Acoustical detection of obstructions in a pipe with a temperature gradient

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Abstract

Exhaust pipes of furnaces and similar equipment (e.g. for gasification) may get clogged up by deposition of sticky particles (ashes, tar, soot) carried by the gas mixture. The possibility of detecting the presence of such an obstruction by the reflection of an acoustic wave is investigated by a mathematical model. An important part of the problem is the considerable temperature variation along the pipe. It is argued that the high frequency components of the wave may be not reliable because of refraction and spurious reflection effects caused by this temperature gradient. Therefore, the reflection of low-frequency (one mode propagating) waves is investigated. It appears that the reflection becomes significant for a blockage area of more than 50%.

1. INTRODUCTION

Mathematics has, historically, its sources of inspiration in applications [1]. Indeed, unexpected questions from practice force one to go off the beaten track. Also it is easier to portray an abstraction with a concrete example at hand. Therefore, most mathematics is applied, or at least applicable or emerging from applications. And if we give up the mathematico-centric point of view there is even more applied mathematics hidden in the various theoretical disciplines of science.

Most of this applied mathematics is now applied to a practical problem as existing mathematical results, rather than mathematics invented for the problem. Mathematics is fed back to the application. However, there is more.

Before mathematics may be applied, the problem must be mathematized: formulated in equations and formula's, to render it amenable to formal manipulations. This is called mathematical modelling. If the problem considered does not have an accepted model, the modelling is evidently very important, and the mathematical discipline with this crucial rôle of modelling is (now) called industrial mathematics [2].

If the problem has a longer history (like from fluid mechanics, acoustics, elasticity, etc.) there is usually a universally accepted general model. Such a model is, however, often too general to be useful and it pays to break it down a bit to a simplified model which contains exactly the essential elements of the problem. In that case mathematics serves also as a language in which we can express precisely the various elements of the model and formulate exactly what we mean with "essential elements".

Finally the main rôle of mathematics comes into play, as the model may be analysed by mathematical techniques, not rarely originating from other scientific disciplines or problems, and thus realizing a fruitful cross-fertilization [3].

This last type of mathematics finds its area of applications mainly in the traditional engineering problems, which is why we call it *Engineering Mathematics*: applied mathematics where the problem is central and the mathematics most important.

In the present paper we present a problem of this type. It is typically engineering: acoustic wave propagating and scattering in an exhaust pipe with temperature gradients. The mathematics is, although not very difficult, not trivial. Furthermore, advantageous for the present introductory purposes, the problem consists of a number of subproblems, each requiring a suitable mathematical technique that heavily leans on the physics of the problem.

2. THE PROBLEM

The exhaust ducting of furnaces and gasification installations transfers a mixture of sometimes highly pressurized gases from the high temperature furnace chamber to an area of lower temperatures. Sticky particles of ashes, tar, and soot are carried away with this gas mixture.

During the design of a new type of industrial gasification furnace doubts were raised as to what extent the liquid particles would be deposited on the (colder) wall. If this really occurs, in spite of the precautions taken, it is necessary to measure it during the process in operation, because the contamination will accumulate at about the same location in the pipe and the resulting constriction may finally block the pipe.

Due to the very polluting and hostile conditions inside the pipe it is very difficult to apply measuring devices directly. A possible alternative proposed is to measure the contamination acoustically. At a distance sufficiently downstream in the pipe a transducer mounted in the wall generates an acoustic wave, which is reflected by the possible obstacle in the pipe. By measuring the reflected wave this obstacle may be detected.

The present paper reports a theoretical investigation to quantify this idea by a mathematical model, and to find out under which conditions a sufficiently discriminative reflection is obtained, not afflicted or spoiled by unwanted other reflections. In particular, the reflection from the pipe-furnace connection, and possible reflections caused by the temperature gradients may mask the reflections from the sought constriction.

To distinguish between incident and reflected wave easily it is probably most convenient to deal with a wave of finite duration. However, if the wave is very much pulse-like, the incident wave form is practically unknown, and the spectrum contains many high frequency components which propagate down the duct mainly via spiralling paths (modes), in contrast to the straight, axial path of the low frequency components [4]. Apart from the longer travel time, these spiralling modes are also very vulnerable to the temperature gradients. If the pitch of the spiral is small enough, a temperature increase will cause a reflection of the sound at a point in the duct determined only by an unfortunate but otherwise fortuitous combination of problem parameters. A reflection which has nothing to do with any obstacle in the duct!

For this reason it may be safer to restrict ourselves to the low frequencies. This, however, has the disadvantage that the open end generates a strong reflection and at the same time any obstacle a weak

reflection, and that the front end of the reflected wave is not well defined. On the other hand, the spatial distribution of the wave is very well known, and measuring the pressure at one point is sufficient to know the wave.

The model we will elaborate will contain as essential elements:

(i)   temperature variation, both axially and radially, in the duct to include the wave form variations attended with it, and to predict possible spurious reflections;

(ii)   reflection and scattering by an annular constriction;

(iii)   reflection by the flanged open end connecting the pipe to the large furnace chamber.

This rather ambitious object to incorporate the extra complexities of a non-uniform medium and reflection effects needs, however, to be accompanied by limitations in other respects to keep sight of the physically essential processes. Only after that we understand the problem well we are ready to turn to a more complete model, using a numerical approach. Therefore, the problem will be modelled to allow an analytical treatment as much as is reasonable.

3. THE MODEL

3.1. Differential equations

As a fluid mechanical phenomenon, sound has to satisfy the conservation laws of fluid motion. Introduction of the usual acoustical approximations then will yield a version of the wave equation pertaining to the present problem. For a simple and uniform, quiescent medium the equation for acoustic perturbations is just the standard wave equation, cited in many textbooks [4,5]. For the present configuration, however, with a varying temperature the result is different and not entirely straightforward. It is therefore instructive to derive the equations in detail and see which assumptions are underlying the model.

In tensor notation we have [4,5,6]:
the equation of mass conservation

$$\frac{\partial}{\partial t}\rho + \nabla \cdot (\rho \mathbf{v}) = 0 \; , \tag{3.1}$$

the equation of momentum conservation

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (-\mathbf{P} + \rho \mathbf{v}\mathbf{v}) = \mathbf{f} \; , \tag{3.2}$$

and the equation of energy conservation

$$\frac{\partial}{\partial t}(\rho \epsilon + \tfrac{1}{2}\rho v^2) + \nabla \cdot (\rho \epsilon \mathbf{v} + \tfrac{1}{2}\rho v^2 \mathbf{v}) =$$

$$\nabla \cdot (\mathbf{v} \cdot \mathbf{P}) - \nabla \cdot \mathbf{q} + \mathbf{f} \cdot \mathbf{v} \; , \tag{3.3}$$

where t is the time, ρ is the density, $\mathbf{v}$ is the velocity, $\mathbf{P}$ is the stress tensor, $\mathbf{f}$ is an external force (like gravity), ϵ is the internal energy, and $\mathbf{q}$ is the heat flux due to heat conduction. The stress tensor is split up into a normal stress and a shear stress component

$$\mathbf{P} = -p\,\mathbf{I} + \boldsymbol{\tau} \tag{3.4}$$

where p is the hydrostatic pressure, $\boldsymbol{\tau}$ the viscous stress tensor and $\mathbf{I}$ denotes the identity tensor with elements δ_{ij}.

