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# A MODIFIED CHEBYSHEV-TAU METHOD FOR A HYDRODYNAMIC STABILITY PROBLEM

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Abstract. In this work we propose a modified Chebyshev-tau spectral method to solve numerically an eigenvalue problem arising in the hydrodynamic stability. This method preserves the accuracy of the specral methods and in addition it is numerically stable and generates sparse matrices. Moreover, the spurious eigenvalues generated usually by the classical tau method are removed.

## 1. Introduction

In this work we propose a modifed Chebyshev-tau spectral method to solve numerically an eigenvalue problem arising in the hydrodynamic stability. Spectral methods have been intensively studied in the last two decades because of their good approximation properties. Since the pioneering paper [8], these methods have been considerred to be extremely efficient for hydrodynamic stability problems, where accurate results are necessary. But the advantages of spectral methods are shadowed by some difficulties generated by this numerical approach. Hence the matrices which arise when a spectral method is applied to discretize a differential operator are usually full and have a large condition number. Therefore, especially for the problems of fourth order, when a discretization with a large number of elements is used, the numerical accuracy of these methods can be lost. Moreover, when applied to eigenvalue problems, the tau spectral method may generate spurious eigenvalues.

There are several works concerned with the problem of efficient implementation of spectral methods of the three existing types (see, e.g., [1], [4] for the tau variant, [3] for the collocation one or [9] for the Galerkin approach - all of these cited only in the connection with the Chebyshev polynomials). Applications of spectral methods to eigenvalue problems have been described, for example, in [7], where the nonlinear appearance of the spectral parameter is treated, or [5] and [6], where the removal of spurious eigenvalues has been considerred. In our work a hydrodynamic stability problem is solved numerically, the method we have adopted being a modified Chebyshev-tau one. This approach does not produce any spurious eigenvalues and it is more stable with respect to roundoff errors. The discretization matrices generated in this way are sparse.

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The paper is organized as follows. In the section 2 the problem we want to solve numerically is presented. Next, some properties of the Chebyshev polynomials are provided and the classical Chebyshev-tau spectral method is described briefly. In the fourth section we propose the new approach. Finally, Section 5 illustrates the efficiency of our method with some numerical results.

## 2. The hydrodynamic stability problem

In this section we describe briefly the problem which motivates our work. All the details will be given in a forthcoming paper. The stability problem given below describes the flow of a thin film of an isothermal liquid and studies the appearance of long wave instability induced by a surface tension gradient. The liquid film flows over a rigid plane inclined at the angle $\beta$ with respect to the horizontal. The film is bounded above by a gas exerting a shear stress on the interface. We introduce the characteristic scales for velocity, pressure, length and time and examine the stability of the flow using a standard linear analysis.

$$\begin{aligned} L\Phi \equiv \quad & \Phi^{(iv)} - 2\alpha^2\Phi'' + \alpha^4\Phi \\ & -i\alpha R\left[(U-\lambda)\left(\Phi'' - \alpha^2\Phi\right) - U''\Phi\right] = 0, \quad \text{in} \quad (0,1), \end{aligned} \tag{2.1}$$

$$\Phi(1) = \Phi'(1) = 0, \tag{2.2}$$

$$\Phi''(0) + \alpha^2\Phi(0) + NU''(0) = 0, \tag{2.3}$$

$$\begin{aligned} & \Phi'''(0) + i\alpha\left[R\left(\lambda - U(0)\right) + 3\alpha i\right]\Phi'(0) \\ & + i\alpha\left[2\cot(\beta) + \alpha^2 Ca + (\lambda - U(0))\, RU'(0)\right] N = 0, \end{aligned} \tag{2.4}$$

where $N = \frac{\Phi(0)}{\lambda - U(0)}$. Here $R$ represents the Reynolds number, $Ca$ the capillary one and $\tau$ the dimensionless surface stress. By $\alpha$ we denote the wave number of the disturbance, $\bar{\lambda}_r = Re(\bar{\lambda})$ gives us the phase speed and $\alpha\bar{\lambda}_i = \alpha Im(\bar{\lambda})$ the growth rate of the instability. Here $\bar{\lambda}$ stands for the most unstable mode, i.e. the eigenvalue having the largest imaginary part. The basic state is given by

