CUT-ON, CUT-OFF TRANSITION OF SOUND IN SLOWLY VARYING FLOW DUCTS

S.W. RIENSTRA

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A contribution to a tribute to David Crighton

Sjoerd W. Rienstra
Department of Mathematics and Computing Science,
Eindhoven University of Technology, The Netherlands

Abstract. The cut-on/cut-off transition of a slowly varying acoustic mode in a slowly varying hard-walled duct with irrotational isentropic mean flow is studied. The usual turning point behaviour was found.

1. Introduction

There is no one who influenced my scientific life more than David Crighton (and it would have been more if I wasn't so head-strong now and then). Of course, as a PhD supervisor, he directed me in a subtle way in exactly the right direction by providing exactly the right problems. Most striking, however, for me, as an applied mathematician educated in a non-British style, was how he taught me not just to solve a problem by mathematics, but really to unravel a problem from inside out by using mathematics as a language. In modern terminology: when solving a real world problem he considered the modelling phase as a crude form of the asymptotic analysis, and used the mathematical language right from the start. (This is, of course, the common way in British applied mathematics). But also later his steering influence was sometimes crucial. A for me important example is the following. When I found the exact multiple scale solution for sound propagation in a slowly varying lined flow duct, I did not consider it important enough for a journal like JFM, and I succeeded to keep my initial publication [4] a well kept secret.
Fortunately, I met David Crighton and Nigel Peake at a Euromech Colloquium in 1997 in Scheveningen. Nigel and I came to talk about the problem of sound transmission in varying ducts, and I mentioned, almost in passing, my solution. I remember David's reaction very vividly, when he was called to our discussion. He approached us with: "And where did this man say he published his solution ... ?!". The mild sarcasm worked well enough. I knew I had to publish it as quickly as possible. I generalized the solution for an annular duct, included some figures, and published it in [5].

Apart from direct aeronautical engine applications [6], one of the opportunities this solution offers is a better understanding of the cut-on/cut-off transition of a mode in a hard-wall flow duct, also known as a turning point problem. This was done for the no-flow case by [3]. Here, we will extend this to include the effect of flow.

2. Physical model

Consider inside the cylinder $r = R(X)$, $X = \varepsilon x$, an isentropic irrotational flow, described by velocity $\tilde{\mathbf{v}}$, potential $\tilde{\phi}$, density $\tilde{\rho}$, pressure $\tilde{p}$ and sound speed $\tilde{c}$. We split it up in a stationary part and acoustic harmonic perturbations

$$\tilde{\mathbf{v}} = \mathbf{V} + \mathbf{v}\,\mathrm{e}^{\mathrm{i}\omega t}, \quad \tilde{\phi} = \Phi + \phi\,\mathrm{e}^{\mathrm{i}\omega t}, \quad \tilde{\rho} = D + \rho\,\mathrm{e}^{\mathrm{i}\omega t},$$
$$\tilde{p} = P + p\,\mathrm{e}^{\mathrm{i}\omega t}, \quad \tilde{c} = C + c\,\mathrm{e}^{\mathrm{i}\omega t}.$$

This yields for mean flow field (in dimensionless form)

$$\nabla\cdot(D\mathbf{V}) = 0, \quad \frac{1}{2}|\mathbf{V}|^2 + \frac{C^2}{\gamma - 1} = E, \quad (1)$$
$$C^2 = \frac{\gamma P}{D} = D^{\gamma-1}$$

(γ is the ratio of specific heats, while E is the Bernoulli constant). For the acoustic field we have

$$p = C^2\rho, \quad \mathrm{i}\omega\rho + \nabla\cdot(D\nabla\phi + \rho\mathbf{V}) = 0, \quad (2)$$
$$\mathrm{i}\omega\phi + \mathbf{V}\cdot\nabla\phi + \frac{p}{D} = 0.$$

[The sound speed perturbations play no role, of course.]
As we are interested in the cut-on/cut-off transition, which in general does not occur for modes in a soft-walled duct, we consider only a solid duct wall, so the normal velocity vanishes at the wall (see [5] for the more general case of a lined wall). The mean flow is assumed to be free of swirl, irrotational and nearly uniform, while duct entrance effects are absent. This implies that the mean flow

field is essentially a function of X, rather than x, and to leading order given by

$$\mathbf{V}(X,r;\varepsilon) = U(X)\mathbf{e}_x + \varepsilon V_1(X,r)\mathbf{e}_y + O(\varepsilon^2)$$

Whenever possible, we ignore in the asymptotic expansions the implied indices. The mean flow pressure, density and sound speed are to leading order also functions of only X [5].