The second law of thermodynamics for reversible processes relates for a fluid element the internal energy change $\mathrm{d}\epsilon$, thermodynamic pressure p_{th} and volume change $\mathrm{d}\rho^{-1}$ to temperature T and entropy change $\mathrm{d}s$

$$T\,\mathrm{d}s = \mathrm{d}\epsilon + p_{th}\,\mathrm{d}\rho^{-1} \; . \tag{3.5}$$

Stokes' hypothesis is that the fluid is locally in thermodynamic equilibrium so that thermodynamic pressure and hydrostatic pressure are equivalent $(p_{th} = p)$, and p, ρ and s are related via a single constitutive equation of state

$$p = p(\rho, s) \ . \tag{3.6}$$

If we neglect for the acoustic perturbations viscous dissipation ($\sim \tau$) and heat transfer ($\sim$ q) then equations (3.2) and (3.3) may be simplified. Equation (3.3) becomes

$$\rho \frac{D}{Dt} \epsilon = -p \nabla \cdot \mathbf{v} \tag{3.7}$$

with convective derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \ . \tag{3.8}$$

It is important to note that since the convective derivative just describes temporal variations of a fluid element travelling with the fluid flow, equation (3.5) is only to be interpreted as

$$T \frac{D}{Dt} s = \frac{D}{Dt} \epsilon + p \frac{D}{Dt} \rho^{-1} \ , \tag{3.9}$$

which reduces, with (3.7) and (3.1), to the equation of isentropy along a streamline, $Ds/Dt = 0$. All in all, we have now

$$\frac{D}{Dt} \rho + \rho \nabla \cdot \mathbf{v} = 0 \tag{3.10}$$

$$\rho \frac{D}{Dt} \mathbf{v} + \nabla p = \mathbf{f} \tag{3.11}$$

$$\frac{D}{Dt} s = 0 \ . \tag{3.12}$$

Equation (3.6) with (3.12) can be written out as

$$\frac{D}{Dt} p = c^2 \frac{D}{Dt} \rho \tag{3.13}$$

where $c^2 = (\partial p/\partial \rho)_s$ is the square of the local sound speed. For an ideal gas is

$$p = \rho RT \tag{3.14}$$

(R the gas constant) and ϵ only dependent on T, so that the specific heat capacities (for constant volume and pressure) are given by

$$c_v = d\epsilon/dT \ , \quad c_p = R + c_v$$

while

$$ds = c_v \frac{dp}{p} + c_p \frac{d\rho^{-1}}{\rho^{-1}}.$$

As a result is

$$c^2 = \gamma p/\rho = \gamma RT \tag{3.15}$$

where $\gamma = c_p/c_v$. So in general c is a function of the temperature alone. In the present problem experiments have shown that the medium behaves like an ideal gas with constant specific heat capacities. This implies that c^2 varies linearly with T.

If we write the variables as a mean stationary component plus an acoustic perturbation

$$p = p_0 + p' \ , \quad \rho = \rho_0 + \rho' \ , \quad \mathbf{v} = \mathbf{v}_0 + \mathbf{v}' \ ,$$

we can linearize the equations. Furthermore, we ignore the mean flow $\mathbf{v}_0$. It is, however, not immediately clear if this is an acceptable simplification, because the mean flow, although much slower than the sound speed c, is not necessarily much slower than the acoustic particle velocity $\mathbf{v}'$. In fact, both are of the order of a few meters per second.

The reason why we indeed can ignore $\mathbf{v}_0$ is because the (small) mean flow only affects the sound field if vorticity is injected from the wall by separation at an edge. In that case the coupling with the vorticity

may result in either a source or a sink of acoustic energy. This source or sink is, however, only of importance if the Strouhal number $\omega b/v_0$, based on circular frequency ω and radius of curvature b of the edge, is of order 1 ([7]). In the present problem we expect no appreciable separation at the duct inlet, and as far as separation occurs at the obstacle, a typical Strouhal number very much larger than 1.

As a result, if we ignore $\mathbf{v_0}$ and assume the external force to be stationary with $\nabla p_0 = \mathbf{f}$, we obtain

$$\rho_0 c_0^2 \nabla \cdot \left(\frac{1}{\rho_0} \nabla p' \right) - \frac{\partial^2}{\partial t^2} p' = 0 , \tag{3.16}$$

$$\rho_0 \frac{\partial}{\partial t} \mathbf{v}' + \nabla p' = 0 , \tag{3.17}$$

with $c_0^2 = \gamma R T_0 = \gamma p_0/\rho_0$. Note that, other than in the constant-c_0 case, the equations would have been different if written in ρ', since $p'_t = c_0^2(\rho'_t + \mathbf{v}' \cdot \nabla \rho_0)$ (eq. (3.13)).

Since gravity will be negligible here, we have $p_0 = $ constant, and (3.16) can be simplified further. Moreover, since we will consider the behaviour of a single frequency wave it is convenient to introduce

$$p'(\mathbf{x}, t) = \text{Re} \left(p(\mathbf{x}) \, e^{i\omega t} \right)$$

$$\mathbf{v}'(\mathbf{x}, t) = \text{Re} \left(\mathbf{v}(\mathbf{x}) \, e^{i\omega t} \right)$$

with circular frequency ω, and $p(\mathbf{x})$ and $\mathbf{v}(\mathbf{x})$ complex functions satisfying

$$\nabla \cdot \left(\frac{1}{k^2} \nabla p \right) + p = 0 \tag{3.18}$$

$$i\omega \rho_0 \mathbf{v} + \nabla p = 0 \tag{3.19}$$

and $k(\mathbf{x}) = \omega/c_0(\mathbf{x})$. The equations (3.18) and (3.19) will be the basis of the analysis to follow.

3.2. Geometry

We will consider a cylindrical hard-walled pipe, in cylindrical coordinates (x, r, θ) given by $r = a$ $(a \sim 0.75$ m$)$ (Figure 1). At $r = a$
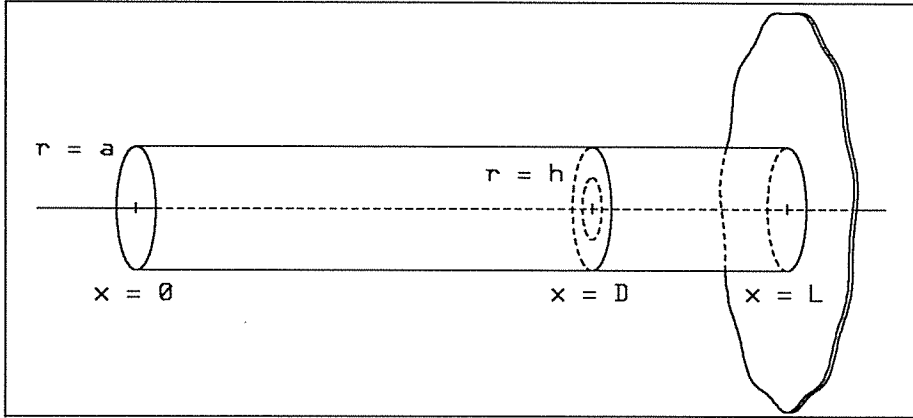

Figure 1. Model geometry

the normal velocity vanishes, $(\mathbf{v} \cdot \mathbf{n}) \sim \partial p/\partial r = 0$. At $x = 0$ a source generates a sound field of circular frequency ω $(\omega \sim 2000/s)$. The frequency is assumed to be so low that only the plane wave is propagating, and no details of the source are to be known. (This requires typically a wavelength larger than a duct diameter.) The part of the pipe $x < 0$ is irrelevant. At $x = L$ $(L \sim 10$ m$)$ the pipe is connected via a flanged opening to the half space $x > L$. The sound speed variation of the mean flow is assumed to be radially symmetric: $c_0 = c_0(x, r)$, $k = k(x, r)$, and constant in $x > L$. Also for the scattering obstacle there is no need to unnecessarily complicate the problem, and we assume an annular hard walled iris at $x = D$ between $r = h$ and $r = a$.

