A stabilized Chebyshev-Galerkin approach for the biharmonic operator

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Dedicated to the 60^{th} anniversary of my beloved mother

Abstract

We propose an efficient implementation of the Chebyshev Galerkin spectral method for the biharmonic operator. This discretization leads to banded matrices which, compared with other methods of the same type, are also better conditioned. The efficiency of the method is illustrated on the Orr-Sommerfeld eigenvalue problem, where an improved convergence can be observed and the spurious eigenvalues are removed.

1 Introduction

Because of being extremely accurate, spectral methods have been intensively studied in the past decades. Mainly three types of spectral methods can be identified, collocative, tau and Galerkin (see, e.g. [5]). The choice of the type of method depends essentially on the application. Collocation methods are suited to nonlinear problems or having complicated coefficients, while Galerkin methods have the advantage of a more convenient analysis and optimal error estimates. The tau method is appliable in the case of complicated (nonlinear) boundary conditions, where a Galerkin approach would be impossible and the collocation extremely tedious.

In any of the cases, the standard approach has significant disadvantages. First of all, the matrices resulting in the discretization process have an increased condition number, so computational roundoff errors deteriorate the expected theoretical exponential accuracy. Moreover, the discretization matrices are generally fully populated, so efficient algebraic solvers are difficult to apply.

These disadvantages are more obvious when solving fourth order problems, where stability and numerical accuracy are lost when applying a higher order approximation.

Several papers try to circumvent these inconvenients by modifying the standard approach. In this sense we mention [4], [6], [9] or [18] for the tau method, [8], [10], [11], [12] or [13] for the collocation variant or [21], [22], [3] for Galerkin schemes. All the above mentioned papers are dealing with Chebyshev methods.

We propose here a stable Chebyshev-Galerkin approach for the biharmonic operator, coupled with homogeneous boundary conditions. The main idea is to consider different shape and test functions, which are based on the Chebyshev polynomials but satisfy the boundary conditions a-priori (see also [20]). A similar approach for second order problems is considered in [19]. The resulting discretization matrices are sparse and better conditioned. After some preliminary considerations concerning Chebyshev spectral methods, the Galerkin approach is presented in Section 3. Some numerical results are given in Section 4, showing that our approach is effective.

2 Chebyshev methods

In what follows we denote by $L^2_\omega(-1,1)$, $H^k_\omega(-1,1)$, $H^k_{\omega,0}(-1,1)$, $(\cdot,\cdot)_{k,\omega}$, $\|\cdot\|_{k,\omega}$, $|\cdot|_{k,\omega}$ the corresponding weighted Sobolev spaces, scalar products, norms and semi-norms on $(-1,1)$, where $\omega(x) = (1-x^2)^{-\frac{1}{2}}$ is the Chebyshev weight. Let $\mathbb{P}_N$ be the space of (real) polynomials of maximal order N and

$$T_k(x) = \cos(k \arccos(x)),\ k \in \mathbb{N} \tag{1}$$

be the k^{th} order Chebyshev polynomial of the first kind. These polynomials satisfy (see, e.g., [5])

$$\begin{aligned} T_{k+1}(x) &= 2xT_k(x) - T_{k-1}(x),\ k > 0, \\ T_k(x)T_p(x) &= \tfrac{1}{2}(T_{k+p} + T_{|k-p|}), \\ T_k(\pm 1) = (\pm 1)^k, & \qquad T'_k(\pm 1) = (\pm 1)^k k^2, \\ (T_n, T_m)_{0,\omega} &= \tfrac{\pi}{2} c_n \delta_{n,m}, \end{aligned} \tag{2}$$

where $\delta_{n,m}$ represents the Kronecker symbol and $c_i := \begin{cases} 0, & \text{if } i < 0 \\ 2, & \text{if } i = 0 \\ 1, & \text{if } i > 0 \end{cases}$.

The above properties are extremely useful for constructing a Chebyshev spectral discretization method for differential equations. Approximating the unknown function by means of a Chebyshev series, a differentiation rule is deduced from the first relation in (2). Thus, if u is expanded as

$$u(x) = \sum_{k=0}^{\infty} a_k T_k(x), \tag{3}$$

its derivative becomes ([5])

$$u'(x) = \sum_{k=0}^{\infty} a_k^{(1)} T_k(x), \tag{4}$$

where

$$a_k^{(1)} = \frac{2}{c_k} \sum_{p=k+1,\ p+k-odd}^{\infty} p a_p. \tag{5}$$

Other operators can be expressed in a similar manner. For example, the fourth order derivative reads

$$(u^N)^{(iv)} = \sum_{k=0}^{\infty} a_k^{(4)} T_k \tag{6}$$

with

$$a_k^{(4)} = \frac{1}{c_k} \sum_{p=k+4,\ p+k-even}^{\infty} p[p^2(p^2-4)^2 - 3k^2p^4 + 3k^4p^2 - k^2(k^2-4)^2]a_p. \tag{7}$$

For obtaining an algebraic finite dimensional problem, the initial spaces are projected onto finite dimensional ones. To do so, we consider truncated series up to an order N in (3), (4) and (6). In the resulting equations the expansion coefficients (a_k, $k = \overline{0, N}$) are the unknowns. Looking now at the discretization matrix produced by the formula in (7), it is easy to see that its upper triangular part is full. Moreover, it has a large condition number, $O(N^8)$ (see, e.g., [5]). Therefore it is rather un-probably to obtain accurate results when working with a large number of expansion functions.