After elimination of p and ρ, the acoustic field is described by

$$\nabla \cdot (D\nabla\phi) - D\left(i\omega + \mathbf{V}\cdot\nabla\right)\left[\frac{1}{C^2}\left(i\omega + \mathbf{V}\cdot\nabla\right)\phi\right] = 0. \quad (3)$$

3. The multiple scales solution

For a uniform flow in a straight duct, the acoustic field is known to be described by circumferential-radial modes. In accordance with WKB theory, the modal spectrum of waves in a slowly varying medium varies along with the slow variations, with only very little intermodal interference if the eigenvalues are well enough separated. Therefore, we introduce for the slowly varying case the WKB-Ansatz

$$\phi(x,r,\theta;\varepsilon) = \quad (4)$$

$$A(X,r;\varepsilon)\exp\left(-im\theta - i\varepsilon^{-1}\int^X \mu(\xi;\varepsilon)\,d\xi\right)$$

where ϕ will be any slowly varying 'mode'. Following [5], we obtain after expanding $A(X,r;\varepsilon) = A(X,r) + O(\varepsilon)$, $\mu(X;\varepsilon) = \mu(X) + O(\varepsilon^2)$ from the leading order equation the solution

$$A(X,r) = N(X)J_m(\alpha(X)r) \quad (5)$$

where J_m is the m-th order Bessel function of the first kind [1]. The boundary condition of vanishing normal velocity yields to leading order the following equation for 'eigenvalue' $\alpha(X)$

$$J_m'(\alpha(X)R(X)) = 0.$$

With j_{mn}' the n-th zero of J_m', we have $\alpha = j_{mn}'/R$. α and μ are related by the dispersion relation

$$C^2(\alpha^2 + \mu^2) = (\omega - \mu U)^2.$$

It is convenient to introduce the reduced axial and radial wave numbers σ and γ, which are really μ and α scaled by ω and without the pure convection effects, such that $\sigma = \sqrt{1-\gamma^2}$, where $\gamma = j_{mn}'\sqrt{C^2 - U^2}/\omega R$, and $\mu = \omega(C\sigma - U)/(C^2 - U^2)$. We define the square root

such that $\text{Re}(\sigma) \geq 0$, and $\text{Im}(\sigma) \leq 0$. For the present hard-wall geometry γ is only real. When $\gamma < 1$, σ is real positive and the mode is called propagating or cut-on. When $\gamma > 1$, σ is negative imaginary and the mode is called evanescent or cut-off.

By various manipulations applied to the higher order equations (see [5] for details), we obtain the following expression for amplitude $N(X)$

$$N = Q\left(\frac{C}{R^2 D\sigma}\right)^{1/2}, \quad (6)$$

where Q is a constant.

4. The turning point problem

An important property of expression (6) for N is that it becomes invalid when $\sigma = 0$. So when the medium and diameter vary in such a way that at some point $X = X_0$ wave number σ vanishes, the present solution breaks down. In a small interval around X_0 the mode does *not* vary slowly and locally a different approximation is necessary. In the terminology of Matched Asymptotic Expansions [2], this is a boundary layer in variable X.
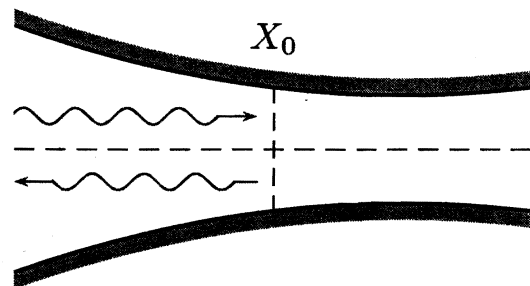

Figure 1. Turning point X_0, where a mode changes from cut-on to cut-off.