$$U(x) = \left(1 - x^2\right) + (1 - x)\,\tau.$$

If $R$ is less than the "critical value", the growth rate of instability is negative, so the flow is lineary stable. Using long wave expansions we obtain for the critical Reynolds number the following estimate

$$R_{cr} = \frac{5}{2}\cot\beta\frac{1}{2+\tau}. \tag{2.5}$$

Numerically, for the phase speed we have obtained

$$\bar{\lambda}_r = 2 + \tau. \tag{2.6}$$

The differential operator in (2.1) is identical to the one in the celebrated Orr-Sommerfeld problem. The difference appears at the last two boundary conditions, where the spectral parameter $\lambda$ appears nonlineary. But this nonlinearity has a special type, which can be easily removed. Concretely, because our basic profile is parabolic, thus $U''(0) \neq 0$, so using (2.3) we can eliminate $N$ from (2.4) and obtain

$$\begin{aligned} &U''(0)\Phi'''(0) + i\alpha \left[R\left(\lambda - U(0)\right) + 3\alpha i\right] U''(0)\Phi'(0) \\ &-i\alpha \left[2\cot(\beta) + \alpha^2 Ca + \left(\lambda - U(0)\right) RU'(0)\right] \left[\Phi''(0) + \alpha^2\Phi(0)\right] = 0. \end{aligned} \tag{2.4$'$}$$

The equation in (2.3) can be rewritten as

$$\left(\lambda - U(0)\right) \left[\Phi''(0) + \alpha^2\Phi(0)\right] + \Phi(0)U''(0) = 0. \tag{2.3$'$}$$

Thus the initial problem has been transformed into a linear one, in which the spectral parameter appears also lineary. In the following sections we will consider only the modified boundary conditions $(2.3)'$ and $(2.4)'$ instead of (2.3) and (2.4).

As mentioned above, an accurate numerical method to compute the eigenvalues of the problem in (2.1) - $(2.4)'$ is needed. For this reason we consider a Chebyshev-spectral one. Although the above modifications simplify our problem, the boundary conditions are still complicated. It is not easy to implement them in a collocative or a Galerkin formulation, while in the tau approach this can be done without any difficulty. This work shows some possibilities to aviod the inconvenients mentioned in the introduction for the Chebyshev-tau method.

## 3. The Chebyshev-tau method

In this section we describe briefly the Chebyshev tau spectral method applied to the problem in (2.1) - $(2.4)'$. Here and below we will denote by $L^2_\omega(-1,1)$, $H^k_\omega(-1,1)$, $H^k_{\omega,0}(-1,1)$, $(\cdot,\cdot)_{k,\omega}$, $\|\cdot\|_{k,\omega}$, $|\cdot|_{k,\omega}$ the corresponding weighted Sobolev spaces, scalar products, norms and seminorms on $(-1,1)$, where $\omega(x) = (1-x^2)^{-\frac{1}{2}}$ is the Chebyshev weight. Let $\mathbb{P}_N$ be the space of (real) polynomials of maximal order N and

$$T_k(x) = \cos(k\arccos(x)), \quad k \in \mathbb{N}$$

be the $k^{th}$ order Chebyshev polynomial of the first kind. Some of the following properties (see, e.g., [2]) will be used in what follows

$$\begin{aligned} T_{k+2}(x) =& 2xT_{k+1}(x) - T_k(x), \\ T_k(x)T_p(x) =& \frac{1}{2}(T_{k+p} + T_{|k-p|}), \\ T_k(\pm 1) = (\pm 1)^k, &\quad T'_k(\pm 1) = (\pm 1)^k k^2, \\ (T_k, T_p)_{0,\omega} =& \frac{\pi}{2} c_k \delta_{k,p}, \end{aligned} \tag{3.1}$$

where $k, p \in \mathbb{N}$, $\delta_{k,p}$ is the Kronecker symbol and $c_i = \begin{cases} 2, & \text{if } i = 0; \\ 1, & \text{if } i > 0. \end{cases}$

