Show that the following rules are sound and complete.

F1′ $X \subseteq R \quad \Rightarrow \quad X \to X$

F2′ $X \to YZ \quad \Rightarrow \quad X \to Y$

F3′ $X \to Y,\ Y \to Z \quad \Rightarrow \quad X \to YZ.$

a) To prove that F1'-F3' are sound we can show that they are deducible from Armstrong's axioms (which are sound – see [Silberschatz]):

F1′: $X \subseteq R \quad \Rightarrow \quad X \subseteq X \quad \Rightarrow \quad X \to X$ (F1)

F2′: $X \to YZ$

$Y \subseteq YZ \quad \Rightarrow \quad YZ \to Y$ (F1)

$\Rightarrow X \to Y$ (F3)

F3′: $X \to Y$

$Y \to Z \quad \Rightarrow \quad Y \to YZ$ (F2)

$\Rightarrow X \to YZ$ (F3)

Alternatively, we can prove soundness according to the definition (see pp. 261-263)

F1′: $\forall\, t_1, t_2 \quad (t_1[X] = t_2[X]) \Rightarrow (t_1[X] = t_2[X])$
which means $X \to X$.

F2′: $X \to YZ$, i.e. $\forall\, t_1, t_2 \quad (t_1[X] = t_2[X]) \Rightarrow (t_1[YZ] = t_2[YZ])$,
then $t_1[Y] = t_2[Y]$.
So if $t_1[X] = t_2[X]$ then $t_1[Y] = t_2[Y]$, which means $X \to Y$.

F3′: $X \to Y$ means that $\forall\, t_1, t_2 \quad (t_1[X] = t_2[X]) \Rightarrow (t_1[Y] = t_2[Y])$
$Y \to Z$ means that $\forall\, t_1, t_2 \quad (t_1[Y] = t_2[Y]) \Rightarrow (t_1[Z] = t_2[Z])$.
Then $\forall\, t_1, t_2 \quad (t_1[X] = t_2[X]) \Rightarrow ((t_1[Y] = t_2[Y]) \,\&\, (t_1[Z] = t_2[Z]))$, i.e.,
$\forall\, t_1, t_2 \quad (t_1[X] = t_2[X]) \Rightarrow (t_1[YZ] = t_2[YZ]))$,
i.e., $X \to YZ$.

b) To prove that F1'-F3' are complete we show that Armstrong's axioms are deducible from the given set of axioms (and Armstrong's axioms are complete, as we know)

F1: $Y \subseteq X \Rightarrow X = Y(X - Y)$
F1′ implies that $X \to Y(X - Y)$. Then (F2′) $X \to Y$.

F2: $X \to Y,\ Z \subseteq R$.
Due to F1′, $XZ \to XZ$.
F2′ implies that $XZ \to X$. Since $X \to Y$, we get (F3′) $XZ \to XY$.
We have, $XZ \to XZ$ and $XZ \to XY$. F3′ implies $XZ \to XYZ$.
Then (F2′) $XZ \to YZ$.

F3: $X \to Y,\ Y \to Z$. F3′ implies that $X \to YZ$, and F2′ implies then that
$X \to Z$.