Parity games

Material: Chapter 3 of "An experimental study of algorithms and optimisations for parity games, with an application to Boolean Equation Systems", MSc thesis, July 2009, Jeroen Keiren

Jeroen J.A. Keiren
jkeiren@win.tue.nl
http://www.win.tue.nl/~jkeiren
HG 6.81

Department of Mathematics and Computer Science
Technische Universiteit Eindhoven

Why parity games?

- BES can blow up exponentially (see e.g. Mader, section 6.4.2)
- Semantics of BES hard to understand
- Alternative model:
  - additional insights
  - other algorithms
  - graph model more intuitive and easier to understand
- Algorithms still exponential

Parity games

- Parity games are graph games played by two players, $Even$ and Odd;
- Each vertex v is owned by either player $Even$ or player Odd;
- Each vertex v is assigned integer priority $p(v)$;
- A player does a step in the game if a token is on a vertex owned by that player;
- A play (denoted π) is an infinite sequence of steps.

Definition (Parity game)

A parity game Γ is a four tuple $(V, E, p, (V_{Even}, V_{Odd}))$, where (V, E) is a directed graph, with vertices V and total edge relation E, $p : V \to \mathbb{N}$ is a priority function, and (V_{Even}, V_{Odd}) is a partitioning of V.

Example of a parity game

Example

Winning a parity game

Definition (Winner)

Let $\pi = v_1 v_2 v_3 \ldots$ be a play, and let $\inf(\pi)$ denote the set of priorities occurring infinitely often in π. Play π is winning for player *Even* iff $\min(\inf(\pi))$ is even.

Example

- Play $s_1 s_2^{\omega}$ won by player ???;
- Play $(s_1 s_2 s_1 s_3)^{\omega}$ won by player ???.

Winning a parity game

Definition (Winner)

Let $\pi = v_1 v_2 v_3 \ldots$ be a play, and let $\inf(\pi)$ denote the set of priorities occurring infinitely often in π. Play π is winning for player $Even$ iff $\min(\inf(\pi))$ is even.

Example

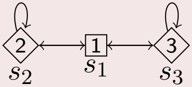
- Play $s_1 s_2^\omega$ won by player $Even$;
- Play $(s_1 s_2 s_1 s_3)^\omega$ won by player ???.

Winning a parity game

Definition (Winner)

Let $\pi = v_1 v_2 v_3 \dots$ be a play, and let $\inf(\pi)$ denote the set of priorities occurring infinitely often in π. Play π is winning for player *Even* iff $\min(\inf(\pi))$ is even.

Example



Legend: *Odd* □ *Even* ◇

- Play $s_1 s_2^\omega$ won by player *Even*;
- Play $(s_1 s_2 s_1 s_3)^\omega$ won by player *Odd*.

Strategies

Definition (Strategy)

A strategy for $Player$ is a partial function $\psi_{Player}{:}V^*V \to V$ that decides the vertex the token is played to based on the history of the vertices that has veen visited.

- A play $\pi = v_1v_2v_3\ldots$ is consistent with strategy ψ_{Player} for $Player$ iff every $v_i \in \pi$ such that $v_i \in V_{Player}$ is immediately followed by $v_{i+1} = \psi_{Player}(v_1 \ldots v_i)$.

Definition (Winning)

Strategy ψ_{Player} is a winning strategy for $Player$ from set $W \subseteq V$ if every play starting from a vertex in W, consistent with ψ_{Player} is winning for $Player$.

Memoryless strategies

Theorem

For finding winning strategies it suffices to look at history free (also memoryless) strategies, i.e. strategies $\psi_{Player}\colon V \rightarrow V$ *in which vertex* v_i *always gets the same successor* v_{i+1}*, independend of the path by which* v_i *is reached.*

Example: Memoryless strategy

Example (Strategy)

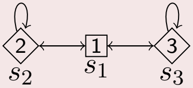
Let $\psi_{Even}(s_2) = s_2, \psi_{Even}(s_3) = s_1$, and $\psi_{Odd}(s_1) = s_3$. Then:

- ψ_{Even} is winning from ???, and
- ψ_{Odd} is winning from ???.

Observe that ψ_{Even} and ψ_{Odd} are memoryless.

Example: Memoryless strategy

Example (Strategy)



Legend: *Odd* □ *Even* ◇

Let $\psi_{Even}(s_2) = s_2, \psi_{Even}(s_3) = s_1$, and $\psi_{Odd}(s_1) = s_3$. Then:

- ψ_{Even} is winning from $\{s_2\}$, and
- ψ_{Odd} is winning from $\{s_1, s_3\}$.

Observe that ψ_{Even} and ψ_{Odd} are memoryless.

