Algorithms for Model Checking (2IW55)

Lecture 6

The μ-Calculus

Chapter 7

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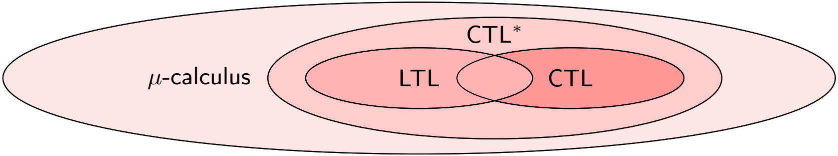
Outline

μ-Calculus: syntax and semantics

Recall: symbolic model checking for CTL was based on fixed points.

Idea of μ-calculus: add fixed point operators as primitives to basic modal logic.

- μ-calculus is very expressive (subsumes CTL, LTL, CTL*).
- μ-calculus is very pure ("assembly language" for modal logic, cf: λ-calculus for functional programming).
- drawback: lack of intuition.
- fragments of the μ-calculus are the basis for practical model checkers, such as μCRL, mCRL2, CADP, Concurrency Workbench.



μ-Calculus: syntax and semantics

Kripke Structures and Labelled Transition Systems

Mix of Kripke Systems and Labelled Transition Systems: $M = \langle S, \mathit{Act}, R, L \rangle$ over a set *AP* of atomic propositions:

- S is a set of states
- *Act* is a set of action labels
- R is a labelled transition relation: $R \subseteq S \times \mathit{Act} \times S$
- L is a labelling: $L \in S \to 2^{AP}$

Notation: $s \xrightarrow{a} t$ denotes $(s, a, t) \in R$

Special cases:

- Kripke Structures: *Act* is a singleton (only one transition relation)
- LTS (process algebra): *AP* is empty (only propositions true and false)

μ-Calculus: syntax and semantics

Let the following sets be given:

- *AP* (atomic propositions),
- *Act* (action labels) and
- *Var* (formal variables).

The syntax of μ-calculus formulae f is defined by the following grammar:

$$f ::= p \mid X \mid \neg f \mid f \wedge f \mid f \vee f \mid [a]f \mid \langle a \rangle f \mid \mu X.f \mid \nu X.f$$

Note:

- $p \in \mathit{AP}, X \in \mathit{Var}, a \in \mathit{Act}$.
- $[a]f$ means "for all direct a-successors, f holds".
- $\langle a \rangle f$ means "for some direct a-successor, f holds".
- We only consider fixed point formulae $\substack{\mu \\ \nu} X.f$ if X occurs under an even number of negations ($\neg$) in f

μ-Calculus: syntax and semantics

Some notation and terminology:

- "X occurs in f only under an even number of $\neg$-symbols" is called the syntactic monotonicity criterion. This criterion ensures the (semantic) existence of fixed points
- An occurrence of X is bound by a surrounding fixed point symbol ${}^{\mu}_{\nu}\ X$ (${}^{\mu}_{\nu} \in \{\mu, \nu\}$). Unbound occurrences of X are called free.
- A formula is closed if it has no free variables, otherwise it is called open
- An environment e interprets the free formal variables X as a set of states
  - Mixed Kripke Structure $M = \langle S, \mathit{Act}, R, L \rangle$
  - $e : \mathit{Var} \rightarrow 2^S$
  - $e[X := V]$ is an environment like e, but X is set to V:

$$e[X := V](Y) := \begin{cases} V & \text{if } Y = X \\ e(Y) & \text{otherwise} \end{cases}$$

μ-Calculus: syntax and semantics

Fix a system: $M = \langle S, \mathit{Act}, R, L\rangle$

- The semantics of a formula is only defined if we know the values of its free variables.
- The semantics of a μ-Calculus formula f in the context of environment e is the set of states where f holds:

$$
\begin{array}{llll}
[\![\neg f]\!]_e & = S \setminus [\![f]\!]_e & & \\
[\![p]\!]_e & = \{s \mid p \in L(s)\} & [\![X]\!]_e & = e(X) \\
[\![f \wedge g]\!]_e & = [\![f]\!]_e \cap [\![g]\!]_e & [\![f \vee g]\!]_e & = [\![f]\!]_e \cup [\![g]\!]_e \\
[\![[a]f]\!]_e & = \{s \mid \forall t.\ s \xrightarrow{a} t \ \Rightarrow\ t \in [\![f]\!]_e\} & [\![\langle a\rangle f]\!]_e & = \{s \mid \exists t.\ s \xrightarrow{a} t \wedge t \in [\![f]\!]_e\} \\
[\![\nu X.f]\!]_e & = gfp(Z \mapsto [\![f]\!]_{e[X:=Z]}) & [\![\mu X.f]\!]_e & = lfp(Z \mapsto [\![f]\!]_{e[X:=Z]})
\end{array}
$$

The semantics immediately gives rise to a naive algorithm for model checking μ-calculus (compute lfp and gfp by iteration).