One observation is important and will be utilized in the analysis: although the variation in r of the temperature is rather steep (from 500 K at the wall to $1000 - 2000$ K at the center) and cannot be ignored or otherwise simplified, the variation in x-direction is relatively smooth. It appears that the ratio between a typical wavelength λ and the typical length (L) associated to substantial variations in k or c_0 is small. This parameter, which we will denote by

$\varepsilon = \lambda/L$ $(\varepsilon \sim 0.1 - 0.2)$, suggests an approach of solution where locally the sound speed is constant in x and the field can be described by a modal expansion. On the larger scale this is to be corrected by slowly varying modal amplitudes and wave numbers.

Where it facilitates the analysis, we will make this slow variation explicit by writing $k = k(\varepsilon x, r)$ without using any special notation.

4. SOLUTION

4.1. Modal expansion

When c_0 is independent of x, equation (3.18) is separable in the cylindrical coordinates (x, r, θ). That means that there exist solutions $p(x, r, \theta) = F(x) \, G(r) \, H(\theta)$, satisfying a uniform boundary condition at the coordinate surface $r = a$. These solutions are called modes [4,5]. Mathematically, these modes are interesting because they form a complete basis by which any other solution can be represented by a so-called modal expansion. Physically, these modes are interesting because the usually complicated behaviour of a general field is easier understood via the simpler properties of its modal elements.

If c_0 is constant the modes of a hardwalled infinite duct are the well-known products of Bessel function and exponential functions

$$d_{m\mu}^{\pm}(x, r, \theta) = J_m(\alpha_{m\mu} r) \, e^{\mp i k_{m\mu} x - i m\theta} \tag{4.1}$$

where J_m is the m-th order ordinary Bessel function of the 1st kind [3], $\alpha_{m\mu} a = j'_{m\mu}$ is the μ-th nonnegative nontrivial zero of J'_m, $k_{m\mu} = \sqrt{k^2 - \alpha_{m\mu}^2}$ is the axial wave number with Re $(k_{m\mu}) \geq 0$, Im $(k_{m\mu}) \leq 0$, $\mu = 1, 2, \ldots$, $m = 0, \pm 1, \pm 2, \ldots$.

Since $\alpha_{m\mu}$ grows without limit both for increasing μ and increasing $|m|$, there are only a *finite* number of real $k_{m\mu}$. The rest is purely imaginary. At the right side of a source the $d_{m\mu}^{+}$ modes are generated, of which the ones with real $k_{m\mu}$ are propagating (*cut on*), the other ones with imaginary $k_{m\mu}$ are exponentially decaying (*cut off*). At the left side of a source the same is true for $d_{m\mu}^{-}$.

In the complex plane, the axial wave numbers are typically located as given in Figure 2. A finite number is cut on, between $-k$ on k, and
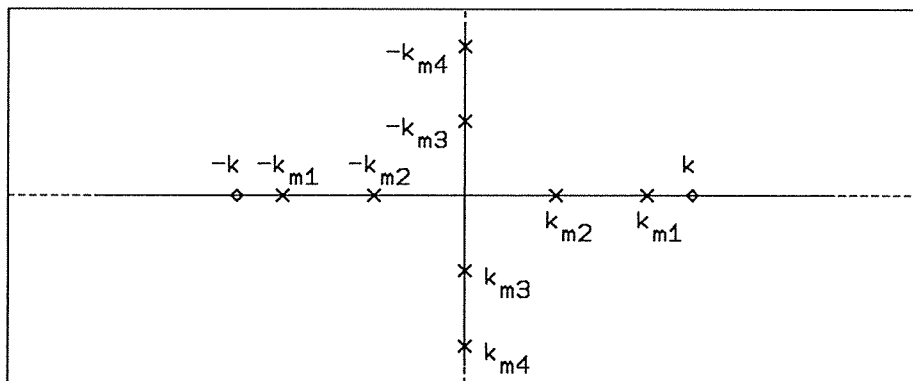
Figure 2. Axial wave numbers.

an infinite number is cut off, along the imaginary axis. It is important to note that if we take k (i.e., ω) small enough, there is only *one* (two if we count left- and right-running separately) mode propagating. This is just what we want in the present problem where we don't want to know details of the source: the other possibly generated modes are exponentially decaying and quickly negligible.

A general solution may be built from the modes $d_{m\mu}^{\pm}$ as the sum (with amplitudes to be determined)

$$p(x, r, \theta) = \sum_{m=-\infty}^{\infty} \sum_{\mu=1}^{\infty} \left(A_{m\mu} d_{m\mu}^{+} + B_{m\mu} d_{m\mu}^{-} \right) . \tag{4.2}$$

Obviously, at the right side of any source or scattering object we have only right-running waves ($B_{m\mu} = 0$), and at the left side only left-running waves ($A_{m\mu} = 0$). Furthermore, if the field is radially symmetric the double series simplifies to a single series for $m = 0$ only.

In the case of a soundspeed depending on r (but *not* on x or θ) the above theory is qualitatively the same. Only the Bessel function becomes now a slightly more general function $\psi_{m\mu}(r)$, defined by the Sturm-Liouville type eigenvalue problem [3]

$$\mathcal{L}_m(\psi; \gamma) = \frac{k^2}{r} \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{r}{k^2} \frac{\mathrm{d}}{\mathrm{d}r} \psi \right) + \left(k^2 - \gamma^2 - \frac{m^2}{r^2} \right) \psi = 0 \qquad (4.3)$$

$$|\psi(0)| < \infty , \quad \partial \psi(a)/\partial r = 0 \qquad (4.4)$$

where $k_{m\mu} = \gamma$ and $\psi_{m\mu} = \psi$. According to the standard theory, the eigenvalues γ^2 form a real sequence, monotonically tending to $\to -\infty$. The eigenfunctions $\psi_{m\mu}$ form a basis orthogonal to the inner product

$$(\psi_{m\mu}, \psi_{m\nu}) = \int_0^a \psi_{m\mu} \psi_{m\nu} \, r k^{-2} \, \mathrm{d}r . \qquad (4.5)$$

$\psi_{m\mu}$ has exactly $\mu - 1$ zeros on $(0, a)$. By integrating (4.3) we find for the 1st eigenvalue

$$\gamma_{m1}^2 = \frac{\int_0^a (1 - m^2/k^2 r^2) \, r \, \psi_{m1} \, \mathrm{d}r}{\int_0^a \psi_{m1} \, r k^{-2} \, \mathrm{d}r} .$$

Since ψ_{m1} has no zeros, $\gamma_{01}^2 > 0$, whereas γ_{m1}^2 for $m \neq 0$ is only positive if k is large enough compared to m.

