# Sound Transmission

# in a Slowly Varying Lined Flow Duct

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Sound transmission through straight circular ducts with a uniform (inviscid) mean flow and a constant acoustic lining (impedance wall) is classically described by a modal expansion. A natural extension for ducts with, in axial direction, slowly varying properties (like diameter, wall impedance, and mean flow) is a multiple-scales solution. It is shown in the present paper that a consistent approximation of boundary condition and mean flow allows the multiple-scales problem to have an exact solution. Turning points and other singularities of this solution are discussed.

## 1. INTRODUCTION

The theory of sound propagation in straight ducts with (constant) impedance type boundary conditions and a homogeneous (stationary) medium is classical and well-established (MORSE and INGARD [7]; PIERCE [14]). Per frequency $\omega$, the sound field, satisfying Helmholtz' equation $(\nabla^2 + \omega^2)\phi = 0$, may be built up by superposition of eigensolutions or modes. These are certain shape-preserving fundamental solutions. The existence of these modes is a consequence of the relatively simple geometry, allowing separation of variables. For cylindrical ducts, the configuration we will consider here further, with associated cylindrical coordinate system $(x, r, \theta)$ the modes are given, in the usual complex notation, by exponentials and Bessel functions: $J_m(\alpha r)\exp(i\omega t - im\theta - ikx)$. The eigenvalue $m$, or circumferential wave number, is, due to the periodicity in $\theta$, an integer; the eigenvalue $\alpha$, or radial wave number, is determined by the appropriate boundary condition at the duct wall $r = 1$, while the axial wave number $k$ is related to $\alpha$ and $\omega$ via a dispersion relation.

If we introduce a mean flow in the duct (motivated by aircraft turbofan engine applications; NAYFEH et al. [10]), the acoustic problem becomes rapidly much more difficult. Spatially varying mean flow velocities produce non-constant coefficients of the acoustic equations, which usually spoils the possibility of a modal expansion. Perhaps the simplest non-trivial mean flow is a uniform flow, in the limit of vanishing viscosity. Then modal solutions are possible, of a form rather similar to the one without flow. A most important problem here is the way the sound field is transmitted through the vanishing mean flow boundary layer at the wall, which thus effectively modifies the impedance

boundary condition at $r = 1$ into an equivalent boundary condition for $r \to 1$. This modified boundary condition was first proposed by INGARD [5], and later on proved by EVERSMAN and BECKEMEYER [2] and TESTER [17] to be indeed the correct limit for a boundary layer which is much smaller than a typical acoustic wave length.

In certain applications the geometry of a cylindrical duct is only an approximate model, and it is therefore of practical interest to consider sound transmission through ducts of varying cross section. In general, this problem is, again, very difficult, and one usually resorts to numerical methods. However, quite often, especially when the duct carries a mean flow, the diameter variations of the duct are only gradual, thus introducing prospects of perturbation solutions. Indeed, several authors have utilized the small parameter related to the slow cross section variations (EISENBERG and KAO [1]; TAM [16]; HUERRE and KARAMCHETI [4]; THOMPSON and SEN [18]). A particularly interesting and systematic approach is the method of multiple scales elaborated by Nayfeh and co-workers, both for ducts without (NAYFEH and TELIONIS [12]) and with flow (NAYFEH, TELIONIS and LEKOUDIS [13]; NAYFEH et al. [11]), and with hard and impedance walls. The multiple-scales technique provides a very natural generalization of modal solutions since a mode of a constant duct is now assumed to vary its shape according to the duct variations, in a way that amplitude and wave numbers are slowly varying functions, rather than constants.

In the present study we will proceed along these lines, and present an explicit multiple-scales solution of a problem, similar to the one considered previously by Nayfeh et al. We will consider a mode propagating in a slowly varying duct with impedance walls and containing almost uniform (inviscid, isentropic, irrotational) mean flow with vanishing boundary layer.

A somewhat puzzling aspect of Nayfeh et al.'s solutions was that without flow the differential equation for the slowly varying amplitude could be solved exactly, whereas with flow this was not the case. Also, in RIENSTRA [15] the amplitude equation for a similar problem of a duct with (slowly varying) porous walls could be solved exactly. We will show that, at least in the present type of problems, an exact solution appears to be the rule rather than an exception, *if* the entire perturbation analysis is consistent at all levels. In the problem under consideration, Nayfeh et al. used an ad hoc mean flow velocity profile (quasi one dimensional with some assumed boundary layer) which is not a solution of the mean flow equations, and, furthermore, in case of a vanishing boundary layer they used an incorrect effective boundary condition, although at that time this was not known. MYERS [8] showed that INGARD's [5] effective boundary condition for an impedance wall with uniform mean flow is to be modified significantly in case of non-uniform mean flow along curved surfaces.