3 A stable Galerkin approach for the biharmonic operator

Here we are interested in differential operators of 4^{th} order, together with homogeneous boundary conditions. Concretely, we look at

$$\begin{cases} u^{(iv)}(x) + \ L.O.T. \ = f(x), \ x \in (-1, 1) \\ u(\pm 1) = u'(\pm 1) = 0, \end{cases} \tag{8}$$

where $L.O.T.$ stands for lower order terms.

For a Chebyshev-Galerkin approximation of the above problem we have to consider the following set

$$H^2_{\omega,0} = \left\{ u/u(\pm 1) = u'(\pm 1) = 0, \ \int_{-1}^{1} \omega(x) \left(u^{(k)}(x)\right)^2 dx < \infty, \ k = \overline{0, 2} \right\},$$

which, endowed with the scalar product

$$(u, v)_{2,\omega} = \sum_{k=0}^{2} \int_{-1}^{1} \omega(x) u^{(k)}(x) v^{(k)}(x) dx$$

forms a Hilbert space. To construct the Chebyshev-Galerkin numerical solution we have to approximate $H^2_{\omega,0}$ by polynomials in $V_N \equiv \mathbb{P}_{N+4} \cap H^2_{\omega,0}$.

After a variational reformulation of the problem in (8), existence, uniqueness and convergence of a Chebyshev-Galerkin approximation is guaranteed by the (nonlinear) Lax-Milgram lemma (see, e.g., [23]) due to the continuity and ellipticity on $H^2_{\omega,0}$ of the resulting forms (see [1], [2], [15]). It is worth here to notice that, by means of the trace theorem in weighted Sobolev spaces ([2]), boundary conditions make sense also in the variational framework.

Our aim here is to describe an efficient implementation of the Chebyshev-Galerkin method. Even though the error estimates do not depend on the choice of the basis functions - as long as these span V_N, this choice is important from practical point of view. Several possibilities are proposed in the literature. in [12] V_N is generated by

$$\Psi_k(x) = (1-x^2)^2 T_k(x), k = \overline{0, N}, \tag{9}$$

whereas Chebyshev polynomials are used as test functions. Another possibility is considered in [21],

$$\Phi_k(x) = T_k(x) - \frac{2(k+2)}{(k+3)} T_{k+2}(x) + \frac{(k+1)}{(k+3)} T_{k+4}(x) k = \overline{0, N}. \tag{10}$$

In both cases the condition number of the 4^{th} order differentiation matrix is reduced to $O(N^4)$, but the differentiation matrices still remain fully populated in the upper diagonal part.

Here we use the polynomials in (9) as shape functions and the ones in (10) as test basis. This approach leads to banded matrices, as resulting from the lemma below.

Lemma 3.1 *For Ψ_k and Φ_k defined as above we have $(\Phi_i, \Psi_j^{(k)})_2 = \frac{2}{\pi} a_{ij}^k$, where*

$$a_{ij}^4 = \begin{cases} c_i(i+1)(i+2)(i+3)(i+4), & \textit{if } j = i, \\ -2i(i+1)(i+2)(i+4), & \textit{if } j = i+2, \\ i(i+1)^2(i+2), & \textit{if } j = i+4, \\ 0, & \textit{otherwise}, \end{cases} \tag{11}$$

$$a_{ij}^2 = \begin{cases} \dfrac{c_{i-2}(i+1)(i+2)}{4}, & \textit{if } j = i-2, \\ -\dfrac{2i(i+3)+4c_i - 3(c_{i-1} - d_{i-1})}{2}, & \textit{if } j = i, \\ \dfrac{3(i+2)(i+1)}{2} + \dfrac{\delta_{i0}}{2}, & \textit{if } j = i+2, \\ -(i+1)(i+2), & \textit{if } j = i+4, \\ \dfrac{(i+1)(i+2)}{4}, & \textit{if } j = i+6, \\ 0, & \textit{otherwise}, \end{cases} \tag{12}$$

N	$cond(A^4)$	$cond(A^4)/N^4$	$cond(PA^4)$	$cond(PA^4)/N^2$
16	$3.85 \cdot 10^3$	0.059	$2.12 \cdot 10^1$	0.083
32	$6.35 \cdot 10^4$	0.060	$7.67 \cdot 10^1$	0.075
64	$1.10 \cdot 10^6$	0.065	$2.88 \cdot 10^2$	0.070
128	$1.89 \cdot 10^7$	0.070	$1.11 \cdot 10^3$	0.068
256	$3.19 \cdot 10^8$	0.074	$4.35 \cdot 10^3$	0.066
512	$5.30 \cdot 10^9$	0.077	$1.72 \cdot 10^4$	0.066

