The Coupling of Acoustical Membrane and Cavity Vibrations

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Abstract

To support the design of kettledrums a mathematical model is made, including a uniform membrane with airloading, a cylindrical cavity, and an infinite field. The model is simple enough to be implemented on a small computer, but yet describes some of the most important effects and acoustical mechanisms in detail, like the generation of a modal spectrum, the free field radiation damping, and the coupling of membrane modes and cavity modes.

Since the solution method breaks down when the system mode becomes equal to a cavity mode (the membrane amplitude vanishes), this phenomenon is further analysed by a one dimensional model.

Introduction

Although the basic principles of most musical instruments are known for more than a century, a rational design based on a complete understanding is still impossible in any practical situation. On the one hand this is due to the inherently rather complex configurations allowing all sorts of coupling and types of vibrational modes, in addition to the naturally limited amount of measurable information. On the other hand the sound is not meant for measuring instruments, but for the extremely sensitive and critical human ears, while at the same time the appreciation of beauty is too subtle and subjective to be represented by a formula. So any experienced designer will primarily rely on his ears, and will use measurements and model calculations only as a guide.

These observations couldn't be more true for the kettledrum, to the design of which a kettledrum factory invited us to give theoretical support by means of a mathematical model. This model had to include some of the most fundamental mechanisms (of the ones that are *supposed* to play a rôle), to allow the designer to order and interprete his experimental data.

The model and solution method we adopted is basically the one introduced by Christian, Davis, Tubis, Anderson, Mills and Rossing [1]. The good agreement they obtained with experimental results apparently justified the simplifications made. We extended the model to introduce the presence of a small vent hole in the bottom (which in the end appeared to have only a little effect). By analysing the necessary numerical calculations carefully, we were able to implement the model on a small computer ([2]).

The method is, however, inherently poor when a mode is close to a cavity mode. Then the coupling between the air vibrations in the cavity and the membrane vibrations disappears,

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and the radition damping vanishes. Therefore, this phenomenon is further analyzed by a very simple 1-D model, allowing a fully analytical solution.

The model

The cavity is modelled as a flanged finite cylinder, in cylindrical coordinates (r, θ, z) described by $r = a$, $0 \leq z \leq L$. The bottom $z = 0$ has a small hole $r \leq d$ ($d << a$), and at the top side $z = L$ the cavity is closed by a uniform membrane of tension T and density σ. The air, with density ρ_a and sound speed c_a, vibrates with (acoustic) excess pressure p and velocity $\vec{v} = \nabla\phi$, while the membrane displacement is given by $z = L + \eta$. As we are interested in the acoustic regime, the equations may be linearized, and we may utilize Fourier analysis and look for a solution per frequency ω. In complex notation we write p, $\vec{v}$, $\eta \sim e^{-i\omega t}$, and then futher ignore the exponential. The prevailing equations are then:

$$\nabla^2 p + k^2 p = 0 \quad , \quad p = i\omega\rho_a\phi \quad , \qquad \text{(in air)} \tag{1}$$

$$T\nabla_0^2\eta + \omega^2\sigma\eta = [\,p\,] \quad , \qquad \text{(at the membrane)} \tag{2}$$

where $k = \omega/c_a$ is the acoustic wave number, ∇_0 denotes ∇ restricted to the membrane surface, and $[\,p\,]$ denotes the pressure difference across the membrane. Note that the propagation speed of (transversal) waves in the membrane is $c_M = (T/\sigma)^{\frac{1}{2}}$. Boundary conditions are a vanishing normal velocity at the hard-walled surfaces: $(\nabla\phi \cdot \vec{n}) = 0$, and a matching membrane surface and air velocity at the membrane: $-i\omega\eta = \phi_z$. The bottom opening is modelled as a small orifice in an infinitely thin wall, so that the diffraction effects are acoustically equivalent to a dipole source, of which the strength is determined by the incompressible flow through the orifice (the inner region in a matched asymptotic expansion formulation). This flow is, on its turn, driven by the pressure of the incident acoustic wave. The resulting relation between the rate of change of volume velocity through and the pressure at the hole is [2]:

$$p(\vec{0}) = -i\omega\frac{\rho_a}{2d}\int_0^{2\pi}\int_0^{d} \phi_z(r,\theta,0)\; r\; dr d\theta \quad (d \to 0) \quad . \tag{3}$$

As there are no sources at infinity we have for $r \to \infty$, $z \to \infty$ Sommerfeld's radiation condition for only outward radiating waves.