When σ^2 changes sign, and σ changes from real into imaginary, the mode changes from cut-on to cut-off. If X_0 is isolated, such that there are no interfering neighbouring points of vanishing σ, no power is transmitted beyond X_0, and the wave has to reflect at X_0. The incident propagating mode is split up into a cut-on reflected mode and a cut-of transmitted mode. Therefore, a point where wave number σ vanishes is called a "turning point".

Assume at $X = X_0$ a transition from cut-on to cut-off, so

$$\sigma_0 = 0, \quad \frac{d\,\sigma_0^2}{dX} < 0, \quad \gamma_0 = 1, \quad \gamma_0' > 0,$$

$$\frac{C_0 C_0' - U_0 U_0'}{C_0^2 - U_0^2} - \frac{R_0'}{R_0} > 0.$$

where subscript "$_0$" indicates evaluation at $X = X_0$.

Consider an incident, reflected and transmitted wave of the type found above. So in $X < X_0$, where σ is real positive, we have the incident and reflected waves

$$\phi = \frac{n(X)}{\sqrt{\sigma(X)}}\, J_m(\alpha r)\, e^{\frac{i}{\varepsilon}\int_{X_0}^{X}\frac{\omega U}{C^2-U^2}dX'} \qquad (7)$$

$$\times \left[e^{-\frac{i}{\varepsilon}\int_{X_0}^{X}\frac{\omega C\sigma}{C^2-U^2}dX'} + \mathcal{R}\, e^{\frac{i}{\varepsilon}\int_{X_0}^{X}\frac{\omega C\sigma}{C^2-U^2}dX'} \right]$$

with reflection coefficient $\mathcal{R}$ to be determined and

$$n(X) = Q\left(\frac{C}{DR^2}\right)^{1/2}.$$

In $X > X_0$, where σ is imaginary negative, we have the transmitted wave

$$\phi = \mathcal{T}\frac{n(X)}{\sqrt{\sigma(X)}}\, J_m(\alpha r)\, e^{\frac{i}{\varepsilon}\int_{X_0}^{X}\frac{\omega U}{C^2-U^2}dX'} \qquad (8)$$

$$\times e^{-\frac{1}{\varepsilon}\int_{X_0}^{X}\frac{\omega C|\sigma|}{C^2-U^2}dX'}$$

with transmission coefficient $\mathcal{T}$ to be determined, while $\sqrt{\sigma} = e^{-\frac{1}{4}\pi i}\sqrt{|\sigma|}$ will be taken.

This set of approximate solutions of equation (3), valid outside the turning point region, constitute the outer solution. Inside the turning point region this approximation breaks down. The approximation is invalid here, because neglected terms of equation (3) are now dominant, and another approximate equation is to be used. This will give us the inner or boundary layer solution. To determine the unknown constants (here: $\mathcal{R}$ and $\mathcal{T}$), inner and outer solution are asymptotically matched.

For the matching it is necessary to determine the asymptotic behaviour of the outer solution in the limit $X \to X_0$, and the boundary layer thickness (i.e., the appropriate local coordinate).

From the limiting behaviour of the outer solution in the turning point region (see below), we can estimate the order of magnitude of the solution. From a balance of terms in the differential equation (3) it transpires that the turning point boundary layer is of thickness $X - X_0 = O(\varepsilon^{2/3})$, leading to a boundary layer variable ξ given by

$$X = X_0 + \varepsilon^{2/3}\lambda^{-1}\xi$$

where λ is introduced for notational convenience later, and is given by

$$\lambda^3 = \frac{2\omega^2 C_0^2}{(C_0^2-U_0^2)^2}\left(\frac{C_0 C_0' - U_0 U_0'}{C_0^2-U_0^2} - \frac{R_0'}{R_0}\right).$$