Both MYERS' [8] boundary condition and a consistent approximation of the mean flow will be seen to be essential for the explicit solution that will be presented here.

## 2. FORMULATION OF THE PROBLEM

We consider a cylindrical duct with slowly varying cross section. Inside this duct we have a compressible inviscid isentropic irrotational mean flow with harmonic perturbations. To the mean flow the duct is hard-walled, but for the acoustic field the duct is lined with an impedance wall.

It is convenient to make dimensionless: spatial dimensions on a typical duct radius $R_\infty$, densities on a reference value $\rho_\infty$, velocities on a reference sound speed $c_\infty$, time on $R_\infty/c_\infty$, pressure on $\rho_\infty c_\infty^2$, and velocity potential on $R_\infty c_\infty$.

We then have in the cylindrical coordinates $(x, r, \theta)$, with unit vectors $e_x$, $e_r$, and $e_\theta$, the duct

$$r = R(X), \quad X = \epsilon x, \quad -\infty < x < \infty, \quad 0 \leqslant \theta < 2\pi$$

where $\epsilon$ is a small parameter, and $R$ is by assumption only dependent on $\epsilon$ through $\epsilon x$. The fluid in the duct is described by (see, for example, PIERCE [14]).

$$\rho'_t + \nabla\cdot(\rho'\mathbf{v}') = 0$$

$$\rho'(\mathbf{v}'_t + \mathbf{v}'\cdot\nabla\mathbf{v}') + \nabla p' = 0$$

$$\gamma p' = \rho'^\gamma$$

$$c'^2 = dp'/d\rho' = \rho'^{\gamma-1}$$

(with boundary and initial conditions), where $\mathbf{v}'$ is particle velocity, $\rho'$ is density, $p'$ is pressure, $c'$ is sound speed, and $\gamma$ is the specific heat ratio, a constant. Since we assumed the flow to be irrotational, we may introduce a potential $\phi'$, with $\mathbf{v}' = \nabla\phi'$.

This flow is split up into a stationary (mean) flow part, and an acoustic perturbation. This acoustic part varies harmonically in time with circular frequency $\omega$, and with small amplitude to allow linearization. To avoid a complicating coupling between the two small parameters ($\epsilon$ and the acoustic amplitude), we assume this acoustic part much smaller than any relevant power of $\epsilon$.

In the usual complex notation we write then

$$\mathbf{v}' = \mathbf{V} + \mathbf{v}e^{i\omega t}, \quad \rho' = D + \rho e^{i\omega t}, \quad p' = P + pe^{i\omega t},$$

$$c' = C + ce^{i\omega t}, \quad \phi' = \Phi + \phi e^{i\omega t}.$$

Substitution and linearization yields:
mean flow field

$$\nabla\cdot(D\mathbf{V}) = 0$$

$$D(\mathbf{V}\cdot\nabla)\mathbf{V} + \nabla P = 0$$

$$\gamma P = D^\gamma \qquad\qquad\qquad (1)$$

$$C^2 = \gamma P/D = D^{\gamma-1}$$

$$\mathbf{V} = \nabla\Phi,$$

acoustic field

$$i\omega\rho + \nabla\cdot(D\ \mathbf{v} + \rho\mathbf{V}) = 0$$

$$D[i\omega\mathbf{v} + (\mathbf{V}\cdot\nabla\mathbf{v}) + (\mathbf{v}\cdot\nabla\mathbf{V})] + \rho(\mathbf{V}\cdot\nabla)\mathbf{V} + \nabla p = 0$$

$$p = C^2\rho \tag{2}$$

$$c = \frac{\gamma-1}{2}p/D^{\frac{\gamma+1}{2}}$$

$$\mathbf{v} = \nabla\phi.$$

For the mean flow the duct wall is solid, so the normal velocity vanishes

$$\mathbf{V}\cdot\mathbf{n} = 0 \ \text{ at } r = R(X) \tag{3}$$

where the outward directed normal vector at the wall is given by

$$\mathbf{n} = (-\epsilon R_X \mathbf{e}_x + \mathbf{e}_r)/(1 + \epsilon^2 R_X^2)^{\frac{1}{2}}.$$