4.2. Slowly varying amplitude and wave number

In the present problem we cannot have a solution built up from modes because the sound speed c_0 varies in x. However, we can borrow the idea of modes. Since c_0 varies relatively slowly in x the solution is locally representable by an approximate modal expansion. For suitably defined modes such a modal representation then remains the same for all x; only modal amplitude, shape and wave number vary slowly with x [8,9].

Since we are primarily interested in the symmetric problem, we consider from here on only $m = 0$, and do not mention the m-dependence.

For the sake of demonstration it is necessary to make the "slow variation" explicit, and we introduce

$$X = \varepsilon x \tag{4.6}$$

where $k = k(X, r)$, $c_0 = c_0(X, r)$, $\rho_0 = \rho_0(X, r)$.

The slowly varying mode is then assumed to have the form

$$p(x, r) = \psi(X, r; \varepsilon) \exp\left(-i \int_0^x \gamma(\varepsilon\xi; \varepsilon) \, d\xi\right) . \tag{4.7}$$

Since the x-derivatives are given by

$$p_x = (-i\gamma\psi + \varepsilon\psi_X) \exp\left(-i \int \gamma \, d\xi\right)$$

$$p_{xx} = (-\gamma^2\psi - i\varepsilon\gamma_X\psi - 2i\varepsilon\gamma\psi_X + \varepsilon^2\psi_{XX}) \exp\left(-i \int \gamma \, d\xi\right)$$

equation (3.18) becomes

$$\mathcal{L}(\psi; \gamma) = i\varepsilon k^2 \psi^{-1} (\gamma\psi^2 k^{-2})_X - \varepsilon^2 k^2 (\psi_X k^{-2})_X \tag{4.8}$$

with $\psi(r = 0) < \infty$, $\partial\psi(r = a) / \partial r = 0$, where we collected certain groups for later convenience. Since ε is small we expand

$$\psi(X, r; \varepsilon) = \psi_0(X, r) + \varepsilon\psi_1(X, r) + \mathcal{O}(\varepsilon^2)$$

$$\gamma(X; \varepsilon) = \gamma_0(X) + \mathcal{O}(\varepsilon^2)$$

and ψ_0 will be found to be a satisfactory approximation for ψ. We find to leading order

$$\mathcal{L}(\psi_0; \gamma_0) = 0 \tag{4.9}$$

which indeed corresponds to the eigenvalue problem (4.3,4), with now x as a parameter, and solutions ψ_μ, γ_μ (Im $\gamma_\mu \leq 0$). The problem that remains is the way ψ_μ varies with x (the amplitude of ψ_μ is still undetermined, but should vary with x if the energy content of a single mode is to remain the same). For this we consider the equation for perturbation ψ_1:

$$\mathcal{L}(\psi_1; \gamma_0) = ik^2 \psi_0^{-1} (\gamma_0 \psi_0^2 k^{-2})_X .$$ (4.10)

We don't have to solve for ψ_1. It is sufficient to make use of the self-adjointness properties of $\mathcal{L}$, and integrate

$$\int_0^a \psi_1 \mathcal{L}(\psi_0; \gamma_0)\, r k^{-2} \mathrm{d}r = \int_0^a \psi_0 \mathcal{L}(\psi_1; \gamma_0)\, r k^{-2} \mathrm{d}r$$

$$= i \int_0^a (\gamma_0 \psi_0^2 k^{-2})_X r\, \mathrm{d}r = 0$$

$$\gamma_0 \int_0^a \psi_0^2 r k^{-2}\, \mathrm{d}r \;=\; \text{constant}$$ (4.11)

to find (4.11), the equation defining ψ_0 as a function of x. For convenience we take here the constant equal to 1 or $-i$ and obtain

$$(\psi_\mu, \psi_\nu) = |\gamma_\mu|^{-1}\, \delta_{\mu\nu} .$$ (4.12)

A very important conclusion to be drawn from equation (4.11) or (4.12) is that the theory is invalid and the mode becomes singular at any position x where $\gamma_\mu = 0$. (This may occur for any eigenvalue other than γ_{01}.) This is to be interpreted as follows. If the soundspeed c_0 varies along the duct in such a way that γ_μ vanishes at, say, $x = x_0$ (which depends on both c_0 and frequency ω), then the mode is locally in resonance and changes from propagating (cut on) into decaying (cut off). Since the energy is conserved the mode cannot just disappear but reflects into its backrunning counterpart. These are the spurious unwanted reflections mentioned in chapter 2. They may, as a result, interfere with the reflections from the obstacle and confuse the signal. Therefore, it is important to select a frequency without a resonance for any mode (also $m \neq 0$) along the whole interval $(0, L)$. In that case we do not have to include this turning-point behaviour in the theory [3].

   Since, by assumption, there is no interaction between the modes

along the smoothly varying interval $0 \leq x \leq D$, we have the general (approximate) solution

$$p(x,r) = \sum_{\mu=1}^{\infty} A_\mu \psi_\mu(X,r) \, \exp(-i \int_0^x \gamma_\mu(\varepsilon\xi) \, d\xi)$$

$$+ B_\mu \psi_\mu(X,r) \, \exp(i \int_0^x \gamma_\mu(\varepsilon\xi) \, d\xi) \ .$$

It is notationally convenient to introduce

$$A_\mu(x) = A_\mu \, \exp(-i \int_0^x \gamma_\mu(\varepsilon\xi) \, d\xi)$$

$$\hspace{8cm} (4.13)$$

$$B_\mu(x) = B_\mu \, \exp(i \int_0^x \gamma_\mu(\varepsilon\xi) \, d\xi)$$

and similarly along $D \leq x \leq L$ for right running modal amplitudes C_μ and left running D_μ. Then we have

$$p(x,r) = \sum_{\mu=1}^{\infty} (A_\mu(x) + B_\mu(x)) \, \psi_\mu(X,r) \ \text{for} \ 0 \leq x < D \ , \hspace{1cm} (4.14)$$

$$p(x,r) = \sum_{\mu=1}^{\infty} (C_\mu(x) + D_\mu(x)) \, \psi_\mu(X,r) \ \text{for} \ D < x < L \ . \hspace{1cm} (4.15)$$

$A_\mu(0)$ are given (the source); $B_\mu(0)$ are to be found; at $x = D$ the incident $A_\mu(D)$ and $D_\mu(D)$ are scattered by the annular obstacle into $B_\mu(D)$ and $C_\mu(D)$; at $x = L$ the incident $C_\mu(L)$ are reflected into $D_\mu(L)$. So to find $B_\mu(0)$ we have to combine reflection and transmission properties of the obstacle at $x = D$ and the open end at $x = L$.