The properties above are the starting point in the construction of a Chebyshev spectral method for differential eqations. The unknown function is approximated by a Chebyshev polynomials series. From the first relation in (3.1) the following differentiation rule can be deduced. If $\Phi(x)=\sum_{j=0}^{\infty} a_j T_j(x)$, its derivative can be written as ([2])

$$\Phi'(x)=\sum_{j=0}^{\infty} a_j^{(1)} T_j(x), \text{ where } a_j^{(1)}=\frac{2}{c_j}\sum_{p=j+1,p+j-odd}^{\infty} p a_p. \tag{3.2}$$

Similar relations can be deduced for other operators. Now, the initial space is approximated with finite dimensional ones. Therefore, a truncated series up to an order $N$ is considerred in (3.2). The result is projected on a finite dimensional space in order to get a finite algebraic system. In the tau approach, the space on which the residual is projected has a lower dimension, and the boundary conditions are imposed explicitely. For the calssical Chebyshev-tau discretization of the problem in (2.1) - (2.4)$'$ we transform it first (lineary) on $(-1,1)$ and then make use of the following spaces and the corresponding bases

$$\begin{aligned} X_N=\mathbb{P}_{N+4} = & span\left\{T_k, \quad k=\overline{0,N+4}\right\} \\ Y_N=\mathbb{P}_N = & span\left\{T_k, \quad k=\overline{0,N}\right\} \end{aligned} \tag{3.3}$$

The projections mentioned above are orthogonal with respect to the $L^2_\omega(-1,1)$ scalar product. The Chebyshev-tau discretization of the problem in (2.1) - (2.4)$'$ reads

$$\begin{cases} \Phi_N \in X_N, \\ (L_t\Phi_N,\varphi)_{0,\omega}=0 \qquad \forall\varphi\in Y_N, \\ B.C.'s, \end{cases} \tag{3.4}$$

where $L_t$ denotes the transformed differential operator on $(-1,1)$ and by $B.C$ we mean an explicit imposal of the boundary conditions in $\pm 1$. Replacing $\varphi$ with $T_i, i=\overline{0,N}$ succesively in the above formulation and taking $\Phi_N=\sum_{j=0}^{N+4} a_j T_j$ we get $N+1$ relations for the coefficients $a_j, j=\overline{0,N+4}$ and the spectral parameter $\lambda$. The remaining four equations are given by the boundary conditions in $\pm 1$. Hence we have obtained a generalized eigenvalue problem of the form $Aa=\lambda Ba$.

The discretization matrices generated by this method are badly conditioned and not sparse. This can be illustrated for example by the fourth order differentiation matrix for $\Phi_N$ ([8])

$$\Phi_N^{(iv)}=\sum_{j=0}^{N+4} a_j^{(4)} T_j(x),$$

where

$$a_j^{(4)}=\frac{1}{c_j}\sum_{p=j+4,p+j-even}^{N+4} p\left[p^2(p^2-4)^2-3j^2p^4+3j^4p^2-j^2(j^2-4)^2\right]a_p. \tag{3.5}$$

Even though this method has theoretically a spectral accuracy, it is strongly affected by roundoff errors. Moreover, for our eigenvalue problem it generates two spurious eigenvalues.

## 4. A modified approach

In this section we describe the modified approach to the Chebyshev-tau method. The difference between the calssical tau method and the modified version consists in the spaces involved in the discretization process. In our approach, the approximation of the solution of the problem in (2.1) - (2.4)$'$ implicitly satisfies the boundary conditions in 1, which are simple and independent on $\lambda$. For this purpose we define the functions

$$\Theta_i(x) = (1-x)^2 T_i(x), \quad i \geq 0 \tag{4.1}$$

and approximate $\Phi$ by $\Phi_N = \sum_{j=0}^{N+2} a_j \Theta_j$. Clearly,

$$\tilde{X}_N = span\left\{\Theta_i, i = \overline{0, N+2}\right\} = \{v \in \mathbb{P}_{N+4} / v(1) = v'(1) = 0\}$$

and therefore $\Phi_N \in \tilde{X}_N$ satisfies the boundary conditions in (2.2) apriori.

