.1. Preliminaries

To study the long-term behaviour we consider an extension to Problem P:

$$(\mathrm{P}') \quad \begin{cases} \beta(u)_t + \operatorname{div}(\bar{F}) = \delta_{x=0} + G(x,t) & \text{in} \quad \mathbb{R}^3, \quad t > 0 \\ u(x,0) = u_0(x) & \text{in} \quad \mathbb{R}^3. \end{cases}$$

Here $G \in L^1(0,T,L^1(\mathbb{R}^3))$.

The notion of weak solutions of (P′) follows along the same lines as above. For (P′) we can state a comparison principle:

Proposition 3.1. *Let u^1 be a subsolution and u^2 a supersolution of* (P′) *with data u_0^1, G_1 and u_0^2, G_2. Then for each $t \in [0,T]$,*

$$\int_{\mathbb{R}^3} [\beta(u^1(t)) - \beta(u^2(t))]_+\,dx \le \int_{\mathbb{R}^3} [\beta(u_0^1) - \beta(u_0^2)]_+\,dx + \int_{E_t} [G_1 - G_2]_+\,dxdt.$$

The proof of Proposition 3.1 is a direct extension of that of Proposition 2.2.

Lemma 3.1. *Let $G \equiv 0$. Then $w(r) = 1 - e^{-\frac{1}{r}}$ is a stationary solution of* (P′) *satisfying*

(i)$0 \le w(r)r \le w(r)^p r^p \le 1$ *for all* $r \ge 0$;

(ii)$\dfrac{1}{1+r} \le w(r) \le \min\left\{\dfrac{2}{1+2r}, 1\right\}$ *for all* $r > 0$.

Proof. We only demonstrate (ii). The function $z(s) = w(1/s)$ satisfies $z' = 1 - z$. The function $y(s) = \frac{s}{s+1}$ satisfies $y' < 1 - y'$ this implies the first inequality. The second follows along the same lines. ∎

To prepare the proof of Theorem 1.B we derive some relevant properties of the solutions of (S).

PROPOSITION 3.2. *Let f be a solution of* (S) *and consider the set* $P_f = \{\eta > 0 \mid f(\eta) > 0\}$. *Then*

(i) $f \in C^\infty(P_f)$;

(ii) $f' < 0$, $f'' > 0$ *on* P_f;

(iii) $f' \to 0$ *as* $\eta \to \infty$;

(iv) $\lim_{\eta \to 0^+} f'(\eta) = -K$ *with* $K \in (0, \infty)$;

(v) $\int_0^\infty \beta_p(f)\eta^{2-p} d\eta = 1$;

(vi) *If* $p = 1$, *then* $0 \le f(\eta) \le Ca^{-1}e^{-\ell\eta^2/4}$ *for* $\eta > a$; *if* $p < 1$, *then* $\sup P_f < \infty$.

Proof. Parts (i–iv) follow from Proposition 2.3 in [3]. Part (v) is a simple integration of the equation in (S). For part (vi), case $p < 1$ we refer to [3]. For the case $p = 1$, (S) has the explicit solution $f(\eta) = \mathrm{erfc}(\frac{\sqrt{\ell}}{2}\eta)$. This implies

$$f(\eta) \le -f'(0)2a^{-1}e^{-\ell\eta^2/4} \quad \text{for} \quad \eta > a.$$

∎

3.2. Proof of Theorem 1.B

We consider Problem P in the radially symmetric form. Let $S_T = \{(r,t) : 0 < r < \infty, 0 < t < T\}$.

Proof (Proof of Theorem 1.B). The proof is based on Proposition 3.1, applied to u and fw. We claim that the following estimates hold:

$$0 \le e^{p\alpha\tau} \int_0^\infty [\beta(u) - \beta(fw)]_+ \eta^2 d\eta \le L_1 e^{-\alpha\tau} + L_3 e^{-\tau} + L\|\varphi_-\|_{L^\infty} e^{-\alpha\gamma\tau} \tag{30}$$

for all $\tau \in \mathbb{R}$, and

$$0 \le e^{p\alpha\tau} \int_0^\infty [\beta(fw) - \beta(u)]_+ \eta^2 d\eta \le L_2 e^{-\alpha\tau} + L\|\varphi_+\|_{L^\infty} e^{-\alpha\gamma\tau} \qquad (31)$$

for all $\tau \in \mathbb{R}$. By $(H_\beta 4)$ the function $\psi(s) := \beta(s) - ms^p$ is non decreasing. Therefore, if $a > b$ we have

$$\beta(a) - \beta(b) = \psi(a) - \psi(b) + m(a^p - b^p).$$

By this observation estimates (30–31) imply (16–17).