Outline

Examples

- Not a μ-calculus formula: $\mu X.\neg X$
- Semantically monotone but not syntactically monotone: $\mu X.\neg X \vee \mathsf{true}$
- Let *Act* $= \{a\}$:
  - E G f .... $\nu X.f \wedge \langle a \rangle X$
  - E $[f$ U $g]$ .... $\mu X.g \vee (f \wedge \langle a \rangle X)$
  - Every p is inevitably followed by a q: $\nu X_1.\ \Big((p \Rightarrow (\mu X_2.\ q \vee [a]X_2)) \wedge [a]X_1 \Big)$
- Special case: X_1 does not occur within the scope of μX_2.
- The last formula can therefore be evaluated "inside-out":

$$\begin{array}{rcl} & X_2^0 & = & \mathsf{false} \\ & X_2^1 & = & q \vee [a]X_2^0 \\ & X_2^2 & = & q \vee [a]X_2^1 \\ \ldots & X_2^\omega & = & q \vee [a]X_2^\omega \end{array} \quad \Longrightarrow \quad \begin{array}{rcl} & X_1^0 & = & \mathsf{true} \\ & X_1^1 & = & (p \Rightarrow X_2^\omega) \wedge [a]X_1^0 \\ & X_1^2 & = & (p \Rightarrow X_2^\omega) \wedge [a]X_1^1 \\ \ldots & X_1^\omega & = & (p \Rightarrow X_2^\omega) \wedge [a]X_1^\omega \end{array}$$

Examples

A more difficult case

- On some path, h holds infinitely often: $\nu X_1.\ \langle a\rangle\big(\mu X_2.\ (X_1 \wedge h) \vee \langle a\rangle X_2\big)$
- Problem: the inner fixed point depends crucially on X_1.

$$
\begin{array}{lllll}
 & X_1^0 & = & \text{true} & \\
 & & & & X_2^{00} = \text{false} \\
 & & & & X_2^{01} = (X_1^0 \wedge h) \vee \langle a\rangle X_2^{00} \\
 & & & & X_2^{02} = (X_1^0 \wedge h) \vee \langle a\rangle X_2^{01} \\
 & & & \ldots & X_2^{0\omega} = (X_1^0 \wedge h) \vee \langle a\rangle X_2^{0\omega} \\
 & X_1^1 & = & \langle a\rangle X_2^{0\omega} & \\
 & & & & X_2^{10} = \text{false} \\
 & & & & X_2^{11} = (X_1^1 \wedge h) \vee \langle a\rangle X_2^{10} \\
 & & & \ldots & X_2^{1\omega} = (X_1^1 \wedge h) \vee \langle a\rangle X_2^{1\omega} \\
 & X_1^2 & = & \langle a\rangle X_2^{1\omega} & \\
\ldots & X_1^\omega & = & \langle a\rangle X_2^{\omega\omega} &
\end{array}
$$

Outline

Complexity

Complexity of naive μ-Calculus algorithm

- We check formula f with at most k nested fixed points on the Kripke Structure $M = \langle S, R, \mathit{Act}, L \rangle$.
- In the previous example:
  - The outermost (greatest) fixed point can decrease at most $|S|$ times (recall that S is finite)
  - In total, the innermost fixed point of formula f is evaluated at most $|S|^2$ times.
- In general: the innermost fixed point of formula f is evaluated at most $|S|^k$ times.
- Each iteration requires up to $|M| \times |f|$ steps.
- Total time complexity of naive algorithm: $\mathcal{O}((|S| + |R|) \times |f| \times |S|^k)$.

A more careful analysis will yield a more optimal treatment for nested fixed points of the same type.