4.3. Scattering by annular obstacle

For a given set of modes incident from $x < D$ (A-modes) and modes

incident from $x > D$ (D-modes) we want to know the reflected and transmitted modes generated in $x < D$ (B-modes) and in $x > D$ (C-modes). If we identify with the modal amplitudes $A_\mu(D)$ the vector $\mathbf{A}(D)$, and similarly $\mathbf{B}(D)$, $\mathbf{C}(D)$ and $\mathbf{D}(D)$, this relation is most conveniently expressed by a reflection matrix R and a transmission matrix T.

$$\mathbf{B} = R\mathbf{A} + T\mathbf{D}$$

$$\mathbf{C} = T\mathbf{A} + R\mathbf{D} \ . \tag{4.16}$$

(Due to symmetry, reflection and transmission from the left is the same as from the right.)

A natural method to determine R and T is the technique of mode matching. By projecting the conditions of continuity along an interface to a suitable modal basis (i.e., taking inner products) the problem may be reduced to one of linear algebra. This method is well-known, also for the present iris problem. However, a rather subtle detail in the numerical realisation is only relatively recently well understood [10]. To make this point clear, we will work right from the start with *truncated* series, and assume that our solution will be represented by N modes.

Since the problem is linear it is sufficient to determine the scattered field of a single mode. So we have for $n = 1, \dots, N$ at

$$x = D- \ : \ p \ = \sum_{\mu=1}^{N} \left(\delta_{\mu n} + R_{\mu n} \right) \psi_\mu \ ,$$

$$ip_x = \sum_{\mu=1}^{N} \left(\delta_{\mu n} - R_{\mu n} \right) \gamma_\mu \psi_\mu \ , \tag{4.17}$$

$$x = D+ \ : \ p \ = \sum_{\mu=1}^{N} T_{\mu n} \, \psi_\mu \ ,$$

$$ip_x = \sum_{\mu=1}^{N} T_{\mu n} \, \gamma_\mu \, \psi_\mu \ , \tag{4.18}$$

for matrices $R_{N \times N}$ and $T_{N \times N}$. For the moment, the circular area $0 \leq r < h$ at $x = D$ is considered as a duct of length $= 0$, with a set of modes $\hat{\psi}_\mu$ defined by exactly the same equation (4.3) as for $\psi_\mu(r, x = D)$ but now with boundary conditions (4.4) applied at $r = h$. The corresponding inner product will be denoted by

$$[\hat{\psi}_\mu, \hat{\psi}_\nu] = \int\limits_0^h \hat{\psi}_\mu \hat{\psi}_\nu r \, k(r, D)^{-2} \, dr \sim \delta_{\mu\nu} \ . \tag{4.19}$$

We have now in the iris a representation by Q modes,

$$x = D \ : \ ip_x = \sum_{\mu=1}^Q U_{\mu n} \hat{\psi}_\mu \ \text{on} \ 0 \leq r \leq h \ ,$$

$$ip_x = 0 \qquad \text{on} \ h \leq r \leq a \tag{4.20}$$

where we introduced an auxiliary matrix $U_{Q \times N}$.

It is reasonable to take in the iris the number of modes, Q, smaller than N, the number in the full duct. Since the number of zeros of the μ-th mode is $\mu - 1$, the typical radial wavelength of ψ_μ is $(\mu - 1)/2a$ and of $\hat{\psi}_\mu$ $(\mu - 1)/2h$ (see Figure 3).

So a balance between representation accuracies in the iris and the full duct requires about the same smallest wave lengths, or

$$Q = [Nh/a] \ . \tag{4.21}$$

Indeed, this choice yields the fastest convergence for $N \to \infty$ to the physical solution we are interested in. As an example, this behaviour is illustrated in Figure 4.

In addition, it should be noted, that if we take N/Q very much different from a/h, we *may* converge to another solution. This is not an artefact of the method. The problem stated has indeed a non-unique solution. The additional condition necessary to select the correct solution is the so-called *edge condition* [10]: the integrated energy of the field in a neighbourhood of the edge $r = h$ must be
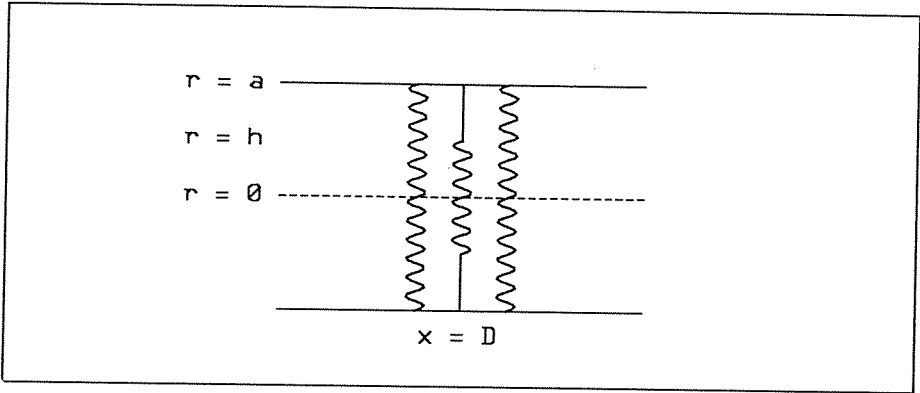
Figure 3. Mode matching through iris.

finite (no source hidden in the edge). Therefore, we may conclude that for the edge condition we should take N/Q not too far from a/h; for reasons of efficiency it is best to take N/Q near a/h as close as possible (eq. (4.21)).

Since p_x is continuous along the *full* interval $0 \le r \le a$ ($p_x = 0$ on both sides of the obstacle along $h \le r \le a$), we have immediately from (4.17) and (4.18) that

$$\delta_{\mu\nu} - R_{\mu\nu} = T_{\mu\nu} \tag{4.22}$$

or in matrix notation $I_{N \times N} - R_{N \times N} = T_{N \times N}$. Using this in the condition of p continuous along $0 \le r \le h$ we find

$$\sum_{\nu=1}^{N} \delta_{\nu n} \psi_\nu = \sum_{\nu=1}^{N} T_{\nu n} \psi_\nu \quad \text{for} \ \ 0 \le r \le h \ .$$

Multiply left- and right-hand side with $\hat{\psi}_\mu r / k^2$ and integrate, to obtain

$$[\hat{\psi}_\mu, \psi_n] = \sum_{\nu=1}^{N} [\hat{\psi}_\mu, \psi_\nu] \, T_{\nu n} \ .$$

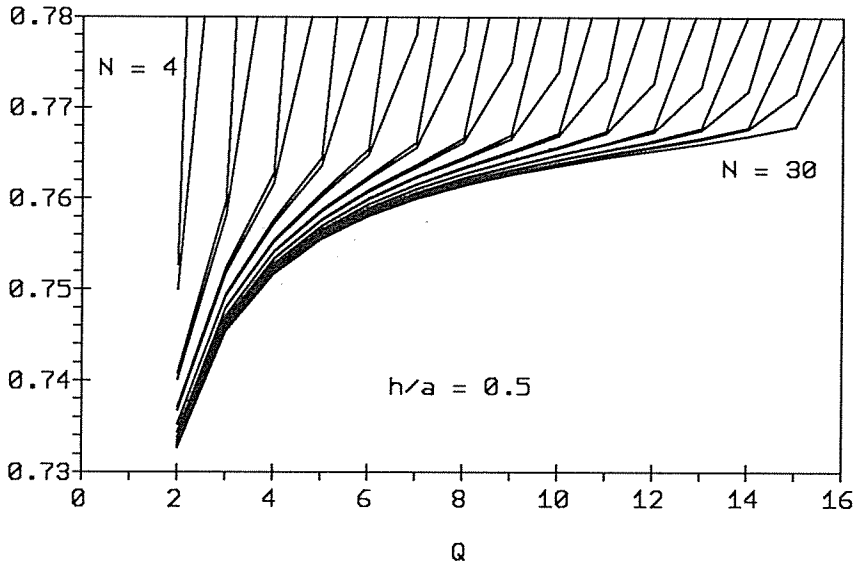**Figure 4.** Re (T_{11}) depending on N for various N/Q rates.