Boolean Equation Systems

Recall the definition of BES:

Definition (Boolean Equation System)

Given a set Var of propositional variables. A Boolean Expression is defined by:

$$f ::= X \mid \text{true} \mid \text{false} \mid f \wedge f \mid f \vee f$$

A Boolean Equation is an equation of the form $\sigma X = f$, where $X \in Var$, $\sigma \in \{\mu, \nu\}$ and f is a Boolean Expression.
A Boolean Equation System is a sequence of Boolean Equations:

$$\mathcal{E} ::= \varepsilon \mid (\sigma X = f)\mathcal{E}$$

Notation

- Lowest rank alway 0 or 1;
- rank(X) indicates in which block of like-signed equations X occurs;
- rank(X) is odd iff X is defined in a μ-equation;
- rank(X) inductively defined on structure of BES;
- op(X) indicates top-level boolean operator of equation for X.

Example (Rank/Op)

$$
\begin{array}{lcl}
\mu X & = & X \wedge (Y \vee Z) \\
\nu Y & = & W \vee (X \wedge Y) \\
\mu Z & = & \text{false} \\
\mu W & = & Z \vee (Z \vee W)
\end{array}
$$

Notation

- Lowest rank alway 0 or 1;
- rank(X) indicates in which block of like-signed equations X occurs;
- rank(X) is odd iff X is defined in a μ-equation;
- rank(X) inductively defined on structure of BES;
- op(X) indicates top-level boolean operator of equation for X.

Example (Rank/Op)

rank(_)	op(_)			
(1)	$\wedge$	μX	$=$	$X \wedge (Y \vee Z)$
(2)	$\vee$	νY	$=$	$W \vee (X \wedge Y)$
(3)	$\perp$	μZ	$=$	false
(3)	$\vee$	μW	$=$	$Z \vee (Z \vee W)$

Boolean Equation Systems in SRF

We say that a BES is in Standard Recursive Form (SRF) if all right hand sides of Boolean Equations adhere to the following syntax:

$$f := X \mid \bigvee F \mid \bigwedge F$$

- X is a proposition variable;
- F is a non-empty set of proposition variables.

Observe that:

- All BESs can be transformed into an equivalent BES in SRF;
- This transformation can be done in polynomial time.

Transformation to SRF

Example

Consider the following BES:

$$\begin{array}{lcl} \mu X & = & X \wedge (Y \vee Z) \\ \nu Y & = & W \vee (X \wedge Y) \\ \mu Z & = & \mathsf{false} \\ \mu W & = & Z \vee (Z \vee W) \end{array}$$

Transformation to SRF

Example

Consider the following BES:

$$\begin{array}{lcl} \mu X & = & X \wedge (Y \vee Z) \\ \nu Y & = & W \vee (X \wedge Y) \\ \mu Z & = & \mathsf{false} \\ \mu W & = & Z \vee (Z \vee W) \end{array}$$

This corresponds to the following BES in SRF:

$$\begin{array}{lcl} \mu X & = & X \wedge X' \\ \mu X' & = & Y \vee Z \\ \nu Y & = & W \vee Y' \\ \nu Y' & = & X \wedge Y \\ \mu Z & = & Z' \\ \mu Z' & = & Z' \\ \mu W & = & Z \vee (Z \vee W) \end{array}$$

Correspondence of parity game to BES

Parity games correspond to closed BESs in SRF.

Definition (Parity game to BES)

Let $(V, E, p, (V_{Even}, V_{Odd}))$ be a parity game. Construct the corresponding closed BES in SRF as follows:

$$\begin{cases} \sigma X_v = \bigwedge\{X_{v'} \mid (v, v') \in E\} & \text{if } v \in V_{Odd} \\ \sigma X_v = \bigvee\{X_{v'} \mid (v, v') \in E\} & \text{if } v \in V_{Even} \end{cases}$$

where $\sigma = \mu$ if $p(v)$ is odd, and $\sigma = \nu$ otherwise. Note that for vertices v and u with $p(v) < p(u)$ it holds that X_v occurs before X_u in the BES.

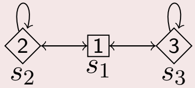
Parity game to BES example

Example (Parity game to BES)

Corresponds to the following BES:

Parity game to BES example

Example (Parity game to BES)



Legend: *Odd* □ *Even* ◇

Corresponds to the following BES:

$$
\begin{aligned}
\mu X_{s_1} &= X_{s_2} \wedge X_{s_3} \\
\nu X_{s_2} &= X_{s_2} \vee X_{s_1} \\
\mu X_{s_3} &= X_{s_1} \vee X_{s_3}
\end{aligned}
$$

Dependency graph of a BES in SRF

Let $\mathcal{E}$ be a BES, then

- bnd($\mathcal{E}$) are all variables occurring at the lhs of an equation in $\mathcal{E}$;
- occ($\mathcal{E}$) are all variables occurring at the rhs of an equation in $\mathcal{E}$;
- occ(f) are all variables occurring in formula f.