Complexity

- A μ-calculus formula is in positive normal form if negations occur only before propositions.
- To transform a formula into positive normal form, negations can be pushed inside using logical dualities:

$\neg\neg f$	$\mapsto$	f
$\neg(f \vee g)$	$\mapsto$	$(\neg f) \wedge (\neg g)$
$\neg(f \wedge g)$	$\mapsto$	$(\neg f) \vee (\neg g)$
$\neg([a]f)$	$\mapsto$	$\langle a\rangle(\neg f)$
$\neg(\langle a\rangle f)$	$\mapsto$	$[a](\neg f)$
$\neg(\mu X.f(X))$	$\mapsto$	$\nu X.\neg f(\neg X)$
$\neg(\nu X.f(X))$	$\mapsto$	$\mu X.\neg f(\neg X)$

- Due to syntactic monotonicity, single negations in front of formal variables cannot arise.
- Hence, the result is a positive normal form.
- Check: the result is logically equivalent.

Complexity

The complexity of a μ-calculus formula depends on the fixed points (*analogue:* the complexity of first-order formulae depends on the universal/existential quantifiers)

- Basic idea: find a syntactic complexity measure that approaches the semantic complexity
- Nesting Depth:
  maximum number of nested fixed points in a positive normal form

$$
\begin{array}{rcll}
ND(f) & := & 0 & \text{for } f \in \{p, \neg p, X\} \\
ND(ⓐ f) & := & ND(f) & \text{for } ⓐ \in \{[a], \langle a \rangle\} \\
ND(f \Box g) & := & max(ND(f), ND(g)) & \text{for } \Box \in \{\wedge, \vee\} \\
ND({}^{\mu}_{\nu}\, X.f) & := & 1 + ND(f) & \text{for } {}^{\mu}_{\nu} \in \{\mu, \nu\}
\end{array}
$$

- Example: $ND\Big((\mu X_1.\ \nu X_2.\ X_1 \vee X_2) \wedge (\mu X_3.\ \mu X_4.\ (X_3 \wedge \mu X_5.\ p \vee X_5))\Big)$

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- Example: $ND\Big((\mu X_1.\ \nu X_2.\ X_1 \vee X_2) \wedge (\mu X_3.\ \mu X_4.\ (X_3 \wedge \mu X_5.\ p \vee X_5))\Big) = 3$
- X_3, X_4 and X_5 have no alternation between fixed point signs

Complexity

- Capture alternation
- Alternation Depth: number of alternating fixed points of a formula in positive normal form.

$AD(f)$	$:=$	0	for $f \in \{p, \neg p, X\}$
$AD(ⓐ f)$	$:=$	$AD(f)$	for $ⓐ \in \{[a], \langle a \rangle\}$
$AD(f \square g)$	$:=$	$max(AD(f), AD(g))$	for $\square \in \{\wedge, \vee\}$
$AD(\mu X.f)$	$:=$	$1 + max\{AD(g) \mid g$ is a ν-subformula of $f\}$	
$AD(\nu X.f)$	$:=$	$1 + max\{AD(g) \mid g$ is a μ-subformula of $f\}$	

- Examples:

$$AD\Big((\mu X_1.\ \nu X_2.\ X_1 \vee X_2) \wedge (\mu X_3.\mu X_4.\ (X_3 \wedge \mu X_5.p \vee X_5))\Big)$$

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\end{array}
$$

- Examples:

$$AD\Big((\mu X_1.\ \nu X_2.\ X_1 \vee X_2) \wedge (\mu X_3.\mu X_4.\ (X_3 \wedge \mu X_5.p \vee X_5))\Big) = 2$$

$$AD\Big((\mu X_1.\ \nu X_2.\ X_1 \vee X_2) \wedge (\mu X_3.\nu X_4.\ (X_3 \wedge \mu X_5.p \vee X_5))\Big) = 3$$

- X_5 does not depend on X_3 and X_4

Complexity

- Dependent Alternation Depth (dAD): number of alternating fixed points, such that the innermost fixed point depends on the outermost.
- The definition of dAD is identical to AD, except for

$$
\begin{array}{lcl}
dAD(\mu X.f) & := & max(dAD(f), \\
 & & \quad 1 + max\{dAD(g) \mid \\
 & & \qquad g \text{ is a } \nu\text{-subformula of } f \text{ and } X \text{ occurs in } g\} \\
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- Examples:

$$dAD\Big((\mu X_1.\ \nu X_2.\ X_1 \vee X_2) \wedge (\mu X_3.\mu X_4.\ (X_3 \wedge \mu X_5.p \vee X_5))\Big) = 2$$