Let $h(r,t) := f(r/t^\alpha)$ for all $(r,t) \in (0,\infty) \times (0,\infty)$. Then h satisfies

$$r^{1-p}\beta_p(h)_t - h_{rr} = 0 \quad \text{in} \quad S_T \qquad (32)$$

Using (32) and $\beta'(s) = \beta_p'(s) + \varphi(s)s^\gamma$, the function $g(r,t) := h(r,t)w(r)$ satisfies (P$'$) with

$$G(r,t) = \varphi(g)pg^{p-1+\gamma}g_t - \beta_p(h)_t w^p(r^{1-p}w^{1-p} - 1) + \frac{((1-rw)r)_r}{r^2}h_r.$$

Writing $G(r,t) := G_{1+}(r,t) + G_{1-}(r,t) + G_2(r,t) + G_3(r,t)$, with

$$G_{1\pm}(r,t) := \mp\frac{\alpha}{p+\gamma}\varphi_\pm(g)pw^{p+\gamma}(f^{p+\gamma})'\frac{r}{t^{\alpha+1}},$$

$$G_2(r,t) := \frac{\alpha\beta_p(f)'w^p r}{t^{\alpha+1}}(r^{1-p}w^{1-p} - 1), \text{ and } G_3(r,t) := \frac{((1-rw)r)_r}{r^2}h_r,$$

we note that $G_{1+} \ge 0$, $G_{1-} \le 0$, $G_2 \ge 0$, and $G_3 \le 0$. These inequalities follows directly from Lemma 3.1 and Proposition 3.2.

Now we compute estimates for the integrals associated with each part of G. For G_{1+}, we have

$$\begin{aligned}\int_0^\infty G_{1+}(r,t)r^2 dr &\le -\frac{C\|\varphi_+\|_{L^\infty}}{t^{\alpha\gamma}}\int_0^\infty (f^{p+\gamma})'\eta^{3-p-\gamma}d\eta \\ &\le t^{-\alpha\gamma}C\|\varphi_+\|_{L^\infty}\int_0^\infty f^{p+\gamma}\eta^{2-p-\gamma}d\eta = L\|\varphi_+\|_{L^\infty}t^{-\alpha\gamma}\end{aligned}$$

since $\gamma < 3-p$. Hence L is a positive constant. We have a similar estimate for G_{1-} replacing $\|\varphi_+\|_{L^\infty}$ by $\|\varphi_-\|_{L^\infty}$.

For G_2 we have two cases. For $p < 1$ we use $1 - r^{1-p}w^{1-p} \le w \le 1/r$ to obtain

$$\int_0^\infty G_2(r,t)r^2 dr \le \frac{\alpha}{t^\alpha}\int_0^\infty (\beta_p(f))'\eta^{2-p}d\eta \le \frac{(2-p)\alpha}{t^\alpha}\int_0^\infty \beta_p(f)\eta^{1-p}d\eta = \frac{L_2}{t^\alpha}.$$

For $p = 1$, we use $1 - r^{1-p}w^{1-p} = 0$, so that $G_2 \equiv 0$.

Computing the integral of G_3, gives

$$-\int_0^\infty G_3(r,t)r^2 dr = \int_0^\infty (r(wr-1))_r h_r dr = \int_0^\infty r(1-wr)h_{rr} dr \le h_r|_0^\infty = \frac{K}{t^\alpha},$$

where K is defined in Proposition 3.2. Here we used Lemma 3.1 (ii). To complete the proof we use the sign of the functions $G_{1\pm}$, G_2, and G_3, and Proposition 3.1. ▌

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