If we do this for $\mu = 1, \ldots, Q$ then we have for the auxiliary matrix $S_{Q \times N}$ with elements $S_{\mu\nu} = [\hat{\psi}_\mu, \psi_\nu]$

$$S_{Q \times N} = S_{Q \times N} T_{N \times N} . \tag{4.23}$$

(Note that this implies that $T_{N \times N}$ has at least Q eigenvalues equal to 1.) Finally, from equality of p_x in (4.18) and (4.20) we have

$$\sum_{\nu=1}^{N} T_{\nu n} \gamma_\nu \psi_\nu = \sum_{\nu=1}^{Q} U_{\nu n} \hat{\psi}_\nu \text{ on } 0 \le r \le h$$

$$= 0 \qquad \text{on } h < r \le a .$$

Multiply left- and right-hand side with $\psi_\mu \, r/k^2$ and integrate, to obtain

$$T_{\mu n} \gamma_\mu (\psi_\mu, \psi_\mu) = \sum_{\nu=1}^{Q} [\psi_\mu, \hat{\psi}_\nu] U_{\nu n} .$$

For the auxiliary matrix $M_{N \times Q}$ with elements

$$M_{\mu\nu} = [\psi_\mu, \hat{\psi}_\nu] \, / \, \gamma_\mu(\psi_\mu, \psi_\mu)$$

we have then

$$T_{N \times N} = M_{N \times Q} \, U_{Q \times N} \, . \tag{4.24}$$

Combining (4.23) with (4.24) and eliminating U gives the final result

$$T_{N \times N} = M_{N \times Q}(S_{Q \times N} \, M_{N \times Q})^{-1} \, S_{Q \times N} \, . \tag{4.25}$$

4.4. Reflection at open end

The reflection of a sound field in a semi-infinite pipe at a flanged open end is (at least for the uniform sound speed as we assumed here in $x \geq L$) classic and goes back to Rayleigh [4,5].

As in the previous chapter the reflection of incident modes $\mathbf{C}(L)$ into backwards running modes $\mathbf{D}(L)$ is most conveniently expressed by an end reflection matrix E as

$$\mathbf{D} = E \, \mathbf{C} \, . \tag{4.26}$$

This matrix E is found as follows.

Using the free field Green's function and its image in the surface $x = L$ we can express the field p in $x > L$ in terms of a normal velocity distribution at the pipe end cross section. For the field just in the pipe opening this *Rayleigh integral* becomes

$$p(L,r) = -\frac{1}{2\pi} \int\limits_0^{2\pi} \int\limits_0^a p_x(L,r') \, \frac{e^{-ik_0\sigma}}{\sigma} \, r'\mathrm{d}r' \, \mathrm{d}\theta' \tag{4.27}$$

where $k_0 = k(x \geq L, r)$ (constant) and $\sigma^2 = r^2 + r'^2 - 2rr' \cos(\theta - \theta')$. Since at $x = L$ ψ_μ is now just a multiple of a Besselfunction (equation (4.1) with normalization (4.12)) we may write

$$p(L, r) = \sum_{\mu=1}^{\infty} (C_\mu(L) + D_\mu(L)) \, \psi_\mu(L, r) = \sum_{\mu=1}^{\infty} q_\mu \, J_0(\alpha_{0\mu} r) \, , \qquad (4.28)$$

$$q_\mu = (C_\mu(L) + D_\mu(L)) \, n_\mu \, , \quad n_\mu = \sqrt{2} \, k_0 / a \, J_0(j'_{0\mu}) \, |k_0^2 - \alpha_{0\mu}^2|^{1/2} \, ,$$

and similarly

$$p_x(L, r) = -\frac{i}{a} \sum_{\mu=1}^{\infty} v_\mu \, J_0(\alpha_{0\mu} r) \, , \qquad (4.29)$$

$$v_\mu = a(C_\mu(L) - D_\mu(L)) \, \gamma_\mu n_\mu \, .$$

Substitute (4.28) and (4.29) into (4.27), and introduce Sonine's integral

$$\frac{e^{-ik_0 \sigma}}{\sigma} = \frac{-i}{a} \int_0^\infty \frac{z \, J_0(z\sigma/a)}{w(z)} \, \mathrm{d}z \, , \qquad (4.30)$$

$$w(z) = \sqrt{(k_0 a)^2 - z^2} \quad \text{with} \quad \text{Im}\,(w) \leq 0 \, ,$$

and Gegenbauer's addition theorem

$$J_0(z\sigma/a) = \sum_{m=-\infty}^{\infty} e^{im(\theta - \theta')} \, J_m(zr/a) \, J_m(zr'/a) \, , \qquad (4.31)$$

so that the r'- and θ'-integrals can be evaluated. We arrive at the equation

$$\sum_{\mu=1}^{\infty} q_\mu \, J_0(\alpha_{0\mu} r) = -\sum_{\mu=1}^{\infty} v_\mu \, J_0(j'_{0\mu}) \int_0^\infty \frac{z^2 \, J_0(zr/a) \, J_0'(z)}{w(z) \, (z^2 - j_{0\mu}'^2)} \, \mathrm{d}z \, . \qquad (4.32)$$

Multiply left- and right-hand side with $J_0(\alpha_{0\nu} r) \, r$ and integrate over $0 \leq r \leq a$ (note orthogonality). Substitute for q_μ and v_μ the original C_μ and D_μ. Then we obtain finally

$$C_\mu(L) + D_\mu(L) = \sum_{\nu=1}^{\infty} Z_{\mu\nu}(C_\nu(L) - D_\nu(L)) \qquad (4.33)$$

with

$$Z_{\mu\nu} = 2a\gamma_\nu \, \frac{n_\nu \, J_0(j'_{0\nu})}{n_\mu \, J_0(j'_{0\mu})} \int_0^\infty \frac{z^3 \, J_0'(z)^2}{w(z)\,(z^2 - j'^2_{0\mu})\,(z^2 - j'^2_{0\nu})} \, \mathrm{d}z \ .$$

In matrix notation this is $\mathbf{C} + \mathbf{D} = Z(\mathbf{C} - \mathbf{D})$, or $\mathbf{D} = (Z + I)^{-1} \, (Z - I)\,\mathbf{C}$. Hence, the reflection matrix is

$$E = (Z + I)^{-1}\,(Z - I) \ . \qquad (4.34)$$

(of course, finally to be truncated to $N \times N$).

Finally, we include an additional reflection matrix to matrix E to allow for a possible discontinuous temperature across $x = L$. The proper transition conditions used are continuity of pressure ($[p] = 0$) and momentum ($[k^{-2}\,p_x] = 0$). For the present purposes this is a minor detail, and will not be further worked out here.