Table 1: Condition number for the A^4 discretization matrix in (11).

$$a^0_{ij} = \begin{cases} \dfrac{c_{i-4}}{16}, & \textit{if } j = i-4, \\ -\dfrac{c_{i-2}(3i+8) - 3(c_{i-3} - d_{i-3})}{8(i+3)}, & \textit{if } j = i-2, \\ \dfrac{15i + 35c_i - 22(c_{i-1} - d_{i-1}) + 5(c_{i-2} - d_{i-2})}{16(i+3)}, & \textit{if } j = i, \\ -\dfrac{5i + 6 + 4c_i - c_{i-1} + d_{i-1}}{4(i+3)}, & \textit{if } j = i+2, \\ \dfrac{15i + 22 + 3c_i}{16(i+3)}, & \textit{if } j = i+4, \\ -\dfrac{3i+4}{8(i+3)}, & \textit{if } j = i+6, \\ \dfrac{i+1}{16(i+3)}, & \textit{if } j = i+8, \\ 0, & \textit{otherwise}, \end{cases} \tag{13}$$

with c_i *defined in (2) and* $d_i := \begin{cases} 0, & \textit{if } i < 0 \\ 1, & \textit{if } i \geq 0 \end{cases}$.

Proof. These formulae and similar ones for other operators can be obtained by direct computation, using the properties mentioned in (2). □

4 Numerical results

In this section we give some numerical results for showing the efficiency of the proposed approach. First we look at the condition number of the 4^{th} order differentiation matrix. As resulting from Table 1, this is $O(N^4)$ (similar to the one provided by a finite element discretization). Compared with the results in [21], the condition number here is almost halved. Even the simplest preconditioner - the diagonal one (denoted by P) - is effective, reducing the conditioning of the problem to $O(N^2)$. Similar results are obtained when considering other operators of 4^{th} order.

The accuracy and stability of the method is tested on the Orr-Sommerfeld

N	$Re(\lambda)$	$Im(\lambda)$	$log_{10}\lvert\lambda - \lambda_{ex}\rvert$
16	0.23 27015601471970	0.00 20071352336258	-2.29
24	0.237 7416510772222	0.003 8390544140477	-3.62
32	0.23752 59345883960	0.00373 31827776226	-5.19
48	0.23752648 75498026	0.0037396 696859990	-8.80
64	0.2375264888204 568	0.00373967062 18966	-11.97
96	0.2375264888204 710	0.0037396706229 786	-14.36
128	0.2375264888204 703	0.0037396706229 756	-14.91
196	0.2375264888204 726	0.0037396706229 762	-14.49
256	0.2375264888204 703	0.0037396706229 711	-14.47
512	0.2375264888204 699	0.0037396706229 751	-15.22
768	0.2375264888204 718	0.0037396706229 809	-14.17
1024	0.2375264888204 691	0.0037396706229 745	*

Table 2: Most unstable mode, $\alpha = 1.00$, $R = 10000$, $\lambda_{ex} =$.

equation

$$\left\{ \begin{array}{ll} & \Phi^{(iv)}(x) - 2\alpha^2\Phi''(x) + \alpha^4\Phi(x) \\ = & i\alpha Re[(U(x)-\lambda)(\Phi''(x) - \alpha^2\Phi(x)) - U''(x)\Phi(x)], \; x \in (-1,1) \\ & \Phi(\pm 1) = \Phi'(\pm 1) = 0. \end{array} \right. \tag{14}$$

This eigenvalue problem occurs in the (linear) stability analysis of shear flows (see, e.g., [7]). Here α stands for the wave number, $i = \sqrt{-1}$, Re denotes the Reynolds number and the eigenvalues λ are the growth rates. Φ denotes the amplitude of the stream function, while for the basic profile we take $U(x) = 1 - x^2$, corresponding to the Poiseuille flow. As pointed out in [17] (see also [6] for a careful analysis of computational aspects), the Chebyshev-tau method generates spurious eigenvalues due to the way the boundary conditions are imposed. Moreover, stability problems may occur when doing computations with a large number of polynomials, resulting in an eventual loss of accuracy.

Several approaches are proposed to alleviate these problems (see [6] and the references therein for spectral methods of Chebyshev type, or [14], [16] for a Galerkin-Legendre approach). Applying the discretization proposed in Lemma 3.1, we have computed the most unstable mode for $\alpha = 1$ at $Re = 10^4$. It is to notice that no spurious eigenvalues have been encountered. As we can see in Table 2, due to the reduced condition number, this approach is stable. A discretization of order $N = 64$ achieves an 10^{-12} accuracy. Moreover, increasing N does not lead to a loss in accuracy, computations until $N = 1024$ yielding errors close to the machine precision (with results for $N = 1024$ as reference values). It is worth here to notice that these computations are in good agreement with the results reported in the cited literature.

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