Since for $\varepsilon \to 0$

$$\sigma^2(X) = \sigma^2(X_0 + \varepsilon^{2/3}\xi) =$$

$$-2\varepsilon^{2/3}\left(\frac{C_0 C_0' - U_0 U_0'}{C_0^2-U_0^2} - \frac{R_0'}{R_0}\right)\lambda^{-1}\xi + O(\varepsilon^{4/3}\xi^2),$$

we have

$$\frac{1}{\varepsilon}\int_{X_0}^{X}\frac{\omega C\sigma}{C^2-U^2}\,dX' = \begin{cases} -\frac{2}{3}(-\xi)^{3/2} = -\zeta, & \text{if } \xi < 0 \\[4pt] -i\frac{2}{3}\xi^{3/2} = -i\zeta, & \text{if } \xi > 0 \end{cases}$$

where we introduced $\zeta = \frac{2}{3}|\xi|^{3/2}$. The limiting behaviour for $X \uparrow X_0$ is now given by

$$\phi \simeq \frac{n_0}{\varepsilon^{1/6}(-\xi)^{1/4}}\left(\frac{\omega C_0}{\lambda(C_0^2-U_0^2)}\right)^{1/2} \qquad (9)$$

$$\times J_m(\alpha_0 r)\, e^{-im\theta}\left(e^{i\zeta} + \mathcal{R}\, e^{-i\zeta}\right),$$

while it is for $X \downarrow X_0$ given by

$$\phi \simeq \mathcal{T}\frac{n_0}{\varepsilon^{1/6}\xi^{1/4}}\left(\frac{\omega C_0}{\lambda(C_0^2-U_0^2)}\right)^{1/2} e^{\frac{1}{4}\pi i} \qquad (10)$$

$$\times J_m(\alpha_0 r)\, e^{-im\theta}\, e^{-\zeta}.$$

Since the boundary layer is relatively thin, also compared to the radial coordinate, the behaviour of the incident mode remains rather unaffected in radial direction, and we can assume in the turning point region

$$\phi(x,r,\theta) = \psi(\xi) J_m(\alpha(X)r)\, e^{-im\theta}\, e^{\frac{i}{\varepsilon}\int_{X_0}^{X}\frac{\omega U}{C^2-U^2}dX'}.$$

Substitution in equation (3), and using the definition of the Bessel equation, we arrive at

$$\left(1 - \frac{U_0^2}{C_0^2}\right)\varepsilon^{2/3}\lambda^2 J(\alpha_0 r)(\psi'' - \xi\psi) = O(\varepsilon).$$

So to leading order we have Airy's equation

$$\frac{d^2\psi}{d\xi^2} - \xi\psi = 0.$$

This has the general solution (Figure 2)

$$\psi(\xi) = a\,\mathrm{Ai}(\xi) + b\,\mathrm{Bi}(\xi),$$

where a and b, parallel with $\mathcal{R}$ and $\mathcal{T}$, are to be determined from matching. Using the asymptotic expressions (Eqs. 13, 14) for Airy functions, we find that for ξ large with $1 \ll \xi \ll \varepsilon^{-2/3}$, equation (10) matches the inner solution if

$$\frac{a}{2\sqrt{\pi}\xi^{1/4}}\, e^{-\zeta} + \frac{b}{\sqrt{\pi}\xi^{1/4}}\, e^{\zeta} \;\sim$$

$$\mathcal{T}\frac{n_0}{\varepsilon^{1/6}\xi^{1/4}}\, e^{\frac{1}{4}\pi i}\left(\frac{\omega C_0}{\lambda(C_0^2-U_0^2)}\right)^{1/2} e^{-\zeta}.$$
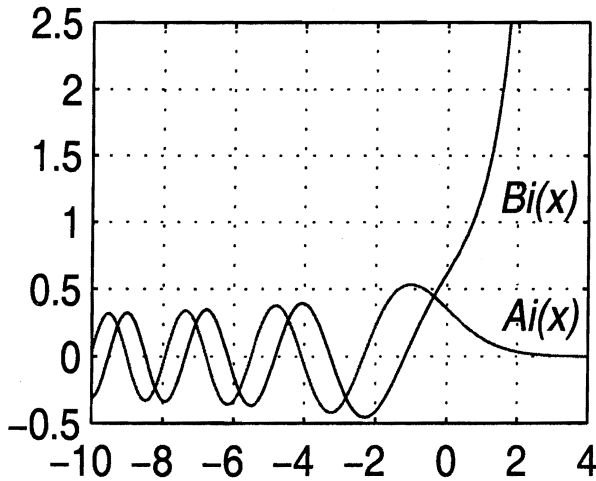
Figure 2. Airy functions