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Outline

Emerson-Lei Algorithm

- Given a finite set S and a monotonic $\tau : 2^S \rightarrow 2^S$ in the partial order $(2^S, \subseteq)$.
- We used to compute the least fixed point from $\emptyset$:

$$\emptyset \subseteq \tau(\emptyset) \subseteq \tau^2(\emptyset) \subseteq \ldots \subseteq \tau^i(\emptyset) = \tau^{i+1}(\emptyset)$$

  then $\mu X.\tau(X) = \tau^i(\emptyset)$

- Actually, instead of $\emptyset$, we can start in any set known to be smaller than the fixed point:

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  then $\mu X.\tau(X) = \tau^i(\emptyset)$
- Actually, instead of $\emptyset$, we can start in any set known to be smaller than the fixed point:
  - Assume $W \subseteq \mu X.\tau(X)$, so we have:

$$\emptyset \subseteq W \subseteq \tau^i(\emptyset)$$

  - By monotonicity and the definition of fixed points:

$$\tau^i(\emptyset) \subseteq \tau^i(W) \subseteq \tau^{2i}(\emptyset) = \tau^i(\emptyset)$$

  - So if $W \subseteq \mu X.\tau(X)$ we compute the least fixed point as:

$$W, \tau(W), \tau^2(W), \ldots, \tau^j(W) = \tau^{j+1}(W)$$

    This converges at some $j \leq i$ (may be $j < i$)

Emerson-Lei Algorithm

- The observations on the previous slide can speed up computations of nested fixed points.
- Consider two nested μ-fixed points: $\mu X_1.f(X_1, \mu X_2.\ g(X_1, X_2))$
- Start approximation of X_1 and X_2 with $X_1^0 = X_2^0 = \text{false}$:

$$
\begin{array}{lllll}
X_1^0 & = \text{false} & & & \\
 & & & X_2^{00} & = \text{false} \\
 & & & X_2^{01} & = g(X_1^0, X_2^{00}) \\
 & & \ldots & X_2^{0\omega} & = g(X_1^0, X_2^{0\omega}) \\
X_1^1 & = f(X_1^0, X_2^{0\omega}) & & &
\end{array}
$$

- Clearly, $X_1^0 \subseteq X_1^1$, so also $X_2^{0\omega} = \mu X_2.g(X_1^0, X_2) \subseteq \mu X_2.g(X_1^1, X_2) = X_2^{1\omega}$. So, approximating X_2 can start at $X_2^{0\omega}$ instead of at false:

$$
\begin{array}{lllll}
 & & & X_2^{10} & = X_2^{0\omega} \\
 & & \ldots & X_2^{1\omega} & = g(X_1^1, X_2^{1\omega}) \\
X_1^2 & = f(X_1^1, X_2^{1\omega}) & & &
\end{array}
$$

Emerson-Lei Algorithm

Given:

- Mixed Kripke Structure: $M = \langle S, R, \mathit{Act}, L \rangle$
- A μ-Calculus formula f and an environment e

Returns: $[\![f]\!]_e$, the set of states in S where f holds.

Idea:

- The function eval(f) proceeds by recursion on f, using iteration for the fixed points.
- The value of the current approximation for variable X_i is stored in array $A[i]$, in order to reuse it in later iterations.
- Reset $A[i]$ only if:
  - a higher X_j of different sign changed, and
  - $^{\mu}_{\nu}\ X_i.f$ contains free variables.