For the definition of the test functions we use the functions below

$$\Psi_i^1(x) = \frac{2}{\pi} d_i^N \left[\frac{2i+3}{4(i+1)} T_i(x) - T_{i+1}(x) + \frac{2i+1}{4(i+1)} T_{i+2}(x)\right], \quad i \geq 0, \tag{4.2$a$}$$

where $d_i^N = 1$ if $0 \leq i \leq N$ and $d_i^N = 0$ otherwise. Now let

$$\Psi_i^{k+1}(x) = \frac{1}{2i+k+2}\left(\Psi_i^k(x) + \Psi_{i+1}^k(x)\right), \quad i \geq 0 \text{ and } k \geq 1. \tag{4.2$b$}$$

The choice of the test function space is justified by the following

**Lemma 4.1.** *For any $k \geq 1$ we have*

$$\tilde{Y}_N = span\left\{\Psi_i^k, i = \overline{0, N}\right\} = \{v \in \mathbb{P}_{N+2} / v(1) = v'(1) = 0\}.$$

*Proof.* The case $k = 1$ is obvious. For $k > 1$ the mathematical induction can be applied easily.

Having defined the spaces for our approach we can proceed with the discretization of the problem in (2.1) - (2.4)$'$. Similar with the formulation in (3.4), the modified version reads

$$\begin{cases} \Phi_N \in \tilde{X}_N, \\ (L_t \Phi_N, \varphi)_{0,\omega} = 0 \qquad \forall \varphi \in \tilde{Y}_N, \\ B.C.'s, \end{cases} \tag{4.3}$$

but now $B.C.$ stands only for the two boundary conditions in $-1$, which still have to be imposed explicitely.

For the discretization of the transformed differential operator $L_t$ we take $\Psi_i^5, i = \overline{0, N}$ as test functions and construct the corresponding discretization matrices. This choice is justified by the lemma below

**Lemma 4.2.** *Let $\Theta_j$ and $\Psi_i^5$ be as defined in (4.1), (4.2b). For all $i = \overline{0, N}$ and $j \geq 0$ we have $(\Psi_i^5, \Theta_j^{(iv)})_{0,\omega} = d_j^{N+2} a_{ij}$, where*

$$a_{ij} = \begin{cases} \dfrac{i+4}{4(2i+5)}, & \textit{if } j = i+2; \\ -\dfrac{(i+2)(i+4)}{(2i+5)(2i+7)}, & \textit{if } j = i+3; \\ \dfrac{i+2}{4(2i+7)}, & \textit{if } j = i+4; \\ 0, & \textit{otherwise} \end{cases} \tag{4.4}$$

*and $d_j^N$ is defined in (4.2).*

*Proof.* The relations above can be obtained from the relations in (3.1). The calculus is quite tedious but not difficult, so we won't reproduce it here.

*Remark 4.1.* In this approach, the discretization matrix for the fourth order derivative is banded. Compared with the one in the classical approach, which, as revealed in (3.5), is only upper traingular and difficult to compute, it is better conditioned and therefore the method features more stability.

*Remark 4.2.* Similar discretization matrices are obtained for lower order derivatives, or other operators.

*Remark 4.3.* The basis adopted here are suited only for the boundary conditions in (2.2). Similar ideas can be applied for the spectral discretization of the problems having other types of (homogenous) boundary conditions.

## 5. Numerical results

In this section we present some numerical results to demonstrate the efficiency of our method. For this purpose we compute the numerical eigenvalues of the problem in (2.1) - (2.4)$'$ at the critical Reynolds number (as given by (2.5)) for a small $\alpha$, namely $10^{-4}$. In the classical approach the spurious eigenvalues are not taken into consideration. We compare the most unstable mode ($\bar{\lambda}$) given by both approaches with the estimate in (2.6), which we consider to be the exact value ($\lambda_{ex}$). In both cases the approximation of the solution $\Phi_N$ was taken in $\mathbb{P}_N$. Two situations were considered, namely $\tau = -1.75$ and $\tau = 1.75$. The estimated critical Reynolds number is 11.9175 in the first case and 0.7945 in the second one. The results obtained here do not depend on the capillary number $Ca$. Table 1 presents the most unstable mode produced by the classical tau approach and the logarithm of the absolute error. Table 2 displays the same for the modified approach. Because of the order of magnitude for of $R_{cr}$ we were able to obtain good results using a few polynolmials. This confirms the accuracy of Chebyshev-spectral methods. But the difference between the results produced by both approaches becomes obvious as $N$ increases. In this case, the classical approach loses its accuracy significantly, while our approach produces the same results. Therefore we can state that the proposed method is more stable.