Ideal modes

In the absence of air ($\rho_a = 0$) the acoustic field vanishes, and the solution reduces to vibrations of the membrane alone:

$$\begin{aligned} \eta &= \eta_{mn}^{(0)}(r,\theta) = J_m(x_{mn}r/a)\; e^{\mathrm{i}m\theta}/a\sqrt{\pi}\; J'_m(x_{mn}) \quad (m = 0,\ 1, \ldots;\ n = 1,\ 2, \ldots) \\ \omega &= \omega_{mn}^{(0)} = x_{mn}\; c_M/a \end{aligned} \tag{4}$$

where J_m is the m-th order Besselfunction of the first kind, and $J_m(x_{mn}) = 0$. Note that $\{\eta^{(0)}_{mn}\}$ forms an orthogonal basis.

When the density σ of the membrane tends to infinity, the deflection η vanishes, and the cavity field is decoupled from the (vanishing) outer field, so the solution reduces to cavity resonances. These are given by (for $d = 0$)

$$\begin{aligned} \phi &= \phi^{(c)}_{mnl}(r,\theta,z) = J_m(y_{mn}r/a)\cos(l\pi z/L)\, e^{im\theta} \\ \omega &= \omega^{(c)}_{mnl} = (y^2_{mn} + l^2\pi^2a^2/L^2)^{\frac{1}{2}}\; c_a/a \end{aligned} \tag{5}$$

where $m = 0,\ 1,\ 2,...,\ n = 1,\ 2,\ 3,...,\ l = 0,\ 1,\ 2,...,$ and $J'_m(y_{mn}) = 0$.

General solution

The approach is as follows. Both inside and outside the cavity the acoustic field is described, via a Green's function representation, as if it were driven by the membrane displacements. At the same time the field just below and above the membrane can be considered as a driving force to the membrane. So we can formulate an integral equation for the membrane displacement η. This equation is then written in matrix form by a suitable expansion in vacuum modes $\eta^{(0)}_{mn}$. Finally, this matrix equation is solved numerically. Note that this approach is obviously inadequate if a mode has a relatively small membrane deflection.

By standard techniques ([1, 2]) we obtain outside

$$\phi(r,\theta,z) = \frac{i\omega}{4\pi}\int_0^{2\pi}\int_0^a \eta(r',\theta')\, G_{\text{out}}(r,\theta,z;r',\ \theta',L)\, r'dr'd\theta' \ , \tag{6}$$

and inside

$$\phi(r,\theta,z) = -\frac{i\omega}{4\pi}\int_0^{2\pi}\int_0^a \eta(r',\theta')\, G_{\text{in}}(r,\theta,z;\ r',\theta',L)\, r'dr'd\theta' + \frac{d}{2\pi}\, G_{\text{in}}(r,\theta,z;\ \vec{0})\, \phi(\vec{0})\ . \tag{7}$$

It may be noted that $G_{\text{in}}(\vec{r};\ \vec{0})$ is symmetric (only $m = 0$ modes), and singular in $\vec{r} = \vec{0}$. Indeed, this expression (7) is, with respect to the orifice effect, approximate and only valid for $d << a$ and $|r| >> d$. Furthermore, since the second term is a correction (as long as η is not small), the value $\phi(\vec{0})$ to be substituted is effectively the one obtained for $d = 0$. So we end up with

$$\begin{aligned} \phi(r,\ \phi,\ z) \ &= -\frac{i\omega}{4\pi}\int_0^{2\pi}\int_0^a \eta(r',\ \theta')\ [G_{\text{in}}(r,\theta,z;r',\theta',L) \\ &+\frac{d}{2\pi}\, G_{\text{in}}(r,\theta,z;\vec{0})\, G_{\text{in}}(\vec{0};r',\theta',L)]\, r'dr'd\theta' \ . \end{aligned} \tag{8}$$