For the acoustic part the duct wall is a locally reacting impedance wall with complex impedance $Z = Z(X)$ - slow variations of $Z$ in $x$ may be included -, meaning that *at* the wall, at a hypothetical point with zero mean flow,

$$p = Z(\mathbf{v}\cdot\mathbf{n}).$$

However, this is not the boundary condition needed here. Since we deal with a fluid of vanishing viscosity, the boundary layer along the wall in which the mean flow tends to zero is of vanishing thickness, and we cannot apply a boundary condition at the wall. The required condition is for a point near the wall but still (just) inside the mean flow. For arbitrary mean flow along a (smoothly) curved wall it was given by MYERS [8], eq. 15):

$$i\omega(\mathbf{v}\cdot\mathbf{n}) = [i\omega + \mathbf{V}\cdot\nabla - \mathbf{n}\cdot(\mathbf{n}\cdot\nabla\mathbf{V})](p/Z) \ \text{ at } r = R(X) \tag{4}$$

with the remark that for simplicity we will exclude here the case $Z = 0$. Moreover, we will assume $Z = O(1)$. The above equations and boundary conditions are evidently still insufficient to define a unique solution, and we need additional conditions for mean flow and sound field. Since we are studying axial variations due to the geometry of the pipe, the natural choice is to consider a mean flow, almost uniform, with axial variations only in $X$, and a sound field consisting of a constant-duct mode perturbed by the $X$-variations. Furthermore, this choice indeed implies the absence of vorticity (apart from the vortex sheet along the wall), allowing the introduction of a potential.

Before turning to the acoustic problem, we will derive in the next section the solution of the mean flow problem as a series expansion in $\epsilon$. As noted before, a consistent mean flow expansion is necessary to obtain the explicit multiple scale solution of the acoustic problem.

## 3. MEAN FLOW

Since we assumed a mean flow, nearly uniform, with axial variations in $X$, we have

$$\mathbf{V} = \nabla\Phi(X,r;\epsilon) = U(X,r;\epsilon)\mathbf{e}_x + V(X,r;\epsilon)\mathbf{e}_r.$$

Integration of the momentum equation of (1) yields Bernoulli's equation

$$\frac{1}{2}(U^2 + V^2) + \frac{\gamma}{\gamma-1}P/D = E, \quad \text{a constant.} \tag{5}$$

A useful relation, to be used later, for the cross-sectional mass flux is obtained by application of Gauss' divergence theorem to the mass equation of (1), with a volume enclosed by the duct walls and two axial cross-sections (e.g., PIERCE [14]). Since the normal velocity at the wall vanishes, we have

$$2\pi \int_0^{R(X)} D(X,r;\epsilon)U(X,r,\epsilon)r\,dr = \pi F, \quad \text{a constant.} \tag{6}$$

Since the variations in $x$ are through $X$ only, we may assume the constants $E$ and $F$ to be independent of $\epsilon$. In addition, from the continuity equation of (1) it follows that the small axial mass variations can only be balanced by small radial variations, so $V = O(\epsilon)$, and hence

$$\Phi(X,r;\epsilon) = \epsilon^{-1}\Phi_{-1}(X) + \epsilon\Phi_1(X,r) + O(\epsilon^3).$$

Therefore

$$U(X,r;\epsilon) = U_0(X) + O(\epsilon^2), \quad V(X,r;\epsilon) = \epsilon V_1(X,r) + O(\epsilon^3)$$

and so, with Bernoulli's equation,

$$P(X,r;\epsilon) = P_0(X) + O(\epsilon^2), \ D(X,r;\epsilon) = D_0(X) + O(\epsilon^2),$$