Table 1. Most unstable mode, classical approach; $\tau = -1.75$.

| $N$ | $Re(\overline{\lambda})$ | $Im(\overline{\lambda})$ | $log_{10}\lvert\overline{\lambda}-\lambda_{ex}\rvert$ |
|---|---|---|---|
| 8 | 0.2499999791 | -0.444E-10 | -7.68 |
| 16 | 0.2499999791 | 0.592E-11 | -7.66 |
| 32 | 0.2499999843 | 0.334E-07 | -7.43 |
| 48 | 0.2499999635 | -0.732E-06 | -6.13 |
| 64 | 0.2499995577 | -0.114E-03 | -3.94 |
| 96 | 0.2498154984 | -0.292E-04 | -3.72 |
| 128 | 0.2492347238 | 0.639E-02 | -2.19 |

*Exact:* $\lambda_{ex} = 0.25$.

Table 2. Most unstable mode, modified approach; $\tau = -1.75$.

| $N$ | $Re(\overline{\lambda})$ | $Im(\overline{\lambda})$ | $log_{10}\lvert\overline{\lambda}-\lambda_{ex}\rvert$ |
|---|---|---|---|
| 8 | 0.2499999791 | -0.111E-10 | -7.68 |
| 16 | 0.2499999791 | -0.423E-11 | -7.68 |
| 32 | 0.2499999791 | -0.257E-10 | -7.68 |
| 48 | 0.2499999791 | -0.286E-10 | -7.68 |
| 64 | 0.2499999791 | -0.208E-09 | -7.68 |
| 96 | 0.2499999787 | -0.737E-09 | -7.67 |
| 128 | 0.2500001534 | 0.123E-05 | –5.90 |

*Exact:* $\lambda_{ex} = 0.25$.

Table 3. Most unstable mode, classical approach; $\tau = 1.75$.

| $N$ | $Re(\overline{\lambda})$ | $Im(\overline{\lambda})$ | $log_{10}\lvert\overline{\lambda}-\lambda_{ex}\rvert$ |
|---|---|---|---|
| 8 | 3.7499999800 | -0.446E-10 | -7.70 |
| 16 | 3.7499999800 | -0.723E-09 | -7.70 |
| 32 | 3.7499999811 | -0.475E-07 | -7.29 |
| 48 | 3.7499999790 | -0.103E-05 | -5.99 |
| 64 | 3.7500000515 | -0.553E-05 | -5.26 |
| 96 | 3.7498189188 | -0.143E-03 | -3.64 |
| 128 | 3.7491599376 | -0.508E-02 | -2.29 |

*Exact:* $\lambda_{ex} = 3.75$.

Table 4. Most unstable mode, modified approach; $\tau = 1.75$.

| $N$ | $Re(\overline{\lambda})$ | $Im(\overline{\lambda})$ | $log_{10}\lvert\overline{\lambda}-\lambda_{ex}\rvert$ |
|---|---|---|---|
| 8 | 3.7499999800 | -0.261E-11 | -7.70 |
| 16 | 3.7499999800 | 0.207E-11 | -7.70 |
| 32 | 3.7499999800 | -0.877E-10 | -7.70 |
| 48 | 3.7499999804 | -0.132E-09 | -7.71 |
| 64 | 3.7499999805 | -0.828E-10 | -7.71 |
| 96 | 3.7499999954 | -0.844E-08 | -8.02 |
| 128 | 3.7500041460 | 0.970E-06 | -5.37 |

*Exact:* $\lambda_{ex} = 3.75$.

**Conclusions.** We have proposed a modified Chebyshev-tau spectral method to solve a hydrodynamic stability problem numerically. This method is more stable as the classical approach and generates sparse matrices. Moreover, the spurious eigenvalues are removed.

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