4.5. Gathering the pieces

Now that we have prepared the building blocks of our complete solution ((i) propagation in the smooth parts of the duct; (ii) scattering by the annular obstacle; (iii) reflection by the open end) it is relatively straightforward to assemble them to one coupled system of reflecting and transmitting acoustical elements. The coupling becomes especially clear if we retain the matrix notation already introduced, and present the solution as a modal amplitude vector $\mathbf{B}(0)$ for given source $\mathbf{A}(0)$.

To describe the modal phase shift from $x = 0$ to D and from D to L we introduce the diagonal matrices

$$F \text{ with elements } F_{\mu\mu} = \exp(-i \int_0^D \gamma_\mu(\xi)\,\mathrm{d}\xi)$$

$$G \text{ with elements } G_{\mu\mu} = \exp(-i \int_D^L \gamma_\mu(\xi)\,\mathrm{d}\xi) \ . \qquad (4.35)$$

Using (4.26) and (4.16) we have now immediately

$$\mathbf{D}(D) = G\,\mathbf{D}(L) = GE\,\mathbf{C}(L) = GEG\,\mathbf{C}(D)$$

$$= GEGT\,\mathbf{A}(D) + GEGR\,\mathbf{D}(D)$$

and so

$$\mathbf{D}(D) = (I - GEGR)^{-1}\,GEGT\,\mathbf{A}(D) = H\,\mathbf{A}(D)\ . \tag{4.36}$$

Finally, the solution we were looking for is

$$\mathbf{B}(0) = F\,\mathbf{B}(D) = FR\,\mathbf{A}(D) + FT\,\mathbf{D}(D)$$

$$= F(R + TH)\,\mathbf{A}(D)$$

$$\mathbf{B}(0) = F(R + TH)F\,\mathbf{A}(0)\ . \tag{4.37}$$

Note that although the exponential decay of the cut-off modes would make it possible to consider only one mode in $[0, D]$, this is not the case along $[D, L]$ because this interval may be relatively short.

5. RESULTS

5.1. Introduction

Any sufficiently complex mathematical model of a serious engineering problem has to be implemented, eventually, as a computer program. Of course, trends, simplified cases, the behaviour near singularities in parameter and variable space, and the character of single isolated effects should be understood analytically as much as possible. But the interaction of various equally important components and the rôle of more general configurations and more realistic geometries can only be studied by numerical simulation.

The solution of the present problem is split up in components for

which the numerical solution is, although not straightforward, standard in the sense that we can utilize the world-wide available well-tested robust public-domain software [11], or routines of commercial libraries of numerical software like NAG [12] or IMSL [13].

The first important problem to be solved numerically is the Sturm-Liouville eigenvalue problem (4.3) and (4.4). In particular the eigenvalues are essential, because we want to design the configuration such that for $m = 0$ only the first eigenvalue γ_{01} is real, and that for $|m| \geq 1$ all eigenvalues are imaginary.

If the temperature (and therefore $k(x, r)$) is independent of r, the solution is just the Bessel function of the first kind J_m, so this can serve as a check for the numerics. The routine we used is the Fortran translation TSTURM by B.S. Garbow (Argonne National Laboratory) of the Algol procedure Tristurm by Peters and Wilkinson ([14]). Details of the application like accuracy and prediction of the number of eigenvalues are necessary but will not be considered here because these evidently depend on the routine chosen.
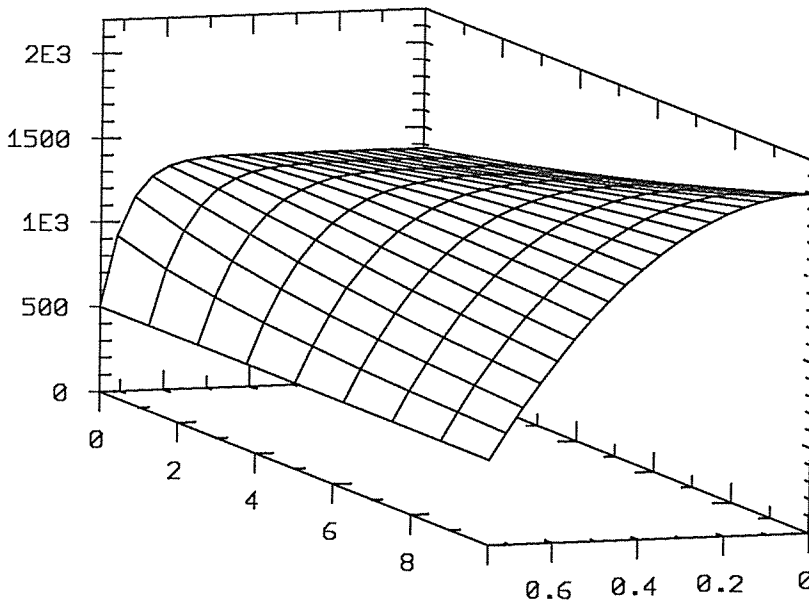

Figure 5. Temperature profile.

For the open end reflection we need to evaluate the integral in each matrix $Z_{\mu\nu}$ element (eq. (4.33)). The convergence rate at infinity of the integrand is rather poor, especially for increasing μ and ν, and we rewrote the integral by using complex contour deformation into a set of numerically more pleasant integrals. One Bessel function is rewritten as a sum of Hankel functions: $J_0' = \frac{1}{2}H_0^{(1)'} + \frac{1}{2}H_0^{(2)'}$, the first of which converges in the upper complex half plane, and the other in the lower half plane. The integral may subsequently be split up in two of which the contours can be deformed to the positive and to the negative imaginary axis. After taking care of the branch cut of square root $w(z)$ and possible residues if $\mu = \nu$, we arrive at two integrals which can be evaluated in a standard way (we used Romberg integration).

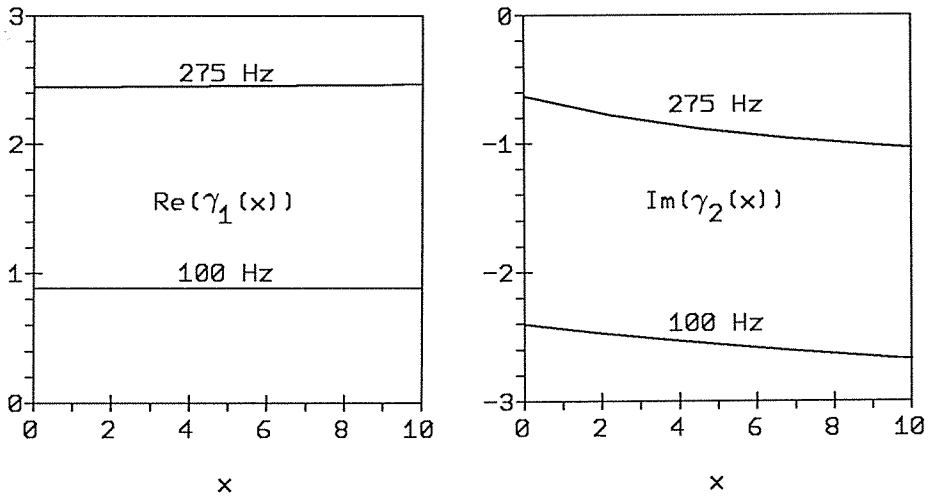The matrix and vector manipulations necessary for the scatter-

Figure 6. First two modes for $f = 100\ Hz$ and $f = 275\ Hz$.

ing problem (eq. [4.22–25]) and the coupling of the various parts (eq. [4.36,37]) are taken from the BLAS/LINPACK package written by Dongarra et al. ([15]).