Emerson-Lei algorithm

Initialisation:

for all variables X_i **do**
  if X_i is bound by a μ **then** $A[i] :=$ false;
  else if X_i is bound by a ν **then** $A[i] :=$ true;
  else $A[i] := e(X_i)$
  end if
end for

Emerson-Lei algorithm

function eval(f)
  if $f = X_i$ **then return** $A[i]$
  else if $f = g_1 \vee g_2$ **then return** eval(g_1) $\cup$ eval(g_2)
  else if $\ldots$ **then** $\ldots$
  else if $f = \mu X_i.g(X_i)$ **then**
    if the surrounding binder of f is a ν **then**
      for all open subformulae of f of the form $\mu X_k.g$ **do** $A[k] := false$
      end for
    end if
    repeat
      $X_{old} := A[i]$; {continue from previous value}
      $A[i] :=$ eval(g);
    until $A[i] = X_{old}$
    return $A[i]$
  end if
end function

Emerson-Lei algorithm

Given a formula $\nu X_1.\nu X_2.\mu X_3.\mu X_4.(X_1 \vee X_2 \vee (\mu X_5.X_5 \wedge p))$

- When computing νX_2, μX_4 and μX_5: no reset is needed because the surrounding binder has the same sign.
- When computing X_3:
  - Reset X_3, X_4: their subformula contains X_1 and X_2 as free variables
  - Do not reset X_5: the subformula $(\mu X_5.X_5 \wedge p)$ is closed

Modifications with respect to the book (p. 105):

- We identified e and $A[i]$ (they play the same role)
- The restriction to reset open formulae only makes the algorithm more efficient. This is essential for CTL (see later).
- The book is wrong: the reset of $A[j]$ should occur *within* the repeat-until loop. It resets the wrong fixed points. We went back to the original Emerson and Lei algorithm (1986).

Emerson-Lei algorithm

Complexity analysis

- Let formula f be given, with dependent alternation depth $dAD(f) = d$.
- Let the Kripke Structure be $\langle S, \mathit{Act}, R, L\rangle$.
- Take a block of fixed points of the same type:
  - its length is at most $|f|$.
  - the value of each fixed point in it can grow/shrink at most $|S|$ times.
- In total, the innermost block will have no more than $(|f| \cdot |S|)^d$ iterations of the repeat-loop.
- Each iteration requires time at most $\mathcal{O}(|f| \cdot (|S| + |R|))$.
- Hence: the overall complexity of the Emerson-Lei algorithm is $\mathcal{O}(|f| \cdot (|S| + |R|) \cdot (|f| \cdot |S|)^d)$

Outline

Embedding CTL-formulae

Again, assume $Act = \{a\}$. Given the fixed point characterisation of CTL, there is a straightforward translation of CTL to the μ-calculus:

- $Tr(p) = p$
- $Tr(\neg f) = \neg Tr(f)$
- $Tr(f \wedge g) = Tr(f) \wedge Tr(g)$
- $Tr(\mathsf{E}\ \mathsf{X}\ f) = \langle a \rangle\ Tr(f)$
- $Tr(\mathsf{E}\ \mathsf{G}\ f) = \nu Y.(Tr(f) \wedge \langle a \rangle\ Y)$
- $Tr(\mathsf{E}\ [f\ \mathsf{U}\ g]) = \mu Y.(Tr(g) \vee (Tr(f) \wedge \langle a \rangle\ Y))$

Note:

- $Tr(f)$ is syntactically monotone
- $Tr(f)$ is a closed μ-calculus formula
- $dAD(Tr(f)) \leq 1$, which is called the alternation free fragment of the μ-calculus
- $AD(Tr(f))$ is not bounded!

Outline

Conclusions

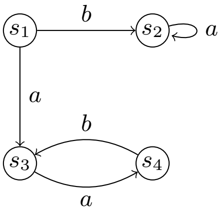
- the μ-calculus incorporates least and greatest fixed points directly in the logic.
- the naive algorithm is exponential in the nesting depth of fixed points.
- a careful analysis leads to an algorithm which is exponential in the (dependent) alternation depth only,
- Hence: alternation free μ-calculus is linear in the Kripke Structure and polynomial in the formula.
- CTL translates into the alternation free fragment of the μ-calculus.
- for the latter we essentially needed the dependent alternation depth.
- fairness constraints typically lead to one extra alternation ($dAD(f) = 2$)

Outline

Exercise

Consider the following μ-calculus formula ϕ and LTS $\mathcal{L}$:

$$\phi := \nu X.\Big([a]X \wedge \nu Y.\mu Z.(\langle b\rangle Y \vee \langle a\rangle Z)\Big)$$



- Compute the set of states where ϕ holds with the naive algorithm (give all intermediate approximations).
- Compute the set of states where ϕ holds with the Emerson-Lei's algorithm (give all intermediate approximations).
- Explain in natural language the meaning of formula ϕ.