Appendix

Related to Bessel functions of order $\frac{1}{3}$ are the Airy functions Ai and Bi, solution of

$$y'' - xy = 0, \tag{12}$$

with the following asymptotic behaviour (introduce $\zeta = \frac{2}{3}|x|^{3/2}$)

$$\mathrm{Ai}(x) \simeq \begin{cases} \frac{\cos(\zeta - \frac{1}{4}\pi)}{\sqrt{\pi}|x|^{1/4}} & (x \to -\infty), \\ \frac{e^{-\zeta}}{2\sqrt{\pi}x^{1/4}} & (x \to \infty), \end{cases} \tag{13}$$

$$\mathrm{Bi}(x) \simeq \begin{cases} \frac{\cos(\zeta + \frac{1}{4}\pi)}{\sqrt{\pi}|x|^{1/4}} & (x \to -\infty), \\ \frac{e^{\zeta}}{\sqrt{\pi}x^{1/4}} & (x \to \infty). \end{cases} \tag{14}$$

Since $e^{\zeta} \to \infty$, we can only have $b = 0$, and thus

$$a = \frac{2n_0\sqrt{\pi}}{\varepsilon^{1/6}} \left(\frac{\omega C_0}{\lambda(C_0^2 - U_0^2)} \right)^{1/2} e^{\frac{1}{4}\pi i} \, \mathcal{T}$$

If $-\xi$ is large with $1 \ll -\xi \ll \varepsilon^{-2/3}$ we use the asymptotic expression (13), and find that equation (9) matches the inner solution if

$$a \frac{\cos(\zeta - \frac{1}{4}\pi)}{\sqrt{\pi}\,(-\xi)^{1/4}} \sim \frac{n_0}{\varepsilon^{1/6}(-\xi)^{1/4}} \left(\frac{\omega C_0}{\lambda(C_0^2 - U_0^2)} \right)^{1/2}$$

$$(e^{i\zeta} + \mathcal{R}\,e^{-i\zeta}),$$

or

$$\mathcal{T}e^{i\zeta} + \mathcal{T}i\,e^{-i\zeta} \sim e^{i\zeta} + \mathcal{R}\,e^{-i\zeta}.$$

So, finally, we have

$$\mathcal{T} = 1, \qquad \mathcal{R} = i. \tag{11}$$

The amplitudes of these reflection and transmission coefficients could of course be guessed by conservation of energy arguments. This is not the case with the phase. It appears that the wave reflects with a phase change of $\frac{1}{2}\pi$, while the transmission is without phase change.

5. Conclusions

David Crighton played a small but crucial role in the final publication of the exact solution of the multiple scales problem of sound transmission in a slowly varying flow duct.

A loose end in this problem was the transition through cut-on/cut-off. By this publication we hope to have amended this omission.

References

1. Abramowitz, M., Stegun, I.A., "Handbook of Mathematical Functions," National Bureau of Standards, Dover Publications, Inc., New York, 1964.
2. Holmes, M.H., "Introduction to Perturbation Methods," Springer-Verlag, New-York, 1995.
3. Nayfeh, A.H., Telionis, D.P., "Acoustic propagation in ducts with varying cross sections," Journal of the Acoustical Society of America, 54(6), 1973, pp. 1654–1661.
4. Rienstra, S.W., "Sound Transmission in a Slowly Varying Lined Flow Duct," Nieuw Archief voor Wiskunde, series 4, Part 6, No. 1–2, 1988, pp. 157–167.
5. Rienstra, S.W., "Sound Transmission in Slowly Varying Circular and Annular Ducts with Flow," Journal of Fluid Mechanics, 380, 1999, pp. 279–296.
6. Rienstra, S.W., Eversman, W., "A Numerical Comparison Between Multiple–Scales and FEM Solution for Sound Propagation in Lined Flow Ducts," accepted for publication Journal of Fluid Mechanics, 2001.