After substitution of these (formal) solutions into eq. (2), we introduce the expansion

$$\eta(r,\theta)=\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}a_{mn}\,\eta_{mn}^{(0)}(r,\theta)\quad,$$

multiply left- and righthand side by $\eta_{m'n'}(r,\theta)^* r$ and integrate over the membrane surface. After utilizing the orthogonality, and some other details, we arrive at the infinite set of equations

$$(\omega^2-\omega_{mn}^{(0)2})\sigma a_{mn}=4\omega^2\rho_a a\sum_{n'=0}^{\infty}a_{mn'}\;x_{mn}\;x_{mn'}\;[C_{mnn'}-\tfrac{1}{2}iI_{mnn'}-\frac{2d}{\pi a}\,\delta_{0,m}\;S_n\;S_{n'}]$$

where, per m, the set of ω's allowing a solution $(a_{m1},\;a_{m2},...)$ is to be found. This set is the sought spectrum. The notation used is

$$C_{mnn'}=\sum_{n''=1}^{\infty}\mathrm{cotg}(\gamma_{mn''}L/a)\;/\gamma_{mn''}(x_{mn}^2-y_{mn''}^2)(x_{mn'}^2-y_{mn''}^2)(1-m^2/y_{mn''}^2)\quad,$$

$$I_{mnn'}=\int_0^{\infty}\lambda J_m(\lambda)^2/\gamma(\lambda)(\lambda^2-x_{mn}^2)(\lambda^2-x_{mn'}^2)\;d\lambda\quad,$$

$$\gamma(\lambda)=i\;\sqrt{i(\lambda-ka)}\;\sqrt{-i(\lambda+ka)},\qquad\gamma_{mn}=\gamma(y_{mn})\quad,$$

$$S_n=\sum_{n''=1}^{\infty}[(x_{0n}^2-y_{0n''}^2)\gamma_{0n''}\;\sin(\gamma_{0n'}L/a)J_0(y_{0n''})]^{-1}\quad.$$

By a suitable contour deformation the integral $I_{mnn'}$ is further prepared for efficient numerical evaluation. The set of equations is solved via iterations over ka after writing the equations, in matrix form, as a quasi-eigenvalue problem. Further details may be found in [2].

Numerical example

To illustrate the present theory, we plotted in fig.1 the $m=1$ spectrum for a typical kettledrum as a function of tension: $a=0.328$ m, $\sigma=0.2653$ kg/m^2, $L=0.4142$ m, $\rho_a=1.21$ kg/m^3, $c_a=344.0$ m/s. (The $m=1$ modes are musically most important. The bottom hole affects only the $m=0$ modes; we found for the present example only a small effect). The dotted lines denote the level of the cavity modes. We see that in general the frequencies increase steadily with tension. Only a mode near a cavity mode is, however, somewhat reluctant to increase. Since the presented frequencies are really complex (the imaginary part is related to the decay time) we plotted one mode in the complex plane (fig.2,3). We see that for certain values of T the mode indeed becomes very close to a cavity mode, as the imaginary part also vanishes. A purely real frequency, however, does not decay, and since this is only possible with a stagnant membrane, we may expect problems with our solution method (which indeed occur) in this regime, and we would like to see additional confirmation of this phenomenon. Furthermore, it would be interesting to see whether it is accidental (for example, because σ was relatively large), or whether it would occur always at some tension.

Therefore we analysed a model, complex enough to include the basic effects, but simple enough to allow a fully analytical approach.

1-D model

A first proposal was a low frequency limit ($ka \to 0$) of the present model, with $kL = O(1)$, since this reduces the cavity field into a simple plane wave, and the radiated field into that of a point source. This is not a convenient limit, however, since the impedance at the cavity opening, even without membrane, becomes infinitely large, implying total reflection and a vanishing outside field. A finite impedance is obtained if the outside space is also reduced to a (semi-infinite) cylinder:

$$\phi_{zz} - c_a^{-2}\phi_{tt} = 0 \ \text{ in } \ 0 < z < L, \ L < z < \infty \ ,$$

$$-Ta^{-2}\eta - \sigma\eta_{tt} = p(L+, \ t) - p(L-, \ t) \ \text{ at } \ z = L \ ,$$

with $\phi_z = 0$ at $z = 0$, $\phi_z = \eta_t$ at $z = L \pm 0$, and outgoing wave for $z \to \infty$.

For ϕ, η, $p \sim e^{-i\omega t}$, finding the solution is standard, and we end up with the eigenvalue equation for $kL = \omega L/c_a$

$$kL + i\frac{\rho_a L}{\sigma} - \left(\frac{c_M L}{c_a a}\right)^2 \frac{1}{kL} = \frac{\rho_a L}{\sigma} \operatorname{cotg}(kL) \ ,$$

which can be solved easily numerically for varying $s = \sqrt{\rho_a L/\sigma}$ and $b = c_M L/c_a a$ (fig.4). Note that $\sin(kL) = 0$, i.e. $kL = n\pi$, correspond to cavity modes. It is seen that indeed for any s these cavity modes are passed for increasing b, while at the same time the contour of fig.3 is very similar to a corresponding one of fig.4. So the $1-D$ model provides qualitative insight into the coupling phenomenon, and indeed it does not seem to be accidental that the modes of the full 3-D model passed through the cavity modes.