5.2. Example
The realistic example we have considered is a duct of length $L =$

10 m, radius $a = 0.75$ m, and a temperature profile as given in Figure 5. The temperature varies along the centerline from 2000 K to 1350 K, and remains along the wall at 500 K, such that the cross sectional average is constant 1250 K. (This is also the temperature in the furnace chamber $x > L$).

We scanned the modal x-variation of two frequencies: $f = 100\ Hz$ and $f = 275\ Hz$. One much lower and the other just low enough for the first mode to be the only one cut-on (Figure 6). Note that the first few modes appear to vary only very slightly in x. This is caused by the temperature varying such that the average is constant. As a result only a few positions in x are necessary.

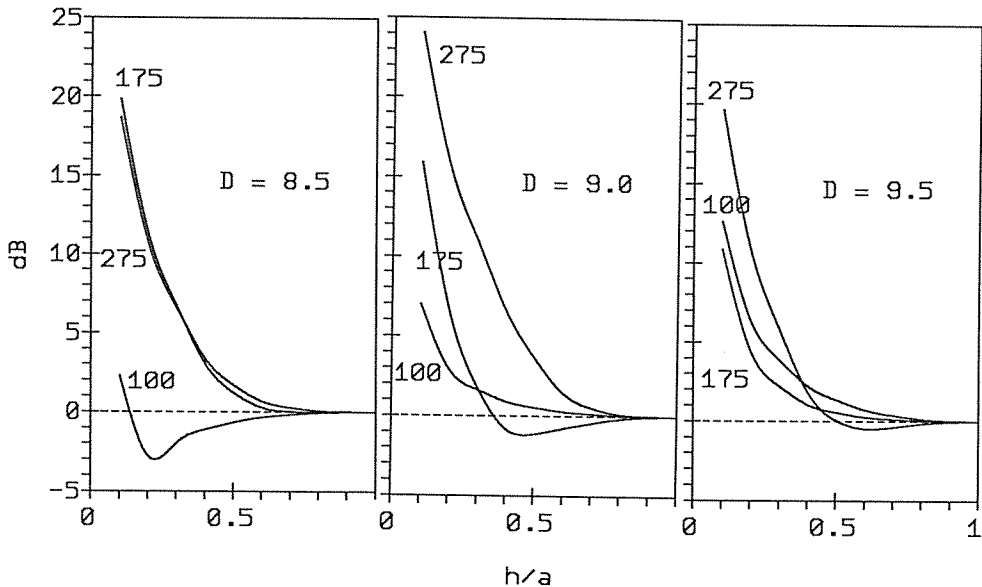The most important question for the engineer involved with the

Figure 7. Relative transmitted power Δ for $D = 8.5,\ 9,\ 9.5$ m at $f = 100,\ 175,\ 275\ Hz$.

contamination problem is of course how much of a constriction of the pipe can be observed from the acoustical reflection. To this end we varied the iris radius h for three frequencies (Figure 7), and the frequency
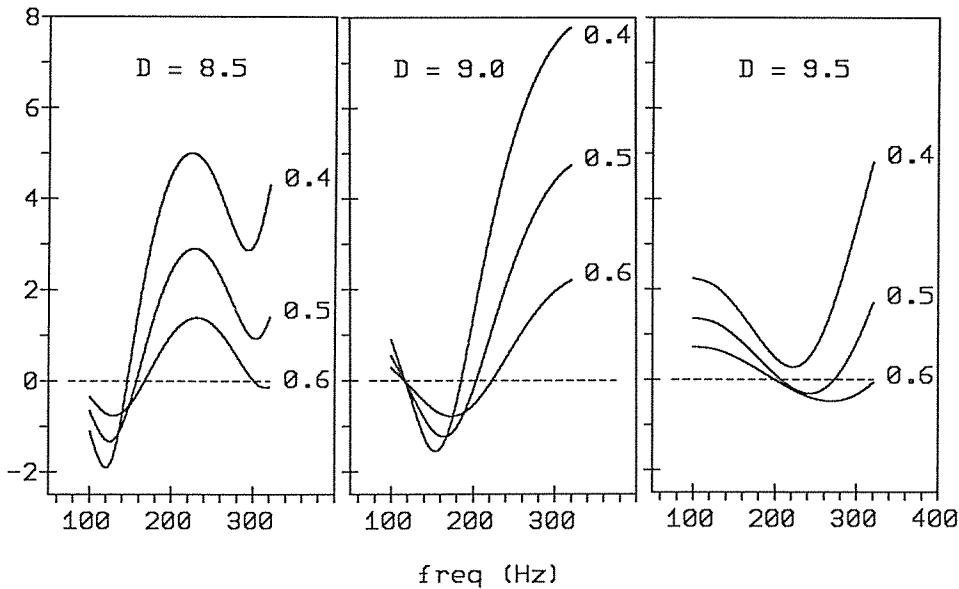
Figure 8. Relative transmitted power Δ for $D = 8.5,\ 9,\ 9.5$ m at $h/a = 0.4,\ 0.5,\ 0.6$.

f for three radii h (Figure 8), at three positions D. We compared the acoustic transmitted power $\mathcal{P} \sim |A(0)|^2 - |B(0)|^2$ [4,5] with ($\mathcal{P}$) and without ($\mathcal{P}_0$) the obstacle. This is commonly expressed in decibels as $\Delta = 10\ \log_{10}(\mathcal{P}_0/\mathcal{P})\,dB$. If the reflected signal is strong enough, $\mathcal{P}$ becomes small, and for a difference larger than, say, $\Delta = 3\,dB$ we can safely say that the obstacle can be detected.

We see that in general a high frequency is favourable for the obstacle to be visible. There is, however, always the inopportune possibility of resonance and the excitation of a standing wave, increasing the net transmission so much that with obstacle more energy is transmitted than without ($\Delta < 0$).

This effect may be considered as a typical example of an interaction effect, adverse to intuition, and only found (usually) by numerical experiments.

Of course, what we now should do is to further investigate the in-

fluence of position D, temperature profile, and consider in detail the behaviour of the reflection matrices R and E, and the sound pressure profile in r. However, this is a bit outside the scope of the present introductory purposes.

We end our discussion of results here with the remark that it is very important for the applied mathematician to understand the physics of his problem well. Apart from the selection of the model and the numerical methods, this is vital for the interpretation and feed back to the model of the final results.

6. CONCLUSIONS

As an example of a mathematical engineering problem we have considered the acoustical detection of constrictions in a pipe due to contamination by liquid particles of ashes in gasified coal. By applying a semi-analytical solution based on slowly varying modes one can select an acoustic frequency such that there are guaranteed no spurious reflections due to the temperature gradients. Although for relatively low frequencies small obstacles are not visible, a constriction of say 50% is well visible for high enough frequencies. For some frequencies the constriction appeared, in some cases, to reduce the reflection and spoil the detection. It is clear that for any diagnosis a complete frequency range is to be taken into consideration.

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