Definition (Dependency graph)

Let $\mathcal{E}$ be a BES. Its dependency graph $\mathcal{G}_{\mathcal{E}}$ is defined as (V, E), where:

- $V = \text{bnd}(\mathcal{E})$
- $(X, Y) \in E$ iff there is $\sigma X = f$ in $\mathcal{E}$ with $Y \in \text{occ}(f)$

Dependency graph example

Example (Dependency graph)

Consider the following BES:

$$
\begin{array}{lcl}
\mu X & = & X \wedge X' \\
\mu X' & = & Y \vee Z \\
\nu Y & = & W \vee Y' \\
\nu Y' & = & X \wedge Y \\
\mu Z & = & Z' \\
\mu Z' & = & Z' \\
\mu W & = & Z \vee (Z \vee W)
\end{array}
$$

Its dependency graph is:

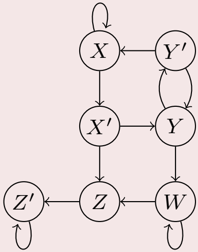
Dependency graph example

Example (Dependency graph)

Consider the following BES:

$$\begin{array}{lcl} \mu X & = & X \wedge X' \\ \mu X' & = & Y \vee Z \\ \nu Y & = & W \vee Y' \\ \nu Y' & = & X \wedge Y \\ \mu Z & = & Z' \\ \mu Z' & = & Z' \\ \mu W & = & Z \vee (Z \vee W) \end{array}$$

Its dependency graph is:



Correspondence of BES to parity game

Closed BESs in SRF correspond to parity games.

Definition (BES to parity game)

Let $\mathcal{E}$ be a closed BES in SRF. This corresponds to the parity game $\Gamma = (V, E, p, (V_{Even}, V_{Odd}))$, where

- (V, E) is the dependency graph $\mathcal{G}_\mathcal{E}$ of $\mathcal{E}$,
- $p(X) = \mathsf{rank}(X)$ for all variables $X \in \mathsf{bnd}(E)$,
- $V_{Odd} = \{X \mid \mathsf{op}(X) = \wedge\}$, so all conjunctive equations are assigned to V_{Odd}, and
- $V_{Even} = V \setminus V_{Odd}$, all other equations are assigned to V_{Even}.

BES to Parity game example

Example (BES to parity game)

Consider the following BES:

$$
\begin{array}{lcl}
\mu X & = & X \wedge X' \\
\mu X' & = & Y \vee Z \\
\nu Y & = & W \vee Y' \\
\nu Y' & = & X \wedge Y \\
\mu Z & = & Z' \\
\mu Z' & = & Z' \\
\mu W & = & Z \vee (Z \vee W)
\end{array}
$$

Its parity game is:

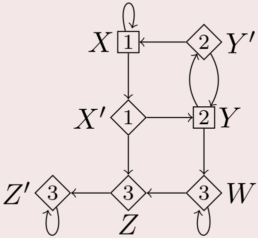
BES to Parity game example

Example (BES to parity game)

Consider the following BES:

$$
\begin{array}{lcl}
\mu X & = & X \wedge X' \\
\mu X' & = & Y \vee Z \\
\nu Y & = & W \vee Y' \\
\nu Y' & = & X \wedge Y \\
\mu Z & = & Z' \\
\mu Z' & = & Z' \\
\mu W & = & Z \vee (Z \vee W)
\end{array}
$$

Its parity game is:



Legend: *Odd* □ *Even* ◇

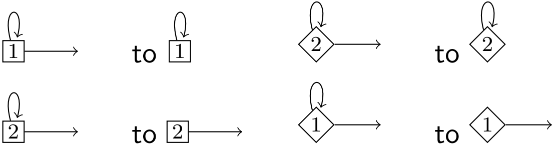
Intuition

Intuitively, the correspondence between BES and parity game is as follows:

- Variable X in BES $\mathcal{E}$ is true, iff player $Even$ has a winning strategy from corresponding vertex V_X in the parity game for $\mathcal{E}$.
- Variable X in BES $\mathcal{E}$ is false, iff player Odd has a winning strategy from corresponding vertex V_X in the parity game for $\mathcal{E}$.

Transformations on parity games

- Self-loop elimination

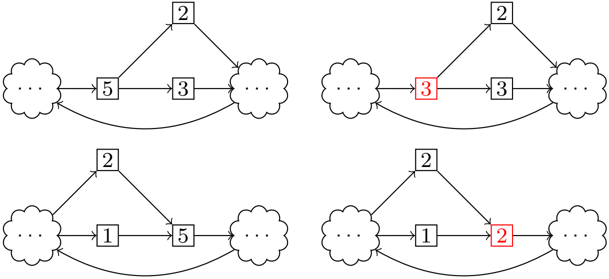


- Priority compaction

In case priority 4 does not occur in the parity game. Evenness must be preserved!

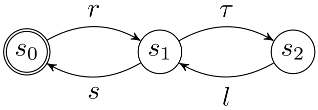
Transformations on parity games

- Priority propagation:



Evenness of priorities may change!

Exercise



Consider the following modal μ-calculus formula f:

$$\phi \equiv \nu X.\mu Y.(([r]X \wedge [s]X \wedge (\nu Z.\langle \overline{s} \rangle Z)) \vee ([r]Y \wedge [s]Y))$$

- Translate the model checking question $M \models f$ to a BES.
- Transform the resulting BES into a parity game.
- Give a winning strategy for player $Even$ from X_{s_0}.