References

1. R.S. Christian, R.E. Davis, A. Tubis, C.A. Anderson, R.I. Mills, T.D. Rossing, "Effects of air loading on timpani membrane vibrations", Journal of the Acoustical Society of America, **76** (5), 1984, 1336-1345.

2. S.W. Rienstra, "The acoustics of a kettledrum", IWDE Report WD 89.10, 1989, Eindhoven University of Technology.

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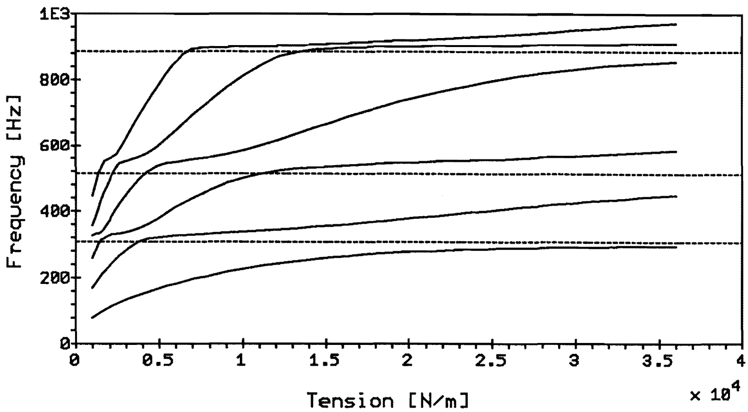

Figure 1. m=1 modes.

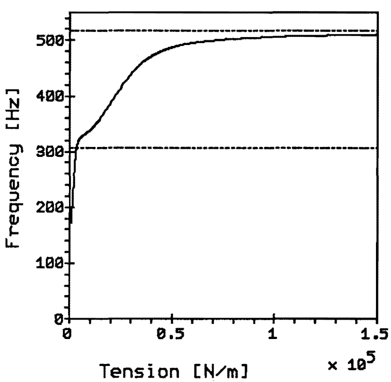

Figure 2. 2nd mode (real part)

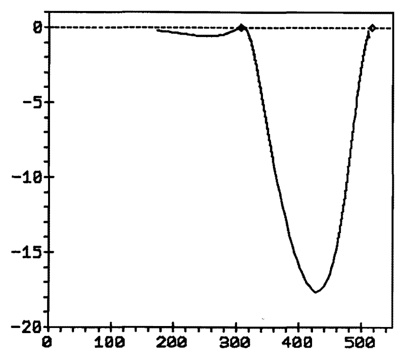

Figure 3. 2nd mode in complex k-plane

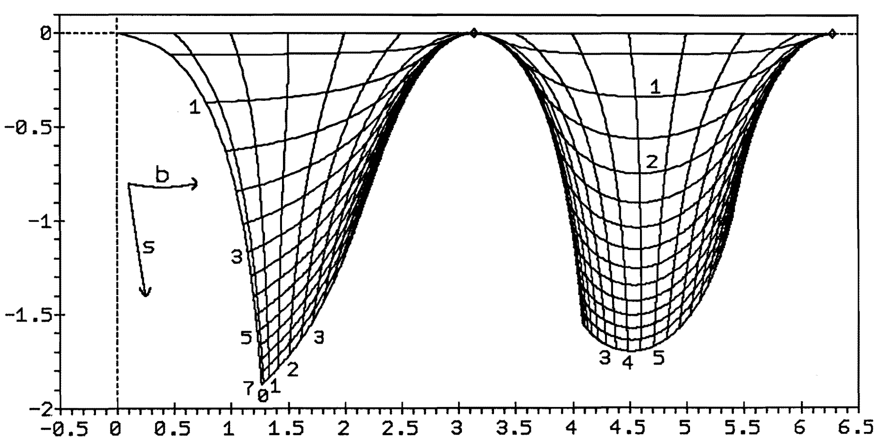

Figure 4. 1-D modes in complex kL-plane; contours for varying s and b.