$$C(X,r;\epsilon) = C_0(X) + O(\epsilon^2).$$

With equations (5) and (6) it follows readily that

$$U_0(X) = F/D_0(X)R^2(X) \tag{7}$$

with $D_0$ given by

$$\frac{1}{2}(F/D_0R)^2 + \frac{1}{\gamma-1}D_0^{\gamma-1} = E \tag{8}$$

which is to be solved numerically, per $X$. Of course,

$$P_0 = \frac{1}{\gamma}D_0^\gamma, \quad C_0 = D_0^{\frac{\gamma-1}{2}}.$$

For $V_1$ we return to the continuity equation, which is to leading order

$$(D_0U_0)_X + (rD_0V_1)_r/r = 0.$$

The boundary condition

$$-R_XU_0 + V_1 = 0 \quad \text{at } r = R(X)$$

is already satisfied, through the application of (6) leading to (7). The (finite) solution is then

$$V_1(X,r) = rU_0(X)R_X(X)/R(X). \tag{9}$$

The above solutions $U_0, P_0, D_0$ are recognized as the well-known one dimensional gas flow equations (e.g., MEYER [6]). It should be stressed, however, that the radial velocity component $V_1$ is essential for a consistent mean flow description, and therefore necessary here.

## 4. THE ACOUSTIC FIELD

In this section we will derive the main result of the present paper: the explicit multiple-scales solution for a mode-like wave described by equation (2) with (4).

Since we introduced a velocity potential, we can integrate the momentum equation of (2). So we have

$$i\omega\rho + \nabla\cdot(D\nabla\phi + \rho\nabla\Phi) = 0$$

$$i\omega\phi + \nabla\Phi\cdot\nabla\phi + p/D = 0 \tag{10}$$

$$p = C^2\rho.$$

(We will ignore from here on the uninteresting equation for $c$.) An unperturbed modal wave form would be a function of $r$ multiplied by a complex exponential in $\theta$ and $x$. Then a mode-like wave is obtained by assuming the amplitude and axial and radial wave numbers to be slowly varying, i.e. depending on $X$ (NAYFEH et al. [12]). So we assume

$$\phi(x,r,\theta;\epsilon) = A(X,r;\epsilon)\exp(-im\theta - i\epsilon^{-1}\int^X \mu(\xi)d\xi) \tag{11}$$

$$\rho(x,r;\theta;\epsilon) = B(X,r;\epsilon)\exp(-im\theta - i\epsilon^{-1}\int^X \mu(\xi)d\xi).$$

Then the partial derivatives to $x$ become formally (suppressing the exponent)

$$\partial_x = -i\mu + \epsilon\partial_X, \quad \partial_{x^2}^2 = -\mu^2 - i\epsilon\mu_X - 2i\epsilon\mu\partial_X + \epsilon^2\partial_{X^2}^2.$$

Substitution in (10), and collecting like powers of $\epsilon$ yield up to order $\epsilon^2$

$$i\hat{\omega}B + D_0(A_{rr} + A_r/r - \mu^2 A - m^2 A/r^2)$$

$$+ \epsilon[(U_0 B)_X + (V_1 B)_r + V_1 B/r - i(\mu D_0)_X A - 2i\mu D_0 A_X] = 0$$

$$i\hat{\omega}A + C_0^2 B/D_0 + \epsilon[U_0 A_X + V_1 A_r] = 0$$

where $\hat{\omega} = \omega - \mu U_0$. It is now convenient to eliminate $B$; the equation for $A$ is then (up to order $\epsilon^2$)

$$D_0\mathcal{L}(A) = i\epsilon A^{-1}[(U_0 D_0\hat{\omega}A^2/C_0^2)_X + (rV_1 D_0\hat{\omega}A^2/C_0^2)_r/r + (\mu D_0 A^2)_X] \tag{12}$$

where the operator $\mathcal{L}$ is defined by

$$\mathcal{L} = \partial_r^2 + \partial_r/r + \hat{\omega}^2/C_0^2 - \mu^2 - m^2/r^2.$$

The boundary condition (4), up to order $\epsilon^2$, is now

$$i\omega(\partial_r + i\epsilon\mu R_X)A = [i\hat{\omega} + \epsilon(U_0\partial_X + V_1\partial_r - V_{1r})](C_0^2 B/Z) \text{ at } r = R(X)$$

which becomes, after eliminating $B$,

$$i\omega A_r - \hat{\omega}^2 D_0 A/Z = \epsilon\omega\mu R_X A + \epsilon A^{-1}[U_0\partial_X + V_1\partial_r - V_{1r}](-i\hat{\omega}D_0 A^2/Z). \quad (13)$$

Now assume

$$A(X,r;\epsilon) = A_0(X,r) + \epsilon A_1(X,r) + \cdots,$$

then $\mathcal{L}(A_0)=0$, which is, up to a radial coordinate stretching, Bessel's equation in $r$, with $X$ acting only as a parameter. The mode-like solution we are looking for is then

$$A_0(X,r) = N(X)J_m(\alpha(X)r)$$

where $J_m$ is the $m$-th order Bessel function of the first kind (finite in $r=0$) (WATSON [19]), the 'eigenvalue' $\alpha$ is a solution, continuous in $X$, of

$$i\omega\alpha J'_m(\alpha R) - \hat{\omega}^2 D_0 J_m(\alpha R)/Z = 0 \qquad (14)$$

and $\alpha$ and $\mu$ are related by the dispersion relation

$$\alpha^2 + \mu^2 = \hat{\omega}^2/C_0^2.$$

The amplitude function $N(X)$ is determined from the condition that there exists a solution $A_1$. This is not trivial since we assumed the solution to behave in a certain way, namely, to depend on $X$ rather than $x$. Now suppose that we would proceed and solve the equation for $A_1$, and subsequently find the necessary form of $N$, then it would appear that we end up with similarly undetermined functions in $A_1$. So this approach looks rather inefficient. Indeed, it is not necessary to work out the equations for $A_1$ in detail. We only need a solvability condition (NAYFEH [9]), sufficient to yield the required equation for $N$.

Since the operator $r\mathcal{L}$ is self-adjoint in $r$, we have

$$-iD_0 \int_0^R A_0\mathcal{L}(A_1)r dr = -iD_0 R(A_0\partial_r A_1 - A_1\partial_r A_0)_{r=R}.$$

Further evaluation of this expression (using (13)) and the corresponding right-hand side of (12) gives finally, after some calculus, the following equation

$$\frac{d}{dX}\left[D_0(\mu + \hat{\omega}U_0/C_0^2)\int_0^R A_0^2 r dr - iF\hat{\omega}D_0 A_0^2/RZ\omega\right] = 0.$$

Use is made of equations (7) and (9), and the identities

$$\int_0^R \partial_X f(X,r)dr = \frac{d}{dX}\int_0^R f(X,r)dr - R_X f(X,R),$$

and

$$U_0\partial_X + V_1\partial_r = U_0\frac{d}{dX} \quad \text{along } r = R(X).$$

The above equation can be integrated immediately, with a constant of integration $N_0^2$. The integral of $A_0^2 r$, finally to be evaluated, is a well-known integral of Bessel functions (WATSON [19] p. 135), with the result (using (14))

$$\int_0^R A_0^2 r\, dr = \tfrac{1}{2}N^2 J_m(\alpha R)^2[R^2 - (m/\alpha)^2 - (RD_0\hat{\omega}^2/\alpha Z\omega)^2].$$

After some further simplifications we thus obtain

$$N(X) = \tag{15}$$

$$N_0/J_m(\alpha R)D_0^{1/2}\left[\tfrac{1}{2}\Omega R^2[1-(m/\alpha R)^2 - (D_0\hat{\omega}^2/\alpha Z\omega)^2] - iF\hat{\omega}/RZ\omega\right]^{1/2}$$

with $\quad \Omega = \mu(1 - U_0^2/C_0^2) + \omega U_0/C_0^2,$

so $\quad \Omega^2 = (\omega/C_0)^2 - \alpha^2(1 - U_0^2/C_0^2).$

An interesting special case is the hard-walled duct, where $Z = \infty$. In that case $\alpha R$ is a constant $> m(\geqslant m$ if $m = 0)$, and we can absorb some constant factors of (15) into $N_0$ to obtain

$$N(X) = N_0/D_0^{1/2}\Omega^{1/2}R \quad (Z = \infty). \tag{16}$$

Expression (15) for $N(X)$ is the principal result of the present paper. In the next section we will discuss qualitatively some of the properties. In particular, we will consider the singularity of the mode at points of a vanishing denominator of $N$.

## 5. DISCUSSION AND CONCLUSIONS

If the multiple-scales solution is valid, the mode-like wave behaves locally like a mode of a straight duct. Rotating with angular velocity $\omega/m$, it propagates in axial direction with or without attenuation (unattenuated or cut on: $\mathrm{Im}\,\mu = 0$; attenuated or cut off: $\mathrm{Im}\,\mu \neq 0$). The more interesting aspects here are, of course, connected to the slow variations in $X$. These are mainly represented by the amplitude function $N$ and the phase function $\mu$.

When $R$ and $Z$ vary with $X$, the mode changes gradually, except at the points where the denominator of $N$ (eq. (15)) vanishes and the approximation breaks down. These points are just found at the double eigenvalues, i.e., where two eigenvalues $\mu$ (or $\alpha$) coalesce. These are given by equation (14) and its partial derivative to $\mu$, which is just equivalent to the denominator of $N$.

Clearly, the approximation breaks down because the two coalescing modes couple (the energy of the incident mode is distributed over the two) in a short region. A local analysis is necessary to determine the resulting amplitudes $N_0$. In general the two modes propagate in the same direction, but in some cases the second mode is running backwards while at the same time the incident mode becomes cut-off in such a way that beyond the point no energy is propagated. Points with this behaviour are usually called turning points (NAYFEH

[9]), since the incident mode is totally reflected into the backward running mode (we assume, of course, the absence of other interfering turning points).

Since it is sometimes instructive to illustrate this behaviour by a more global, energy-like, quantity, we interrupt the discussion to introduce the acoustic power $\mathcal{P}$ of a single (slowly varying) mode at a duct cross section. Following GOLDSTEIN [3], we define the acoustic power at a surface $S$

$$\mathcal{P} = \int_S \mathbf{I} \cdot \mathbf{n} ds$$

where $\mathbf{I}$ is the time-averaged acoustic intensity or energy flux, here given by

$$\mathbf{I} = \tfrac{1}{2} \text{Re}[(p/D + \nabla \Phi \cdot \nabla \phi)(D \nabla \phi + \rho \nabla \Phi)^*]$$

with $^*$ denoting the complex conjugate. Considering here for $S$ a duct cross section, we need the axial component of $\mathbf{I}$, which is to leading order (see expression (11))

$$I_x = \tfrac{1}{2} \omega D_0 \text{Re}(\Omega) |A_0|^2 \exp(2\epsilon^{-1} \int^X \text{Im}\,\mu(\xi) d\xi). \qquad (17)$$

We will now consider a few examples of turning points in detail. Evidently important is the case of a duct with hard walls ($Z = \infty$), where a real $\Omega$ tends to zero to become pure imaginary ($\alpha$ is always real). At $\Omega = 0$, $N$ is singular (eq. (16)), and the incident mode couples to a backwards running mode. For real $\Omega$ we have

$$\mathcal{P} = \tfrac{1}{2} \pi \omega |N_0|^2 (1 - m^2/\alpha^2 R^2) J_m(\alpha R)^2$$

whereas for pure imaginary $\Omega$

$$\mathcal{P} = 0$$

so the mode must reflect indeed. Note that this behaviour is irrespective of the presence of mean flow. This is not the case in the next example. If $Z$ is pure imaginary ($Z = iY$) and $U_0 = 0$ we have again a turning point at $\Omega = \mu = 0$ (equation (15) with $F = 0$). Using the fact that $\alpha$ remains real when $\mu$ changes from real into imaginary, we find for real $\mu$

$$\mathcal{P} = \pi \omega |N_0|^2$$

and for imaginary $\mu$

$$\mathcal{P} = 0$$

which is similar to the hard wall case. However, with flow ($U_0 \neq 0$), we have the interesting result that $\Omega = 0$ is *not* a turning point. Indeed, the mode changes from cut-on ($\Omega$ real) to cut-off ($\Omega$ complex), but there is no reflection possible now because the backward running mode becomes cut-on at another point. Also from energy considerations there is no need for a reflection, because the power remains non-zero for the cut-off mode: from equation (14) we see that a cut-on mode with $\alpha$, $\Omega$, and $\mu$ real cannot become cut-off ($\mu$

complex) with $\alpha$ real; as a result Re $\Omega \neq 0$, and so $\mathscr{P} \neq 0$ (eq. (17)).

As a conclusion, we may summarize that the approximation breaks down at the double eigenvalues when the mode couples with other modes. In particular, if $Z = \infty$, or $Z = iY$ with $U_0 = 0$, the mode reflects at the double eigenvalue $\Omega = 0$. If $Z = iY$ and $U_0 \neq 0$, $\Omega = 0$ is not a double eigenvalue, and the mode changes smoothly and without reflection from propagating to dissipating.

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